ON A SHOCK PROBLEM INVOLVING A NONLINEAR VISCOELASTIC BAR

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We treat an initial boundary value problem for a nonlinear wave equation $u_{tt} - u_{xx} + K|u|^{\alpha}u + \lambda|u_t|^{\beta}u_t = f(x,t)$ in the domain $0 < x < 1$, $0 < t < T$. The boundary condition at the boundary point $x = 0$ of the domain for a solution u involves a time convolution term of the boundary value of u at $x = 0$, whereas the boundary condition at the other boundary point is of the form $u_x(1,t) + K_1u(1,t) + \lambda_1u_t(1,t) = 0$ with K_1 and λ_1 given nonnegative constants. We prove existence of a unique solution of such a problem in classical Sobolev spaces. The proof is based on a Galerkin-type approximation, various energy estimates, and compactness arguments. In the case of $\alpha = \beta = 0$, the regularity of solutions is studied also. Finally, we obtain an asymptotic expansion of the solution (u,P) of this problem up to order $N+1$ in two small parameters K, λ.

1. Introduction

Given $T > 0$, we consider the problem to find a pair of functions (u,P) such that

$$\begin{gathered}
u_{tt} - u_{xx} + F(u,u_t) = f(x,t), \quad 0 < x < 1,\ 0 < t < T, \\
u_x(0,t) = P(t), \\
u_x(1,t) + K_1u(1,t) + \lambda_1u_t(1,t) = 0, \\
u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x),
\end{gathered} \tag{1.1}$$

where

- $F(u,u_t) = K|u|^{\alpha}u + \lambda|u_t|^{\beta}u_t$,
- u_0, u_1, f are given functions,
- K, K_1, α, β, λ and $\lambda_1 \geq 0$ are given constants

and the unknown function $u(x,t)$ and the unknown boundary value $P(t)$ satisfy the following Cauchy problem for ordinary differential equation

$$\begin{gathered}
P^{//}(t) + \omega^2P(t) = hu_{tt}(0,t), \quad 0 < t < T, \\
P(0) = P_0, \qquad P^{/}(0) = P_1,
\end{gathered} \tag{1.2}$$

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where $\omega > 0$, $h \geq 0$, P_0, P_1 are given constants. Problem (1.1)–(1.2) describes the shock between a solid body and a nonlinear viscoelastic bar resting on a viscoelastic base with nonlinear elastic constraints at the side, constraints associated with a viscous frictional resistance.

In [1], An and Trieu studied a special case of problem (1.1)–(1.2) with $\alpha = \beta = 0$ and f, u_0, u_1 and P_0 vanishing, associated with the homogeneous boundary condition $u(1,t) = 0$ instead of $(1.1)_3$ being a mathematical model describing the shock of a rigid body and a linear visoelastic bar resting on a rigid base.

From (1.2), solving the equation ordinary differential of second order, we get

$$P(t) = g(t) + hu(0,t) - \int_0^t k(t-s)u(0,s)ds, \tag{1.3}$$

where

$$\begin{gathered} g(t) = (P_0 - hu_0(0))\cos\omega t + \frac{1}{\omega}(P_1 - hu_1(0))\sin\omega t, \\ k(t) = h\omega\sin\omega t. \end{gathered} \tag{1.4}$$

This observation motivates to consider problem (1.1) with a more general boundary term of the form

$$P(t) = g(t) + hu(0,t) - \int_0^t k(t-s)u(0,s)ds, \tag{1.5}$$

which we will do henceforth.

In [9, 10], Dinh and Long studied problem $(1.1)_{1,2,4}$ and (1.5) with Dirichlet boundary condition at boundary point $x = 1$ in [10] extending an earlier result of theirs for $k = 0$ in [9].

In [15], Santos has studied the following problem

$$\begin{gathered} u_{tt} - \mu(t)u_{xx} = 0, \quad 0 < x < 1,\ t > 0, \\ u(0,t) = 0, \\ u(1,t) + \int_0^t G(t-s)\mu(s)u_x(1,s)ds = 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \end{gathered} \tag{1.6}$$

The integral in $(1.6)_3$ is a boundary condition which includes the memory effect. Here, by u we denote the displacement and by G the relaxation function. The function $\mu \in W^{1,\infty}_{\text{loc}}(\mathbb{R}_+)$ with $\mu(t) \geq \mu_0 > 0$ and $\mu'(t) \leq 0$ for all $t \geq 0$. Frictional dissipative boundary condition for the wave equation was studied by several authors, see for example [4, 5, 6, 11, 16, 17, 18, 19] and the references therein. In these works, existence of solutions and exponential stabilization were proved for linear and for nonlinear equations. In contrast with the large literature for frictional dissipative, for boundary condition with memory, we have only a few works as for example [12, 13, 14].

Applying the Volterra's inverse operator, Santos [15] transformed $(1.6)_3$ into

$$
\begin{aligned}
-\mu(t)u_x(1,t) = {} & \frac{1}{G(0)}K(t)u_0(1) \\
& + \frac{1}{G(0)}u_t(1,t) + \frac{G'(0)}{G^2(0)}u(1,t) \\
& + \frac{1}{G(0)}\int_0^t K'(t-s)u(1,s)ds,
\end{aligned}
\tag{1.7}
$$

where the resolvent kernel satisfies

$$
K(t) + \frac{1}{G(0)}\int_0^t G'(t-s)K(s)ds = \frac{-1}{G(0)}G'(t). \tag{1.8}
$$

The present paper consists of three main sections. In Section 2, we prove a theorem of global existence and uniqueness of a weak solution u of problem (1.1), (1.5). The proof is based on a Galerkin-type approximation in conjunction with various energy estimates, weak convergence compactness arguments. The main difficulty encountered here is the boundary condition at $x = 1$. In order to solve this particular difficulty, stronger assumptions on the initial conditions u_0 and u_1 will be made. We remark that the linearization method in the papers [3, 8] cannot be used in [2, 9, 10]. In the case of $\alpha = \beta = 0$, Section 3 is devoted to the study of the regularity of the solution u. Finally, in Section 4 we obtain an asymptotic expansion of the solution (u,P) of the problem (1.1), (1.5) up to order $N+1$ in two small parameters K, λ. The results obtained here may be considered as generalizations of those in An and Trieu [1] and in Long and Dinh [2, 3, 8, 9, 10].

2. The existence and uniqueness theorem

Put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$. We omit the definitions of the usual function spaces: $C^m(\overline{\Omega})$, $L^p(\Omega)$ and $W^{m,p}(\Omega)$ and denote $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$ and $H^m = W^{m,2}(\Omega)$, $1 \le p \le \infty$, $m \in IN$. The norm in L^2 is denoted by $\|\cdot\|$. Also, we denote by $\langle\cdot,\cdot\rangle$ the scalar product in L^2 or the dual pairing between continuous linear functionals and elements of a function space, by $\|\cdot\|_X$ the norm of a Banach space X, by X' its dual space, and by $L^p(0,T;X)$, $1 \le p \le \infty$ the Banach space of real measurable functions $u : (0,T) \to X$ such that

$$
\begin{aligned}
\|u\|_{L^p(0,T;X)} &= \left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p} < \infty \quad \text{for } 1 \le p < \infty, \\
\|u\|_{L^\infty(0,T;X)} &= \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \quad \text{for } p = \infty.
\end{aligned}
\tag{2.1}
$$

At last, denote $u(t) = u(x,t)$, $u'(t) = u_t(t) = (\partial u/\partial t)(x,t)$, $u''(t) = u_{tt}(t) = (\partial^2 u/\partial t^2)(x,t)$, $u^{(r)}(t) = (\partial^r u/\partial t^r)(x,t)$, $u_x(t) = (\partial u/\partial x)(x,t)$, $u_{xx}(t) = (\partial^2 u/\partial x^2)(x,t)$.

Further, we make the following assumptions:

(H_0) $\alpha \ge 0$, $\beta \ge 0$, $K \ge 0$, $\lambda \ge 0$,

(H_1) $h \ge 0$, $K_1 \ge 0$, $K_1 + h > 0$ and $\lambda_1 > 0$,

(H_2) $u_0 \in H^2$ and $u_1 \in H^1$,
(H_3) $f,\ f_t \in L^2(0,T;L^2)$,
(H_4) $k \in H^1(0,T) \cap W^{2,1}(0,T)$,
(H_5) $g \in H^2(0,T)$.
Then we have the following theorem.

Theorem 2.1. *Let assumptions* (H_0)–(H_5) *be satisfied. Then there exists a unique weak solution u of problem (1.1), (1.5) such that*

$$\begin{gathered} u \in L^\infty(0,T;H^2), \qquad u_t \in L^\infty(0,T;H^1), \qquad u_{tt} \in L^\infty(0,T;L^2), \\ u(0,\cdot) \in W^{1,\infty}(0,T), \qquad u(1,\cdot) \in H^2(0,T) \cap W^{1,\infty}(0,T), \\ P \in W^{1,\infty}(0,T). \end{gathered} \tag{2.2}$$

Remark 2.2. It follows from (2.2) that the component u in the weak solution (u,P) of problem (1.1), (1.5) satisfies

$$u \in C^0(0,T;H^1) \cap C^1(0,T;L^2) \cap L^\infty(0,T;H^2). \tag{2.3}$$

Proof of Theorem 2.1. The proof consists of Steps 1–5.
Step 1 (Galerkin approximation). Let $\{w_j\}$ be an enumeration of a basis of H^2. We find the approximate solution of problem (1.1), (1.5) in the form

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j, \tag{2.4}$$

where the coefficient functions c_{mj} satisfy the ordinary differential equation problem

$$\begin{gathered} \langle u_m''(t), w_j\rangle + \langle u_{mx}(t), w_{jx}\rangle + P_m(t)w_j(0) + Q_m(t)w_j(1) + \langle F(u_m(t),u_m'(t)), w_j\rangle \\ = \langle f(t), w_j\rangle, \quad 1 \le j \le m, \\ P_m(t) = g(t) + hu_m(0,t) - \int_0^t k(t-s)u_m(0,s)\,ds, \\ Q_m(t) = K_1 u_m(1,t) + \lambda_1 u_m'(1,t), \\ u_m(0) = u_{0m} = \sum_{j=1}^{m} \alpha_{mj} w_j \longrightarrow u_0 \quad \text{strongly in } H^2, \\ u_m'(0) = u_{1m} = \sum_{j=1}^{m} \beta_{mj} w_j \longrightarrow u_1 \quad \text{strongly in } H^1. \end{gathered} \tag{2.5}$$

From the assumptions of Theorem 2.1, this problem has a solution $\{(u_m,P_m,Q_m)\}$ on some interval $[0,T_m]$. The following estimates allow one to take $T_m = T$ for all m.
Step 2 (a priori estimates I). Substituting $(2.5)_{2\text{–}3}$ into $(2.5)_1$, then multiplying the jth equation of $(2.5)_1$ by c_{mj}', summing up with respect to j and afterwards integrating with

respect to the time variable from 0 to t, we get

$$S_m(t) = S_m(0) - 2\int_0^t g(s)u'_m(0,s)ds + 2\int_0^t u'_m(0,s)ds\int_0^s k(s-\tau)u_m(0,\tau)d\tau + 2\int_0^t \langle f(s), u'_m(s)\rangle ds, \tag{2.6}$$

where

$$S_m(t) = \|u'_m(t)\|^2 + \|u_{mx}(t)\|^2 + \frac{2K}{\alpha+2}\|u_m(t)\|_{L^{\alpha+2}}^{\alpha+2} + hu_m^2(0,t) + K_1u_m^2(1,t) + 2\lambda\int_0^t \|u'_m(s)\|_{L^{\beta+2}}^{\beta+2}ds + 2\lambda_1\int_0^t |u'_m(1,s)|^2 ds. \tag{2.7}$$

Using assumptions (H_4)–(H_5) and then integrating by parts with respect to the time variable, we get

$$\begin{aligned} S_m(t) = {} & S_m(0) + 2g(0)u_{0m}(0) - 2g(t)u_m(0,t) + 2\int_0^t g'(s)u_m(0,s)ds \\ & + 2u_m(0,t)\int_0^t k(t-\tau)u_m(0,\tau)d\tau - 2k(0)\int_0^t u_m^2(0,s)ds \\ & - 2\int_0^t u_m(0,s)ds\int_0^s k'(s-\tau)u_m(0,\tau)d\tau + 2\int_0^t \langle f(s), u'_m(s)\rangle ds. \end{aligned} \tag{2.8}$$

Then, using $(2.5)_{4\text{–}5}$ and (2.7) we get

$$S_m(0) + 2|g(0)u_{0m}(0)| \le C_1 \quad \forall m \ge 1, \tag{2.9}$$

where C_1 is a constant independent of m. Using the inequality $2ab \le \varepsilon a^2 + (1/\varepsilon)b^2$ for all $a, b \in \mathbb{R}$ and for all $\varepsilon > 0$, it follows that

$$\begin{aligned} S_m(t) \le {} & C_1 + \frac{1}{\varepsilon}g^2(t) + \varepsilon u_m^2(0,t) + \frac{1}{\varepsilon}\int_0^t |g'(s)|^2 ds + \varepsilon\int_0^t u_m^2(0,s)ds \\ & + \varepsilon u_m^2(0,t) + \frac{1}{\varepsilon}\left|\int_0^t k(t-\tau)u_m(0,\tau)d\tau\right|^2 + 2|k(0)|\int_0^t u_m^2(0,s)ds \\ & + \int_0^t \left[\varepsilon u_m^2(0,s) + \frac{1}{\varepsilon}\left|\int_0^s k'(s-\tau)u_m(0,\tau)d\tau\right|^2\right]ds \\ & + \frac{1}{\varepsilon}\int_0^t \|f(s)\|^2 ds + \varepsilon\int_0^t \|u'_m(s)\|^2 ds \end{aligned}$$

$$= C_1 + \frac{1}{\varepsilon}\left[g^2(t) + \int_0^t |g'(s)|^2 ds + \int_0^t \|f(s)\|^2 ds\right]$$
$$+ 2\varepsilon u_m^2(0,t) + 2(\varepsilon + |k(0)|)\int_0^t u_m^2(0,s)ds$$
$$+ \varepsilon \int_0^t \|u_m'(s)\|^2 ds + \frac{1}{\varepsilon}\left|\int_0^t k(t-\tau)u_m(0,\tau)d\tau\right|^2$$
$$+ \frac{1}{\varepsilon}\int_0^t ds\left|\int_0^s k'(s-\tau)u_m(0,\tau)d\tau\right|^2. \tag{2.10}$$

On the other hand, noticing $K_1 + h > 0$,

$$\|v_x\|^2 + hv^2(0) + K_1 v^2(1) \geq \widetilde{C}\|v\|_{H^1}^2 \quad \forall v \in H^1, \tag{2.11}$$

where $\widetilde{C} > 0$ is a constant depending only on K_1 and h, and on the other hand, by $H^1(\Omega) \hookrightarrow C^0(\overline{\Omega})$, we have

$$\|v\|_{C^0(\overline{\Omega})} \leq C_0\|v\|_{H^1} \quad \forall v \in H^1, \tag{2.12}$$

for some constant $C_0 > 0$. Hence it follows from (2.7) that

$$|u_m(0,t)| \leq \|u_m(t)\|_{C^0(\overline{\Omega})} \leq C_0\|u_m(t)\|_{H^1} \leq \frac{C_0}{\sqrt{\widetilde{C}}}\sqrt{S_m(t)} \equiv \widetilde{C}_0\sqrt{S_m(t)}. \tag{2.13}$$

Now, using the Cauchy-Schwarz inequality, we estimate in the right-hand side of (2.10) the last but one integral as

$$\frac{1}{\varepsilon}\left|\int_0^t k(t-\tau)u_m(0,\tau)d\tau\right|^2 \leq \frac{1}{\varepsilon}\int_0^t k^2(\theta)d\theta\int_0^t u_m^2(0,\tau)d\tau \leq \frac{\widetilde{C}_0^2}{\varepsilon}\int_0^t k^2(\theta)d\theta\int_0^t S_m(\tau)d\tau, \tag{2.14}$$

and the last integral as

$$\begin{aligned} &\frac{1}{\varepsilon}\int_0^t ds\left|\int_0^s k'(s-\tau)u_m(0,\tau)d\tau\right|^2 \\ &\quad \leq \frac{1}{\varepsilon}t\int_0^t |k'(\theta)|^2 d\theta\int_0^t u_m^2(0,\tau)d\tau \leq \frac{\widetilde{C}_0^2}{\varepsilon}t\int_0^t |k'(\theta)|^2 d\theta\int_0^t S_m(\tau)d\tau. \end{aligned} \tag{2.15}$$

Choosing ε so that $0 < 2\varepsilon\widetilde{C}_0^2 \leq 1/2$ and using both these estimates, it follows from (2.10) and (2.13) that

$$S_m(t) \leq G_1(t) + G_2(t)\int_0^t S_m(\tau)d\tau, \tag{2.16}$$

where

$$G_1(t) = 2C_1 + \frac{2}{\varepsilon}\left[g^2(t) + \int_0^t |g'(s)|^2 ds + \int_0^t \|f(s)\|^2 ds\right],$$
$$G_2(t) = 2\varepsilon + 4\widetilde{C}_0^2(\varepsilon + |k(0)|) + \frac{2\widetilde{C}_0^2}{\varepsilon}\left(\int_0^t k^2(\theta)d\theta + t\int_0^t |k'(\theta)|^2 d\theta\right). \tag{2.17}$$

Since $H^1(0,T) \hookrightarrow C^0([0,T])$, from assumptions ($H_3$)–($H_5$) we deduce that

$$|G_i(t)| \leq M_T^{(i)}, \quad \text{a.e. on } t \in [0,T],\ i = 1,2, \tag{2.18}$$

where the constants $M_T^{(i)}$ are depending on T only. Therefore

$$S_m(t) \leq M_T^{(1)} + M_T^{(2)} \int_0^t S_m(\tau)d\tau, \quad 0 \leq t \leq T_m \leq T, \tag{2.19}$$

which implies by Gronwall's lemma

$$S_m(t) \leq M_T^{(1)} \exp\left(tM_T^{(2)}\right) \leq M_T \quad \forall t \in [0,T]. \tag{2.20}$$

Step 3 (a priori estimates II). Now differentiating $(2.5)_1$ with respect to t we get

$$\langle u_m'''(t), w_j\rangle + \langle u_{mx}'(t), w_{jx}\rangle + P_m'(t)w_j(0) + Q_m'(t)w_j(1) + K(\alpha+1)\langle |u_m|^\alpha u_m'(t), w_j\rangle$$
$$+ \lambda(\beta+1)\langle |u_m'(t)|^\beta u_m''(t), w_j\rangle = \langle f'(t), w_j\rangle, \quad \forall 1 \leq j \leq m. \tag{2.21}$$

Multiplying the jth equation herein by c_{mj}'', summing up with respect to j and then integrating with respect to the time variable from 0 to t, after some rearrangements we get

$$\begin{aligned} X_m(t) &= X_m(0) \\ &\quad - 2\int_0^t g'(s)u_m''(0,s)ds + 2\int_0^t \left[k(0)u_m(0,\tau) + \int_0^\tau k'(\tau - s)u_m(0,s)ds\right]u_m''(0,\tau)d\tau \\ &\quad + 2K(\alpha+1)\int_0^t d\tau \int_0^1 |u_m(x,\tau)|^\alpha u_m'(x,\tau)u_m''(x,\tau)dx + 2\int_0^t \langle f'(s), u_m''(s)\rangle ds, \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} X_m(t) &= \|u_m''(t)\|^2 + \|u_{mx}'(t)\|^2 + h|u_m'(0,t)|^2 + K_1|u_m'(1,t)|^2 + 2\lambda_1\int_0^t |u_m''(1,\tau)|^2 d\tau \\ &\quad + 2\lambda(\beta+1)\int_0^t d\tau\int_0^1 |u_m'(x,\tau)|^\beta |u_m''(x,\tau)|^2 dx \\ &= \|u_m''(t)\|^2 + \|u_{mx}'(t)\|^2 + h|u_m'(0,t)|^2 + K_1|u_m'(1,t)|^2 + 2\lambda_1\int_0^t |u_m''(1,\tau)|^2 d\tau \\ &\quad + \frac{8\lambda}{(\beta+2)^2}(\beta+1)\int_0^t d\tau \int_0^1 \left|\frac{d}{d\tau}\left(|u_m'(x,\tau)|^{(\beta+2)/2}\right)\right|^2 dx. \end{aligned} \tag{2.23}$$

Integrating by parts in the integrals of the right-hand side of (2.22), we get

$$\begin{aligned} X_m(t) &= X_m(0)+2g'(0)u_{1m}(0)-2g'(t)u'_m(0,t)+2\int_0^t g''(s)u'_m(0,s)ds \\ &\quad +2\left[k(0)u_m(0,t)+\int_0^t k'(t-s)u_m(0,s)ds\right]u'_m(0,t)-2k(0)u_{0m}(0)u_{1m}(0) \\ &\quad -2\int_0^t\left[k(0)u'_m(0,\tau)+k'(0)u_m(0,\tau)+\int_0^\tau k''(\tau-s)u_m(0,s)ds\right]u'_m(0,\tau)d\tau \\ &\quad +2K(\alpha+1)\int_0^t d\tau\int_0^1 |u_m(x,\tau)|^\alpha u'_m(x,\tau)u''_m(x,\tau)dx+2\int_0^t\langle f'(s),u''_m(s)\rangle ds \\ &= X_m(0)+2g'(0)u_{1m}(0)-2k(0)u_{0m}(0)u_{1m}(0)+k'(0)u_{0m}^2(0) \\ &\quad -k'(0)u_m^2(0,t)-2g'(t)u'_m(0,t)+2k(0)u_m(0,t)u'_m(0,t) \\ &\quad +2\int_0^t g''(s)u'_m(0,s)ds-2k(0)\int_0^t|u'_m(0,\tau)|^2d\tau \\ &\quad +2\int_0^t k'(t-s)u_m(0,s)ds\cdot u'_m(0,t)-2\int_0^t u'_m(0,\tau)d\tau\int_0^\tau k''(\tau-s)u_m(0,s)ds \\ &\quad +2K(\alpha+1)\int_0^t d\tau\int_0^1|u_m(x,\tau)|^\alpha u'_m(x,\tau)u''_m(x,\tau)dx+2\int_0^t\langle f'(s),u''_m(s)\rangle ds. \end{aligned} \tag{2.24}$$

First, we deduce from $(2.5)_3$, (2.23) and assumptions (H_4)–(H_5) that

$$|X_m(0)+2g'(0)u_{1m}(0)-2k(0)u_{0m}(0)u_{1m}(0)+k'(0)u_{0m}^2(0)| \le C_2+||u''_m(0)||^2, \tag{2.25}$$

where $C_2>0$ is a constant depending only on u_0, u_1, g, k, K, K_1, h only. But by $(2.5)_{1\text{–}3}$ we have

$$||u''_m(0)||^2-\langle u_{0mxx},u''_m(0)\rangle+\langle F(u_{0m},u_{1m}),u''_m(0)\rangle=\langle f(0),u''_m(0)\rangle. \tag{2.26}$$

Therefore

$$||u''_m(0)||\le ||u_{0mxx}||+||F(u_{0m},u_{1m})||+||f(0)|| \tag{2.27}$$

and by means of $(2.5)_4$ we deduce that

$$||u''_m(0)||\le C_3, \tag{2.28}$$

where $C_3>0$ is a constant depending on u_0, u_1, f, K, λ only.

On the other hand, it follows from (2.11)–(2.13) that

$$||u'_m(t)||_{C^0(\overline{\Omega})}\le C_0||u'_m(t)||_{H^1}\le \widetilde{C}_0\sqrt{X_m(t)}. \tag{2.29}$$

Then, by means of (2.13), (2.20), and (2.29) we deduce that

$$\begin{aligned}
&2K(\alpha+1)\int_0^t d\tau\int_0^1 |u_m(x,\tau)|^\alpha u_m'(x,\tau)u_m''(x,\tau)dx\\
&\quad\le 2K(\alpha+1)\left(\widetilde{C}_0\sqrt{M_T}\right)^\alpha\int_0^t \|u_m'(\tau)\|\,\|u_m''(\tau)\|d\tau\\
&\quad\le 2K(\alpha+1)\widetilde{C}_0\left(\widetilde{C}_0\sqrt{M_T}\right)^\alpha\int_0^t X_m(\tau)d\tau
\end{aligned}\tag{2.30}$$

and from here and (2.22)–(2.28) we obtain

$$\begin{aligned}
X_m(t) &\le C_2+C_3^2+|k'(0)|u_m^2(0,t)+2|g'(t)u_m'(0,t)|+2|k(0)u_m(0,t)u_m'(0,t)|\\
&\quad+2\int_0^t|g''(s)u_m'(0,s)|ds+2|k(0)|\int_0^t|u_m'(0,\tau)|^2d\tau\\
&\quad+2\int_0^t|k'(t-s)u_m(0,s)|ds\cdot|u_m'(0,t)|\\
&\quad+2\int_0^t|u_m'(0,\tau)|d\tau\int_0^\tau|k''(\tau-s)u_m(0,s)|ds\\
&\quad+2K(\alpha+1)\int_0^t d\tau\int_0^1|u_m(x,\tau)|^\alpha|u_m'(x,\tau)u_m''(x,\tau)|dx+2\int_0^t|\langle f'(s),u_m''(s)\rangle|ds\\
&\le C_2+C_3^2+|k'(0)|\widetilde{C}_0^2M_T+2|g'(t)|\widetilde{C}_0\sqrt{X_m(t)}\\
&\quad+2|k(0)|\widetilde{C}_0^2\sqrt{M_T}\sqrt{X_m(t)}+2\widetilde{C}_0\int_0^t|g''(s)|\sqrt{X_m(s)}ds\\
&\quad+2|k(0)|\widetilde{C}_0^2\int_0^t X_m(\tau)d\tau+2\widetilde{C}_0^2\sqrt{M_T}\int_0^t|k'(\theta)|d\theta\sqrt{X_m(t)}\\
&\quad+2\widetilde{C}_0^2\sqrt{M_T}\int_0^t|k''(\theta)|d\theta\int_0^t\sqrt{X_m(\tau)}d\tau\\
&\quad+2K(\alpha+1)\widetilde{C}_0\left(\widetilde{C}_0\sqrt{M_T}\right)^\alpha\int_0^t X_m(\tau)d\tau+\int_0^t\|f'(s)\|^2ds+\int_0^t X_m(s)ds.
\end{aligned}\tag{2.31}$$

We again use the inequality $2ab\le\varepsilon a^2+(1/\varepsilon)b^2\ \forall a,b\in R$, $\forall\varepsilon>0$ with $\varepsilon=(1/4)$. Then it follows that

$$\begin{aligned}
X_m(t) &\le C_2+C_3^2+|k'(0)|\widetilde{C}_0^2M_T+2|g'(t)|\widetilde{C}_0\sqrt{X_m(t)}\\
&\quad+4|k(0)|\widetilde{C}_0^2\sqrt{M_T}\sqrt{X_m(t)}+2\widetilde{C}_0\int_0^t|g''(s)|\sqrt{X_m(s)}ds\\
&\quad+2|k(0)|\widetilde{C}_0^2\int_0^t X_m(\tau)d\tau+2\widetilde{C}_0^2\sqrt{M_T}\int_0^t|k'(\theta)|d\theta\sqrt{X_m(t)}\\
&\quad+\widetilde{C}_0^2\sqrt{M_T}\int_0^t|k''(\theta)|d\theta\int_0^t\sqrt{X_m(\tau)}d\tau+2K(\alpha+1)\widetilde{C}_0\left(\widetilde{C}_0\sqrt{M_T}\right)^\alpha\int_0^t X_m(\tau)d\tau\\
&\quad+\int_0^t\|f'(s)\|^2ds+\int_0^t X_m(s)ds
\end{aligned}$$

$$
\begin{aligned}
&\leq C_2 + C_3^2 + |k'(0)|\widetilde{C}_0^2 M_T + 4(|g'(t)|\widetilde{C}_0)^2 + \frac{1}{4}X_m(t) \\
&\quad + 4k^2(0)\widetilde{C}_0^4 M_T + \frac{1}{4}X_m(t) + \widetilde{C}_0^2 \int_0^t |g''(s)|^2 ds \\
&\quad + \int_0^t X_m(s)ds + 2|k(0)|\widetilde{C}_0^2 \int_0^t X_m(\tau)d\tau + 4\widetilde{C}_0^4 M_T \left(\int_0^t |k'(\theta)|d\theta\right)^2 + \frac{1}{4}X_m(t) \\
&\quad + \widetilde{C}_0^4 M_T t \left(\int_0^t |k''(\theta)|d\theta\right)^2 + \int_0^t X_m(\tau)d\tau + 2K(\alpha+1)\widetilde{C}_0\left(\widetilde{C}_0\sqrt{M_T}\right)^\alpha \int_0^t X_m(\tau)d\tau \\
&\quad + \int_0^t \|f'(s)\|^2 ds + \int_0^t X_m(s)ds.
\end{aligned}
\tag{2.32}
$$

Noting the embedding $H^1(0,T) \hookrightarrow C^0([0,T])$, it follows from assumptions (H_3)–(H_5) that

$$
X_m(t) \leq M_T^{(3)} + M_T^{(4)} \int_0^t X_m(\tau)d\tau \quad \forall t \in [0,T], \tag{2.33}
$$

where

$$
M_T^{(4)} = 12 + 8\widetilde{C}_0^2|k(0)| + 8K(\alpha+1)\widetilde{C}_0\left(\widetilde{C}_0\sqrt{M_T}\right)^\alpha \tag{2.34}
$$

and $M_T^{(3)}$ is a constant depending on T, f, g, k, C_2, C_3, $\widetilde{C}_0$, and M_T only. By Gronwall's lemma we deduce that

$$
X_m(t) \leq M_T^{(3)} \exp\left(tM_T^{(4)}\right) \leq \widetilde{M}_T \quad \forall t \in [0,T]. \tag{2.35}
$$

On the other hand, we deduce from $(2.5)_{2-3}$, (2.7), (2.20), (2.23), and (2.35) that

$$
\begin{gathered}
\|P_m\|_{W^{1,\infty}(0,T)} \leq M_T^{(5)}, \\
\|Q_m\|_{H^1(0,T)} \leq M_T^{(6)}, \\
\left\||u_m'|^\beta u_m'\right\|_{L^{(\beta+2)'}(Q_T)}^{(\beta+2)'} = \|u_m'\|_{L^{\beta+2}(Q_T)}^{\beta+2} \leq M_T^{(7)}, \\
\left\|\frac{\partial}{\partial t}\left(|u_m'|^{(\beta+2)/2}\right)\right\|_{L^2(Q_T)}^2 \leq X_m(t) \leq \widetilde{M}_T,
\end{gathered}
\tag{2.36}
$$

$$
\begin{aligned}
\left\|\frac{\partial}{\partial x}\left(|u_m'|^{(\beta+2)/2}\right)\right\|_{L^2(Q_T)}^2 &= \frac{1}{4}(\beta+2)^2 \int_0^T dt \int_0^1 |u_m'(x,t)|^\beta |u_{mx}'(x,t)|^2 dx \\
&\leq \frac{1}{4}(\beta+2)^2 \int_0^T \left(\widetilde{C}_0\sqrt{X_m(t)}\right)^\beta dt \int_0^1 |u_{mx}'(x,t)|^2 dx \\
&\leq \frac{1}{4}(\beta+2)^2 \int_0^T \left(\widetilde{C}_0\sqrt{X_m(t)}\right)^\beta X_m(t)dt \\
&\leq \frac{1}{4}(\beta+2)^2 T\left(\widetilde{C}_0\sqrt{\widetilde{M}_T}\right)^\beta \widetilde{M}_T \leq M_T^{(8)},
\end{aligned}
\tag{2.37}
$$

for all $T > 0$ and $(\beta+2)' = (\beta+2)/(\beta+1)$.

Step 4 (limiting process). From (2.7), (2.20), (2.23), (2.35), and $(2.36)_{1\text{–}3}$ we deduce the existence of a subsequence of $\{(u_m, P_m, Q_m)\}$, still also so denoted, such that

$$\begin{aligned}
u_m &\longrightarrow u \quad \text{in } L^\infty(0,T;H^1) \text{ weak}*,\\
u'_m &\longrightarrow u' \quad \text{in } L^\infty(0,T;H^1) \text{ weak}*,\\
u'_m &\longrightarrow u' \quad \text{in } L^{\beta+2}(Q_T) \text{ weakly},\\
u''_m &\longrightarrow u'' \quad \text{in } L^\infty(0,T;L^2) \text{ weak}*,\\
u_m(0,\cdot) &\longrightarrow u(0,\cdot) \quad \text{in } W^{1,\infty}(0,T) \text{ weak}*,\\
u_m(1,\cdot) &\longrightarrow u(1,\cdot) \quad \text{in } W^{1,\infty}(0,T) \text{ weak}*,\\
u_m(1,\cdot) &\longrightarrow u(1,\cdot) \quad \text{in } H^2(0,T) \text{ weakly},\\
P_m &\longrightarrow \widetilde{P} \quad \text{in } W^{1,\infty}(0,T) \text{ weak}*,\\
Q_m &\longrightarrow \widetilde{Q} \quad \text{in } H^1(0,T) \text{ weakly},\\
|u'_m|^\beta u'_m &\longrightarrow \chi \quad \text{in } L^{(\beta+2)'}(Q_T) \text{ weakly}.
\end{aligned} \tag{2.38}$$

By the compactness lemma of Lions [7, page 57] we can deduce from $(2.36)_4$, (2.37), and $(2.38)_{1,2,4\text{–}6}$ the existence of a subsequence still denoted by $\{u_m\}$ such that

$$\begin{aligned}
u_m &\longrightarrow u \quad \text{strongly in } L^2(Q_T),\\
u'_m &\longrightarrow u' \quad \text{strongly in } L^2(Q_T),\\
|u'_m|^{(\beta+2)/2} &\longrightarrow \chi_1 \quad \text{strongly in } H^1(Q_T),\\
u_m(0,\cdot) &\longrightarrow u(0,\cdot) \quad \text{strongly in } C^0([0,T]),\\
u_m(1,\cdot) &\longrightarrow u(1,\cdot) \quad \text{strongly in } H^1(0,T),\\
u'_m(1,\cdot) &\longrightarrow u'(1,\cdot) \quad \text{strongly in } C^0([0,T]).
\end{aligned} \tag{2.39}$$

From $(2.5)_{2\text{–}3}$ and $(2.39)_{4\text{–}6}$ we have

$$\begin{aligned}
P_m(t) &\longrightarrow g(t) + hu(0,t) - \int_0^t k(t-s)u(0,s)ds \equiv P(t),\\
Q_m(t) &\longrightarrow K_1 u(1,t) + \lambda_1 u'(1,t) \equiv Q(t)
\end{aligned} \tag{2.40}$$

strongly in $C^0([0,T])$ from where with $(2.38)_{8\text{–}9}$

$$P(t) = \widetilde{P}(t), \qquad Q(t) = \widetilde{Q}(t) \tag{2.41}$$

can be deduced. Using the inequality

$$\big||x|^\alpha x - |y|^\alpha y\big| \le (\alpha+1)R^\alpha |x-y| \quad \forall x,y \in [-R,R], \tag{2.42}$$

for all $R > 0$ and all $\alpha \ge 0$ it follows from (2.13), (2.20), and $(2.39)_1$ that

$$|u_m|^\alpha u_m \longrightarrow |u|^\alpha u \quad \text{strongly in } L^2(Q_T). \tag{2.43}$$

Similarly, we can also obtain from (2.29), (2.35), $(2.39)_2$ and inequality (2.42) with $\alpha = \beta$, that

$$|u_m'|^{\beta} u_m' \longrightarrow |u'|^{\beta} u' \quad \text{strongly in } L^2(Q_T). \tag{2.44}$$

Hence, because of (2.43),

$$F(u_m, u_m') \longrightarrow F(u, u') \quad \text{strongly in } L^2(Q_T). \tag{2.45}$$

Passing to the limit in $(2.5)_{1,4\text{–}5}$, by $(2.38)_{1,2,4}$ and (2.40)–(2.41) and (2.45) we have u satisfying the problem

$$\begin{gathered} \langle u''(t), v\rangle + \langle u_x(t), v_x\rangle + P(t)v(0) + Q(t)v(1) + \langle F(u(t), u'(t)), v\rangle = \langle f(t), v\rangle, \\ u(0) = u_0, \qquad u'(0) = u_1 \end{gathered} \tag{2.46}$$

weak in $L^2(0,T)$ weak, for all $v \in H^1$. On the other hand, we have from (2.18)–(2.20) and assumption (H_3) that

$$u_{xx} = u'' + F(u, u') - f \in L^\infty(0, T; L^2(0,1)). \tag{2.47}$$

Hence $u \in L^\infty(0,T;H^2)$ and the existence proof is completed.
Step 5 (uniqueness of the solution). Let (u_i, P_i), $i = 1,2$ be two weak solutions of problem (1.1), (1.5) such that

$$\begin{gathered} u_i \in L^\infty(0,T;H^2), \qquad u_i' \in L^\infty(0,T;H^1), \qquad u_i'' \in L^\infty(0,T;L^2), \\ u_i(0,\cdot) \in W^{1,\infty}(0,T), \qquad u_i(1,\cdot) \in H^2(0,T) \cap W^{1,\infty}(0,T), \\ P_i \in W^{1,\infty}(0,T). \end{gathered} \tag{2.48}$$

Then (u,P) with $u = u_1 - u_2$ and $P = P_1 - P_2$ satisfies the variational problem

$$\begin{gathered} \langle u''(t), v\rangle + \langle u_x(t), v_x\rangle + P(t)v(0) + Q(t)v(1) + K\langle |u_1|^{\alpha} u_1 - |u_2|^{\alpha} u_2, v\rangle \\ + \lambda\langle |u_1'|^{\beta} u_1' - |u_2'|^{\beta} u_2', v\rangle = 0 \quad \forall v \in H^1, \\ u(0) = u'(0) = 0, \end{gathered} \tag{2.49}$$

where

$$\begin{gathered} P(t) = hu(0,t) - \int_0^t k(t-s)u(0,s)ds, \\ Q(t) = K_1 u(1,t) + \lambda_1 u'(1,t). \end{gathered} \tag{2.50}$$

We take $v = u'$ in $(2.36)_1$, afterwards integrating in t, we get

$$Z(t) = -2K\int_0^t \Big\langle |u_1|^{\alpha} u_1 - |u_2|^{\alpha} u_2, u'\Big\rangle d\tau + 2\int_0^t u'(0,\tau)d\tau \int_0^\tau k(\tau - s)u(0,s)ds, \tag{2.51}$$

where

$$Z(t) = \|u'(t)\|^2 + \|u_x(t)\|^2 + hu^2(0,t) + K_1 u^2(1,t) + 2\lambda_1 \int_0^t |u'(1,s)|^2 ds + 2\lambda \int_0^t \left\langle |u_1'|^\beta u_1' - |u_2'|^\beta u_2', u' \right\rangle d\tau. \tag{2.52}$$

Using inequality (2.42), the first term of the right-hand side of (2.51) can be estimated as

$$\begin{aligned} &2K \left| \int_0^t \left\langle |u_1|^\alpha u_1 - |u_2|^\alpha u_2, u' \right\rangle d\tau \right| \\ &\quad \le 2K(\alpha+1)R^\alpha \int_0^t \|u(\tau)\| \|u'(\tau)\| d\tau \le K(\alpha+1)R^\alpha \int_0^t Z(\tau) d\tau, \end{aligned} \tag{2.53}$$

with $R = \max_{i=1,2} \|u_i\|_{L^\infty(0,T;H^1)}$. Using integration by parts in the last integral of (2.51), we get

$$\begin{aligned} J &\equiv 2\int_0^t u'(0,\tau) d\tau \int_0^\tau k(\tau - s) u(0,s) ds = 2u(0,t) \int_0^t k(t-s) u(0,s) ds \\ &\quad - 2k(0) \int_0^t u^2(0,\tau) d\tau - 2\int_0^t u(0,\tau) d\tau \int_0^\tau k'(\tau - s) u(0,s) ds. \end{aligned} \tag{2.54}$$

On the other hand, it follows from (2.11)-(2.12) and (2.52) that

$$|u(0,t)| \le \|u(t)\|_{C^0(\overline{\Omega})} \le C_0 \|u(t)\|_{H^1} \le \frac{C_0}{\sqrt{\widetilde{C}}} \sqrt{Z(t)} \equiv \widetilde{C}_0 \sqrt{Z(t)}. \tag{2.55}$$

Thus

$$\begin{aligned} |J| &\le 2\widetilde{C}_0^2 \sqrt{Z(t)} \int_0^t |k(t-s)| \sqrt{Z(s)} ds \\ &\quad + 2|k(0)| \widetilde{C}_0^2 \int_0^t Z(\tau) d\tau + 2\widetilde{C}_0^2 \int_0^t \sqrt{Z(\tau)} d\tau \int_0^\tau |k'(\tau - s)| \sqrt{Z(s)} ds \\ &\le \frac{1}{2} Z(t) + 2\widetilde{C}_0^4 \int_0^t k^2(\theta) d\theta \int_0^t Z(s) ds + 2|k(0)| \widetilde{C}_0^2 \int_0^t Z(\tau) d\tau \\ &\quad + 2\widetilde{C}_0^2 \sqrt{t} \left(\int_0^t |k'(\theta)|^2 d\theta \right)^{1/2} \int_0^t Z(s) ds \end{aligned} \tag{2.56}$$

can be deduced. It follows from (2.51) and (2.53)–(2.56) that

$$Z(t) \le m_T \int_0^t Z(s) ds \quad \forall t \in [0,T], \tag{2.57}$$

where

$$m_T = 2K(\alpha+1)R^\alpha + 4\widetilde{C}_0^4 \int_0^T k^2(\theta)d\theta + 4|k(0)|\widetilde{C}_0^2 + 4\widetilde{C}_0^2\sqrt{T}\left(\int_0^T |k'(\theta)|^2 d\theta\right)^{1/2}. \tag{2.58}$$

By Gronwall's lemma, we deduce that $Z \equiv 0$ and Theorem 2.1 is completely proved. □

3. Regularity of solutions

In this section, we study the regularity of solution of problem (1.1), (1.5) corresponding to $\alpha = \beta = 0$. From here, we assume that $(h, K, K_1, \lambda, \lambda_1)$ satisfy assumptions (H_0), (H_1). Henceforth, we will impose the following stronger assumptions:

$(H_1^{[1]})$ $u_0 \in H^3$ and $u_1 \in H^2$,
$(H_2^{[1]})$ f, f_t, $f_{tt} \in L^2(0,T;L^2)$ and $f(\cdot,0) \in H^1$,
$(H_3^{[1]})$ $g \in H^3(0,T)$,
$(H_4^{[1]})$ $k \in H^2(0,T)$.

Formally differentiating problem (1.1) with respect to time and letting $\hat{u} = \hat{u}_t$ and $\hat{P} = P'$ we are led to consider the solution $\hat{u}$ of problem $(\hat{Q})$:

$$\begin{gathered} L\hat{u} \equiv \hat{u}_{tt} - \hat{u}_{xx} + F(\hat{u},\hat{u}_t) = \hat{f}(x,t), \quad (x,t) \in Q_T, \\ \hat{u}_x(0,t) = \hat{P}(t), \\ B_1\hat{u} \equiv \hat{u}_x(1,t) + K_1\hat{u}(1,t) + \lambda_1\hat{u}_t(1,t) = 0, \\ \hat{u}(x,0) = \hat{u}_0(x), \qquad \hat{u}_t(x,0) = \hat{u}_1(x), \\ \hat{P}(t) = \hat{g}(t) + h\hat{u}(0,t) - \int_0^t k(t-s)\hat{u}(0,s)ds, \end{gathered} \tag{3.1}$$

where

$$\begin{gathered} F(u,u_t) = Ku + \lambda u_t, \qquad \hat{f} = f_t, \qquad \hat{g}(t) = g'(t) - k(t)u_0(0), \\ \hat{u}_0 = u_1, \qquad \hat{u}_1 = u_{0xx} - F(u_0,u_1) + f(x,0). \end{gathered} \tag{3.2}$$

Let u_0, u_1, f, g, k satisfy assumptions $(H_1^{[1]})$–$(H_4^{[1]})$. Then $\hat{u}_0$, $\hat{u}_1$, $\hat{f}$, $\hat{g}$, k satisfy assumptions (H_1)–(H_4) and by Theorem 2.1 for problem $(\hat{Q})$ there exists a unique weak solution $(\hat{u},\hat{P})$ such that

$$\begin{gathered} \hat{u} \in C^0(0,T;H^1) \cap C^1(0,T;L^2) \cap L^\infty(0,T;H^2), \\ \hat{u}_t \in L^\infty(0,T;H^1), \qquad \hat{u}_{tt} \in L^\infty(0,T;L^2), \\ \hat{u}(0,\cdot) \in W^{1,\infty}(0,T), \qquad \hat{u}(1,\cdot) \in H^2(0,T) \cap W^{1,\infty}(0,T), \\ \hat{P} \in W^{1,\infty}(0,T). \end{gathered} \tag{3.3}$$

Moreover, from the uniqueness of weak solution we have

$$\hat{u} = u_t, \qquad \hat{P} = P'. \tag{3.4}$$

It follows from (3.3)–(3.4) that

$$\begin{gathered} u \in C^0(0,T;H^2) \cap C^1(0,T;H^1) \cap C^2(0,T;L^2), \\ u_t \in L^\infty(0,T;H^2), \qquad u_{tt} \in L^\infty(0,T;H^1), \qquad u_{ttt} \in L^\infty(0,T;L^2), \\ u(0,\cdot) \in W^{2,\infty}(0,T), \qquad u(1,\cdot) \in H^3(0,T) \cap W^{2,\infty}(0,T), \\ P \in W^{2,\infty}(0,T). \end{gathered} \tag{3.5}$$

We then have the following theorem.

Theorem 3.1. *Let $\alpha = \beta = 0$ and let assumptions (H_0), (H_1) and $(H_1^{[1]})$–$(H_4^{[1]})$ hold. Then there exists a unique weak solution (u,P) of problem (1.1), (1.5) satisfying (3.5).*

Similarly, formally differentiating problem (1.1) with respect to time up to order r and letting $u^{[r]} = \partial^r u/\partial t^r$ and $P^{[r]} = d^r P/dt^r$ we are led to consider the solution $u^{[r]}$ of problem $(Q^{[r]})$:

$$\begin{gathered} Lu^{[r]} = f^{[r]}(x,t), \quad (x,t) \in (0,1)\times(0,T), \\ u_x^{[r]}(0,t) = P^{[r]}(t), \\ B_1 u^{[r]} = 0, \\ u^{[r]}(x,0) = u_0^{[r]}(x), \qquad u_t^{[r]}(x,0) = u_1^{[r]}(x), \\ P^{[r]}(t) = g^{[r]}(t) + hu^{[r]}(0,t) - \int_0^t k(t-s)u^{[r]}(0,s)\,ds, \end{gathered} \tag{3.6}$$

where the functions $u_0^{[r]}$ and $u_1^{[r]}$ are defined by the recurrence formulas

$$\begin{gathered} u_0^{[0]} = u_0, \qquad u_0^{[r]} = u_1^{[r-1]}, \quad r \ge 1, \\ u_1^{[0]} = u_1, \qquad u_1^{[r]} = u_{0xx}^{[r-1]} - F\left(u_0^{[r-1]}, u_1^{[r-1]}\right) + \frac{\partial^{r-1} f}{\partial t^{r-1}}(x,0), \quad r \ge 1, \\ f^{[r]} = \frac{\partial^r f}{\partial t^r}, \\ g^{[0]} = g, \qquad g^{[r]} = \frac{d^r g}{dt^r} - \sum_{\nu=0}^{r-1} u_0^{(r-1-\nu)}(0)\frac{d^\nu k}{dt^\nu}, \quad r \ge 1. \end{gathered} \tag{3.7}$$

Assume that the data u_0, u_1, f, g, k satisfy the following conditions:

$(H_1^{[r]})$ $u_0 \in H^{r+2}$ and $u_1 \in H^{r+1}$,
$(H_2^{[r]})$ $\partial^\nu f/\partial t^\nu \in L^2(0,T;L^2)$, $0 \le \nu \le r+1$, and $(\partial^\mu f/\partial t^\mu)(\cdot,0) \in H^1$, $0 \le \mu \le r-1$,
$(H_3^{[r]})$ $g \in H^{r+2}(0,T)$,
$(H_4^{[r]})$ $k \in H^{r+1}(0,T)$, $r \ge 1$.

Then $u_0^{[r]}$, $u_1^{[r]}$, $f^{[r]}$, $g^{[r]}$, k satisfy (H_1)–(H_4). Applying again Theorem 2.1 for problem $(Q^{[r]})$, there exists a unique weak solution $u^{[r]}$ satisfying (2.2) and the inclusion from

Remark 2.2, that is, such that

$$
\begin{gathered}
u^{[r]} \in C^0(0,T;H^1) \cap C^1(0,T;L^2) \cap L^\infty(0,T;H^2),\\
u_t^{[r]} \in L^\infty(0,T;H^1), \qquad u_{tt}^{[r]} \in L^\infty(0,T;L^2),\\
u^{[r]}(0,\cdot) \in W^{1,\infty}(0,T), \qquad u^{[r]}(1,\cdot) \in H^2(0,T) \cap W^{1,\infty}(0,T),\\
P^{[r]} \in W^{1,\infty}(0,T).
\end{gathered}
\tag{3.8}
$$

Moreover, from the uniqueness of weak solution we have $(u^{[r]},P^{[r]}) = (\partial^r u/\partial t^r, d^r P/dt^r)$. Hence we obtain from (3.8) that

$$
\begin{gathered}
u \in C^{r-1}(0,T;H^2) \cap C^r(0,T;H^1) \cap C^{r+1}(0,T;L^2),\\
u(0,\cdot) \in W^{r+1,\infty}(0,T), \qquad u(1,\cdot) \in H^{r+2}(0,T) \cap W^{r+1,\infty}(0,T),\\
P \in W^{r+1,\infty}(0,T).
\end{gathered}
\tag{3.9}
$$

We then have the following theorem.

Theorem 3.2. *Let $\alpha = \beta = 0$ and let assumptions (H_1) and $(H_1^{[r]})$–$(H_4^{[r]})$ hold. Then there exists a unique weak solution (u,P) of problem (1.1), (1.5) satisfying (3.9) and*

$$
\begin{gathered}
\frac{\partial^r u}{\partial t^r} \in L^\infty(0,T;H^2),\\
\frac{\partial^{r+1} u}{\partial t^{r+1}} \in L^\infty(0,T;H^1),\\
\frac{\partial^{r+2} u}{\partial t^{r+2}} \in L^\infty(0,T;L^2).
\end{gathered}
\tag{3.10}
$$

4. Asymptotic expansion of solutions

In this section, we assume that $\alpha = \beta = 0$ and $(h, K_1, \lambda_1, f, g, k)$ satisfy the assumptions (H_1)–(H_5).

We consider the following perturbed problem $(\tilde{Q}_{K,\lambda})$, where $K \geq 0$, $\lambda \geq 0$ are small parameters:

$$
\begin{gathered}
Lu \equiv u_{tt} - u_{xx} = -Ku - \lambda u_t + f(x,t), \quad 0 < x < 1,\ 0 < t < T,\\
B_0 u \equiv u_x(0,t) = P(t),\\
B_1 u \equiv u_x(1,t) + K_1 u(1,t) + \lambda_1 u_t(1,t) = 0,\\
u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x),\\
P(t) = g(t) + hu(0,t) - \int_0^t k(t-s)u(0,s)ds.
\end{gathered}
\tag{$\tilde{Q}_{K,\lambda}$}
$$

Let $(u_{0,0}, P_{0,0})$ be a unique weak solution of problem $(\widetilde{Q}_{0,0})$ as in Theorem 2.1, corresponding to $(K,\lambda) = (0,0)$, that is,

$$\begin{gathered} Lu_{0,0} = \widetilde{H}_{0,0} \equiv f(x,t), \quad 0 < x < 1,\ 0 < t < T, \\ B_0 u_{0,0} = P_{0,0}(t), \qquad B_1 u_{0,0} = 0, \\ u_{0,0}(x,0) = u_0(x), \qquad u'_{0,0}(x,0) = u_1(x), \\ P_{0,0}(t) = g(t) + hu_{0,0}(0,t) - \int_0^t k(t-s) u_{0,0}(0,s)\,ds, \\ u_{0,0} \in C^0(0,T;H^1) \cap C^1(0,T;L^2) \cap L^\infty(0,T;H^2), \\ u'_{0,0} \in L^\infty(0,T;H^1), \qquad u''_{0,0} \in L^\infty(0,T;L^2), \\ u_{0,0}(0,\cdot) \in W^{1,\infty}(0,T), \qquad u_{0,0}(1,\cdot) \in H^2(0,T) \cap W^{1,\infty}(0,T), \\ P_{0,0} \in W^{1,\infty}(0,T). \end{gathered} \tag{$\widetilde{Q}_{0,0}$}$$

Let us consider the sequence of weak solutions $(u_{\gamma_1,\gamma_2}, P_{\gamma_1,\gamma_2})$, $(\gamma_1,\gamma_2) \in \mathbb{Z}_+^2$, $1 \le \gamma_1 + \gamma_2 \le N$, defined by the following problems:

$$\begin{gathered} Lu_{\gamma_1,\gamma_2} = \widetilde{H}_{\gamma_1,\gamma_2}, \quad 0 < x < 1,\ 0 < t < T, \\ B_0 u_{\gamma_1,\gamma_2} = P_{\gamma_1,\gamma_2}(t), \qquad B_1 u_{\gamma_1,\gamma_2} = 0, \\ u_{\gamma_1,\gamma_2}(x,0) = u'_{\gamma_1,\gamma_2}(x,0) = 0, \\ P_{\gamma_1,\gamma_2}(t) = hu_{\gamma_1,\gamma_2}(0,t) - \int_0^t k(t-s) u_{\gamma_1,\gamma_2}(0,s)\,ds, \\ u_{\gamma_1,\gamma_2} \in C^0(0,T;H^1) \cap C^1(0,T;L^2) \cap L^\infty(0,T;H^2), \\ u'_{\gamma_1,\gamma_2} \in L^\infty(0,T;H^1), \qquad u''_{\gamma_1,\gamma_2} \in L^\infty(0,T;L^2), \\ u_{\gamma_1,\gamma_2}(0,\cdot) \in W^{1,\infty}(0,T), \qquad u_{\gamma_1,\gamma_2}(1,\cdot) \in H^2(0,T) \cap W^{1,\infty}(0,T), \\ P_{\gamma_1,\gamma_2} \in W^{1,\infty}(0,T), \end{gathered} \tag{$\widetilde{Q}_{\gamma 1,\gamma 2}$}$$

where

$$\begin{gathered} \widetilde{H}_{1,0} = -u_{0,0}, \qquad \widetilde{H}_{0,1} = -u'_{0,0}, \\ \widetilde{H}_{\gamma_1,\gamma_2} = -u_{\gamma_1-1,\gamma_2} - u'_{\gamma_1,\gamma_2-1}, \quad (\gamma_1,\gamma_2) \in \mathbb{Z}_+^2,\ 2 \le \gamma_1 + \gamma_2 \le N. \end{gathered} \tag{4.1}$$

Let $(u,P) = (u_{K,\lambda}, P_{K,\lambda})$ be a unique weak solution of problem $(\widetilde{Q}_{K,\lambda})$. Then (v,R), with

$$v = u_{K,\lambda} - \sum_{0 \le \gamma_1+\gamma_2 \le N} u_{\gamma_1,\gamma_2} K^{\gamma_1} \lambda^{\gamma_2}, \qquad R = P_{K,\lambda} - \sum_{0 \le \gamma_1+\gamma_2 \le N} P_{\gamma_1,\gamma_2} K^{\gamma_1} \lambda^{\gamma_2}, \tag{4.2}$$

satisfies the problem

$$\begin{gathered}
Lv = -Kv - \lambda v_t + e_{N,K,\lambda}(x,t), \quad 0 < x < 1,\ 0 < t < T, \\
B_0 v = R(t), \\
B_1 v = 0, \\
v(x,0) = v_t(x,0) = 0, \\
R(t) = hv(0,t) - \int_0^t k(t-s)v(0,s)ds, \\
v \in C^0(0,T;H^1) \cap C^1(0,T;L^2) \cap L^\infty(0,T;H^2), \\
v' \in L^\infty(0,T;H^1), \qquad v'' \in L^\infty(0,T;L^2), \\
v(0,\cdot) \in W^{1,\infty}(0,T), \qquad v(1,\cdot) \in H^2(0,T) \cap W^{1,\infty}(0,T), \\
R \in W^{1,\infty}(0,T),
\end{gathered} \tag{4.3}$$

where

$$e_{N,K,\lambda} = -\sum_{\gamma_1+\gamma_2=N+1} \left(u_{\gamma_1-1,\gamma_2} + u'_{\gamma_1,\gamma_2-1}\right) K^{\gamma_1}\lambda^{\gamma_2}. \tag{4.4}$$

Then, we have the following lemma.

Lemma 4.1. *Let $\alpha = \beta = 0$ and let assumptions (H_1)–(H_5) be satisfied. Then*

$$\|e_{N,K,\lambda}\|_{L^\infty(0,T;L^2)} \le \widetilde{C}_N \left(\sqrt{K^2+\lambda^2}\right)^{N+1}, \tag{4.5}$$

where $\widetilde{C}_N$ is a constant depending only on the constants

$$\|u_{\gamma_1-1,\gamma_2}\|_{L^\infty(0,T;H^1)}, \quad \|u'_{\gamma_1,\gamma_2-1}\|_{L^\infty(0,T;H^1)}, \quad (\gamma_1,\gamma_2) \in \mathbb{Z}_+^2,\ \gamma_1+\gamma_2 = N+1. \tag{4.6}$$

Proof. By the boundedness of the functions u_{γ_1-1,γ_2}, u'_{γ_1,γ_2-1}, $(\gamma_1,\gamma_2) \in \mathbb{Z}_+^2$, $\gamma_1+\gamma_2 = N+1$ in the function space $L^\infty(0,T;H^1)$, we obtain from (4.4), that

$$\|e_{N,K,\lambda}\|_{L^\infty(0,T;L^2)} \le \sum_{\gamma_1+\gamma_2=N+1} \left(\|u_{\gamma_1-1,\gamma_2}\|_{L^\infty(0,T;H^1)} + \|u'_{\gamma_1,\gamma_2-1}\|_{L^\infty(0,T;H^1)}\right) K^{\gamma_1}\lambda^{\gamma_2}. \tag{4.7}$$

On the other hand, using the Hölder's inequality $ab \le (1/p)a^p + (1/q)b^q$, $1/p+1/q = 1$, $\forall a,b \ge 0$, $\forall p,q > 1$ with $a = K^{2\gamma_1/(N+1)}$, $b = \lambda^{2\gamma_2/(N+1)}$, $p = (N+1)/\gamma_1$, $q = (N+1)/\gamma_2$, we obtain

$$K^{\gamma_1}\lambda^{\gamma_2} = \left(K^{2\gamma_1/(N+1)}\lambda^{2\gamma_2/(N+1)}\right)^{(N+1)/2} \le \left(K^2+\lambda^2\right)^{(N+1)/2}, \tag{4.8}$$

for all $(\gamma_1,\gamma_2) \in \mathbb{Z}_+^2$, $\gamma_1+\gamma_2 = N+1$.

Finally, by the estimates (4.7), (4.8), we deduce that (4.5) holds, with

$$\widetilde{C}_N = \sum_{\gamma_1+\gamma_2=N+1} \left(||u_{\gamma_1-1,\gamma_2}||_{L^\infty(0,T;H^1)} + ||u'_{\gamma_1,\gamma_2-1}||_{L^\infty(0,T;H^1)} \right). \tag{4.9}$$

The proof of Lemma 4.1 is completed. □

Next, we obtain the following theorem.

THEOREM 4.2. *Let $\alpha = \beta = 0$ and let assumptions (H_1)–(H_5) be satisfied. Then, for every $K \geq 0$, $\lambda \geq 0$, problem $(\widetilde{Q}_{K,\lambda})$ has a unique weak solution $(u, P) = (u_{K,\lambda}, P_{K,\lambda})$ satisfying the asymptotic estimations up to order $N+1$ as follows*

$$\begin{aligned} &\left\| u'_{K,\lambda} - \sum_{0\leq\gamma_1+\gamma_2\leq N} u'_{\gamma_1,\gamma_2} K^{\gamma_1}\lambda^{\gamma_2} \right\|_{L^\infty(0,T;L^2)} + \left\| u_{K,\lambda} - \sum_{0\leq\gamma_1+\gamma_2\leq N} u_{\gamma_1,\gamma_2} K^{\gamma_1}\lambda^{\gamma_2} \right\|_{L^\infty(0,T;H^1)} \\ &\quad + \left\| u'_{K,\lambda}(1,\cdot) - \sum_{0\leq\gamma_1+\gamma_2\leq N} u'_{\gamma_1,\gamma_2}(1,\cdot) K^{\gamma_1}\lambda^{\gamma_2} \right\|_{L^2(0,T)} \leq \widetilde{C}_N^{*} \left(\sqrt{K^2+\lambda^2} \right)^{N+1}, \end{aligned} \tag{4.10}$$

$$\left\| P_{K,\lambda} - \sum_{0\leq\gamma_1+\gamma_2\leq N} P_{\gamma_1,\gamma_2} K^{\gamma_1}\lambda^{\gamma_2} \right\|_{C^0([0,T])} \leq \widetilde{C}_N^{**} \left(\sqrt{K^2+\lambda^2} \right)^{N+1}, \tag{4.11}$$

for all $K \geq 0$, $\lambda \geq 0$, the functions $(u_{\gamma_1,\gamma_2}, P_{\gamma_1,\gamma_2})$ being the weak solutions of problems $(\widetilde{Q}_{\gamma 1,\gamma 2})$, $(\gamma_1,\gamma_2) \in \mathbb{Z}_+^2$, $\gamma_1+\gamma_2 \leq N$.

Proof. By multiplying the two sides of $(4.3)_1$ with v', and after integration in t, we obtain

$$z(t) = 2\int_0^t \langle e_{N,K,\lambda}, v' \rangle d\tau + 2\int_0^t v'(0,\tau)d\tau \int_0^\tau k(\tau-s)v(0,s)ds, \tag{4.12}$$

where

$$\begin{aligned} z(t) = {} & ||v'(t)||^2 + ||v_x(t)||^2 + hv^2(0,t) + K_1v^2(1,t) + K||v(t)||^2 \\ & + 2\lambda\int_0^t ||v'(\tau)||^2 d\tau + 2\lambda_1 \int_0^t |v'(1,s)|^2 ds. \end{aligned} \tag{4.13}$$

Noting that

$$\begin{aligned} z(t) &\geq ||v'(t)||^2 + ||v_x(t)||^2 + hv^2(0,t) + K_1v^2(1,t) + 2\lambda_1\int_0^t |v'(1,s)|^2 ds \\ &\geq ||v'(t)||^2 + \widetilde{C}||v(t)||_{H^1}^2 + 2\lambda_1\int_0^t |v'(1,s)|^2 ds, \end{aligned} \tag{4.14}$$

$$|v(0,t)| \leq ||v(t)||_{C^0(\overline{\Omega})} \leq \widetilde{C}_0\sqrt{z(t)},$$

where the constants $\widetilde{C}$, $\widetilde{C}_0$ are defined by (2.11), (2.13), respectively. Then, we prove, in a manner similar to the above part, that

$$
\begin{aligned}
z(t) \le{} & T\|e_{N,K,\lambda}\|^2_{L^\infty(0,T;L^2)} + \int_0^t z(s)ds + \varepsilon z(t) + \frac{1}{\varepsilon}\widetilde{C}_0^4 \int_0^t k^2(\theta)d\theta \int_0^t z(s)ds \\
& + 2|k(0)|\widetilde{C}_0^2 \int_0^t z(s)ds + 2\widetilde{C}_0^2\sqrt{t}\left(\int_0^t |k'(\theta)|^2 d\theta\right)^{1/2} \int_0^t z(s)ds
\end{aligned}
\tag{4.15}
$$

for all $t \in [0,T]$ and $\varepsilon > 0$. Choosing $\varepsilon > 0$, such that $\varepsilon \le 1/2$, we obtain from (4.5), (4.15), that

$$
z(t) \le 2T\widetilde{C}_N^2 (K^2+\lambda^2)^{N+1} + \rho_T \int_0^t z(s)ds, \tag{4.16}
$$

where

$$
\rho_T = 2 + 4|k(0)|\widetilde{C}_0^2 + 4\widetilde{C}_0^4 \int_0^T k^2(\theta)d\theta + \frac{2}{\varepsilon}\widetilde{C}_0^2\sqrt{T}\left(\int_0^T |k'(\theta)|^2 d\theta\right)^{1/2}. \tag{4.17}
$$

By Gronwall's lemma, it follows from (4.16), (4.17), that

$$
z(t) \le 2T\widetilde{C}_N^2 (K^2+\lambda^2)^{N+1} \exp(T\rho_T). \tag{4.18}
$$

It follows from (4.14), that

$$
\begin{aligned}
& \|v'(t)\|^2 + \widetilde{C}\|v(t)\|^2_{H^1} + 2\lambda_1 \int_0^t |v'(1,s)|^2 ds \\
& \quad \le z(t) \le 2T\widetilde{C}_N^2 (K^2+\lambda^2)^{N+1} \exp(T\rho_T).
\end{aligned}
\tag{4.19}
$$

Hence

$$
\|v'\|_{L^\infty(0,T;L^2)} + \|v\|_{L^\infty(0,T;H^1)} + \|v'(1,\cdot)\|_{L^2(0,T)} \le \widetilde{C}_N^*\left(\sqrt{K^2+\lambda^2}\right)^{N+1}, \tag{4.20}
$$

or

$$
\begin{aligned}
& \left\|u'_{K,\lambda} - \sum_{0\le\gamma_1+\gamma_2\le N} u'_{\gamma_1,\gamma_2} K^{\gamma_1}\lambda^{\gamma_2}\right\|_{L^\infty(0,T;L^2)} + \left\|u_{K,\lambda} - \sum_{0\le\gamma_1+\gamma_2\le N} u_{\gamma_1,\gamma_2} K^{\gamma_1}\lambda^{\gamma_2}\right\|_{L^\infty(0,T;H^1)} \\
& \quad + \left\|u'_{K,\lambda}(1,\cdot) - \sum_{0\le\gamma_1+\gamma_2\le N} u'_{\gamma_1,\gamma_2}(1,\cdot) K^{\gamma_1}\lambda^{\gamma_2}\right\|_{L^2(0,T)} \le \widetilde{C}_N^*\left(\sqrt{K^2+\lambda^2}\right)^{N+1}.
\end{aligned}
\tag{4.21}
$$

On the other hand, it follows from $(4.3)_5$, (4.20), that

$$\begin{aligned}\|R\|_{C^0([0,T])} &\leq \left(h+\int_0^T |k(\theta)|\,d\theta\right)\|v\|_{L^\infty(0,T;H^1)}\\ &\leq \left(h+\int_0^T |k(\theta)|\,d\theta\right)\widetilde{C}_N^*\left(\sqrt{K^2+\lambda^2}\right)^{N+1}\\ &= \widetilde{C}_N^{**}\left(\sqrt{K^2+\lambda^2}\right)^{N+1},\end{aligned} \tag{4.22}$$

or

$$\left\|P_{K,\lambda}-\sum_{0\leq\gamma_1+\gamma_2\leq N} P_{\gamma_1,\gamma_2}K^{\gamma_1}\lambda^{\gamma_2}\right\|_{C^0([0,T])} \leq \widetilde{C}_N^{**}\left(\sqrt{K^2+\lambda^2}\right)^{N+1}. \tag{4.23}$$

The proof of Theorem 4.2 is completed. □

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