MULTIPLE POSITIVE SOLUTIONS OF SINGULAR p-LAPLACIAN PROBLEMS BY VARIATIONAL METHODS

KANISHKA PERERA AND ZHITAO ZHANG

Received 20 July 2004

We obtain multiple positive solutions of singular p-Laplacian problems using variational methods. The techniques are applicable to other types of singular problems as well.

1. Introduction

We consider the singular quasilinear elliptic boundary value problem

$$\begin{aligned} -\Delta_p u &= a(x)u^{-\gamma} + \lambda f(x,u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded C^2 domain in $\mathbb{R}^n, n \geq 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $1 < p < \infty$, $a \geq 0$ is a nontrivial measurable function, $\gamma > 0$ is a constant, $\lambda > 0$ is a parameter, and f is a Carathéodory function on $\Omega \times [0,\infty)$ satisfying

$$\sup_{(x,t)\in\Omega\times[0,T]} |f(x,t)| < \infty \quad \forall T > 0. \tag{1.2}$$

The semilinear case $p = 2$ with $\gamma < 1$ and $f = 0$ has been studied extensively in both bounded and unbounded domains (see [5, 6, 7, 10, 11, 12, 14, 20] and their references). In particular, Lair and Shaker [11] showed the existence of a unique (weak) solution when Ω is bounded and $a \in L^2(\Omega)$. Their result was extended to the sublinear case $f(t) = t^\beta$, $0 < \beta \leq 1$ by Shi and Yao [15] and Wiegner [18]. In the superlinear case $1 < \beta < 2^* - 1$ and for small λ, Coclite and Palmieri [4] obtained a solution when $a = 1$ and Sun et al. [16] obtained two solutions using the Ekeland's variational principle for more general a's. Zhang [19] extended their multiplicity result to more general superlinear terms $f(t) \geq 0$ using critical point theory on closed convex sets. The ODE case $n = 1$ was studied by Agarwal and O'Regan [1] using fixed point theory and by Agarwal et al. [2] using variational methods. The purpose of the present paper is to treat the general quasilinear case $p \in (1,\infty)$, $\gamma \in (0,\infty)$, and f is allowed to change sign. We use a simple cutoff argument and only the basic critical point theory. Our results seem to be new even for $p = 2$.

Boundary Value Problems 2005:3 (2005) 377–382
DOI: 10.1155/BVP.2005.377

First we assume

(H_1) $\exists\varphi \geq 0$ in $C_0^1(\overline{\Omega})$ and $q > n$ such that $a\varphi^{-\gamma} \in L^q(\Omega)$.

This does not require $\gamma < 1$ as usually assumed in the literature. For example, when Ω is the unit ball, $a(x) = (1 - |x|^2)^\sigma$, $\sigma \geq 0$, and $\gamma < \sigma + 1/n$, we can take $\varphi(x) = 1 - |x|^2$ and $q < 1/(\gamma - \sigma)$ (resp., q with no additional restrictions) if $\gamma > \sigma$ (resp., $\gamma \leq \sigma$).

Theorem 1.1. *If (H_1) and (1.2) hold and $f \geq 0$, then $\exists\lambda_0 > 0$ such that problem (1.1) has a solution $\forall\lambda \in (0,\lambda_0)$.*

Corollary 1.2. *Problem (1.1) with $f = 0$ has a solution if (H_1) holds.*

Next we allow f to change sign, but strengthen (H_1) to

(H_2) $a \in L^\infty(\Omega)$ with $a_0 := \inf_\Omega a > 0$ and $\gamma < 1/n$.

This implies that $a\varphi^{-\gamma} \in L^q(\Omega)$ for any φ whose interior normal derivative $\partial\varphi/\partial\nu > 0$ on $\partial\Omega$ and $q < 1/\gamma$.

Theorem 1.3. *If (H_2) and (1.2) hold, then $\exists\lambda_0 > 0$ such that problem (1.1) has a solution $\forall\lambda \in (0,\lambda_0)$.*

Finally we assume that f is C^1 in t, satisfies

$$|f_t(x,t)| \leq C(t^{r-2} + 1) \tag{1.3}$$

for some $2 \leq r < p^*$, and p-superlinear:

$$0 < \theta F(x,t) \leq t f(x,t), \quad t \text{ large} \tag{1.4}$$

for some $\theta > p$. Here $p^* = np/(n-p)$ (resp., ∞) if $p < n$ (resp., $p \geq n$) is the critical Sobolev exponent and C denotes a generic positive constant.

Theorem 1.4. *If $p \geq 2$, (H_1), (1.3), and (1.4) hold, and $f \geq 0$, then $\exists\lambda_0 > 0$ such that problem (1.1) has two solutions $\forall\lambda \in (0,\lambda_0)$.*

Theorem 1.5. *If $p \geq 2$ and (H_2), (1.3), and (1.4) hold, then $\exists\lambda_0 > 0$ such that problem (1.1) has two solutions $\forall\lambda \in (0,\lambda_0)$.*

2. Preliminaries on the p-Laplacian

Consider the problem

$$\begin{aligned} -\Delta_p u = g(x) \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Proposition 2.1. *If $g \in L^q(\Omega)$ for some $q > n$, then (2.1) has a unique weak solution $u \in C_0^1(\overline{\Omega})$. If, in addition, $g \geq 0$ is nontrivial, then*

$$u > 0 \quad \text{in } \Omega, \qquad \partial u/\partial\nu > 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

Proof. The existence of a unique solution $u \in W_0^{1,p}(\Omega)$ is well-known. The problem

$$\begin{gathered} -\Delta v = g(x) \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.3}$$

has a unique solution $v \in W^{2,q}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$, $\alpha = 1 - n/q$. Then u satisfies

$$\begin{gathered} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u - G(x)\right) = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.4}$$

where $G = \nabla v \in C^{\alpha}(\overline{\Omega})$, and u is bounded by Guedda and Véron [8] since $q > n/p$ if $p \le n$, so $u \in C_0^1(\overline{\Omega})$ by Lieberman [13]. The rest now follows from Vázquez [17]. □

3. Proofs of Theorems 1.1 and 1.3

Proof of Theorem 1.1. Since $a \in L^q(\Omega)$ by (H_1), the problem

$$\begin{gathered} -\Delta_p v = a(x) \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.1}$$

has a unique positive solution $v \in C_0^1(\overline{\Omega})$ with $\partial v/\partial \nu > 0$ on $\partial\Omega$ by Proposition 2.1. Then $\inf_{\Omega}(v/\varphi) > 0$ and hence $av^{-\gamma} \in L^q(\Omega)$. Fix $0 < \varepsilon \le 1$ so small that $\underline{u} := \varepsilon^{1/(p-1)} v \le 1$. Then

$$-\Delta_p \underline{u} - a(x)\underline{u}^{-\gamma} - \lambda f(x,\underline{u}) \le -(1-\varepsilon)a(x) \le 0, \tag{3.2}$$

so $\underline{u}$ is a subsolution of (1.1).

Since $a\underline{u}^{-\gamma} \in L^q(\Omega)$, the problem

$$\begin{gathered} -\Delta_p u = a(x)\underline{u}(x)^{-\gamma} + 1 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.3}$$

has a unique solution $\overline{u} \in C_0^1(\overline{\Omega})$ by Proposition 2.1, and $\overline{u} \ge \underline{u}$ since

$$-\Delta_p \overline{u} \ge a(x) \ge \varepsilon a(x) = -\Delta_p \underline{u}. \tag{3.4}$$

Then

$$-\Delta_p \overline{u} - a(x)\overline{u}^{-\gamma} - \lambda f(x,\overline{u}) \ge 1 - \lambda \sup_{x\in\Omega,\, t \le \max_{\Omega} \overline{u}} f(x,t), \tag{3.5}$$

so $\exists \lambda_0 > 0$ such that $\overline{u}$ is a supersolution of (1.1) $\forall \lambda \in (0,\lambda_0)$ by (1.2).

Let

$$g_{\lambda,\overline{u}}(x,t) = \begin{cases} a(x)\overline{u}(x)^{-\gamma} + \lambda f(x,\overline{u}(x)), & t > \overline{u}(x) \\ a(x)t^{-\gamma} + \lambda f(x,t), & \underline{u}(x) \le t \le \overline{u}(x) \\ a(x)\underline{u}(x)^{-\gamma} + \lambda f(x,\underline{u}(x)), & t < \underline{u}(x), \end{cases}$$
$$G_{\lambda,\overline{u}}(x,t) = \int_0^t g_{\lambda,\overline{u}}(x,s)\,ds,$$
$$\Phi_{\lambda,\overline{u}}(u) = \int_\Omega |\nabla u|^p - pG_{\lambda,\overline{u}}(x,u), \quad u \in W_0^{1,p}(\Omega). \tag{3.6}$$

Since

$$0 \le g_{\lambda,\overline{u}}(x,t) \le a(x)\underline{u}(x)^{-\gamma} + \lambda \sup_{x\in\Omega, t\le \max_\Omega \overline{u}} f(x,t), \quad \forall (x,t) \in \Omega\times\mathbb{R}, \tag{3.7}$$

and $a\underline{u}^{-\gamma} \in L^q(\Omega)$, $\Phi_{\lambda,\overline{u}}$ is bounded from below and has a global minimizer u_0, which then is a solution of (1.1) in the order interval $[\underline{u},\overline{u}]$. □

Proof of Theorem 1.3. The problem

$$-\Delta_p v = a_0 \quad \text{in } \Omega,$$
$$v = 0 \quad \text{on } \partial\Omega \tag{3.8}$$

has a unique positive solution $v \in C_0^1(\overline{\Omega})$ with $\partial v/\partial \nu > 0$ on $\partial\Omega$. Fix $0 < \varepsilon < 1$ so small that $\underline{u} := \varepsilon^{1/(p-1)}\, v \le 1$. Then

$$-\Delta_p \underline{u} - a(x)\underline{u}^{-\gamma} - \lambda f(x,\underline{u}) \le -(1-\varepsilon)a_0 + \lambda \sup_{x\in\Omega, t\le\max_\Omega \underline{u}} |f(x,t)|, \tag{3.9}$$

so $\exists \lambda_0 > 0$ such that $\underline{u}$ is a subsolution of (1.1) $\forall \lambda \in (0,\lambda_0)$. The rest of the proof now proceeds as above. □

4. Proofs of Theorems 1.4 and 1.5

Proof of Theorem 1.4. Define a Carathéodory function on $\Omega \times \mathbb{R}$ by

$$g_\lambda(x,t) = \begin{cases} a(x)t^{-\gamma} + \lambda f(x,t), & t \ge \underline{u}(x) \\ a(x)\underline{u}(x)^{-\gamma} + \lambda f(x,\underline{u}(x)), & t < \underline{u}(x) \end{cases} \tag{4.1}$$

and consider the problem

$$-\Delta_p u = g_\lambda(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega. \tag{4.2}$$

Every solution of (4.2) is $\ge \underline{u}$ and hence also a solution of (1.1). By (1.3),

$$0 \le g_\lambda(x,t) \le a(x)\underline{u}(x)^{-\gamma} + \lambda C\Big(\big(t^+\big)^{r-1} + 1\Big), \quad \forall (x,t) \in \Omega\times\mathbb{R} \tag{4.3}$$

so solutions of (4.2) are the critical points of the C^1 functional

$$\Phi_\lambda(u) = \int_\Omega |\nabla u|^p - pG_\lambda(x,u), \quad u \in W_0^{1,p}(\Omega), \tag{4.4}$$

where $G_\lambda(x,t) = \int_0^t g_\lambda(x,s)ds$.

Since u_0 solves

$$\begin{gathered} -\Delta_p u = g_{\lambda,\overline{u}}(x,u_0(x)) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{4.5}$$

and $g_{\lambda,\overline{u}}(\cdot,u_0(\cdot)) \in L^q(\Omega)$ by (3.7), $u_0 \in C_0^1(\overline{\Omega})$ by Proposition 2.1. Note that, with a possibly smaller λ_0, $2\overline{u}$ is also a supersolution of (1.1) $\forall \lambda \in (0,\lambda_0)$. We assume that u_0 is the global minimizer of the corresponding functional $\Phi_{\lambda,2\overline{u}}$ also, for otherwise we are done. Since

$$u_0 \le \overline{u} < 2\overline{u} \quad \text{in } \Omega, \qquad \partial u_0/\partial\nu \le \partial\overline{u}/\partial\nu < \partial(2\overline{u})/\partial\nu \quad \text{on } \partial\Omega, \tag{4.6}$$

$\Phi_{\lambda,2\overline{u}} = \Phi_\lambda$ in a $C_0^1(\overline{\Omega})$-neighborhood of u_0, so u_0 is a local minimizer of $\Phi_\lambda|_{C_0^1(\overline{\Omega})}$, and hence also of Φ_λ by Brezis and Nirenberg [3] for $p = 2$ and by Guo and Zhang [9] for $p > 2$. The mountain pass lemma now gives a second critical point as (1.4) implies that Φ_λ satisfies the (PS) condition and $\Phi_\lambda(t\underline{u}) \to -\infty$ as $t \to \infty$. □

Proof of Theorem 1.5 is similar and therefore omitted.

Acknowledgments

We would like to thank Professor Marco Degiovanni for showing us the proof of Proposition 2.1 and Professor Mabel Cuesta and Professor Jean-Pierre Gossez for their helpful comments about the p-Laplacian. The first author was supported in part by the National Science Foundation. The second author was supported in part by the National Natural Science Foundation of China, Ky and Yu-Fen Fan Endowment of the AMS, Florida Institute of Technology, and the Humboldt Foundation.

References

[1] R. P. Agarwal and D. O'Regan, *Singular boundary value problems for superlinear second order ordinary and delay differential equations*, J. Differential Equations **130** (1996), no. 2, 333–355.

[2] R. P. Agarwal, K. Perera, and D. O'Regan, *Multiple positive solutions of singular problems by variational methods*, to appear in Proc. Amer. Math. Soc.

[3] H. Brezis and L. Nirenberg, *H^1 versus C^1 local minimizers*, C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), no. 5, 465–472.

[4] M. M. Coclite and G. Palmieri, *On a singular nonlinear Dirichlet problem*, Comm. Partial Differential Equations **14** (1989), no. 10, 1315–1327.

[5] M. G. Crandall, P. H. Rabinowitz, and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations **2** (1977), no. 2, 193–222.

[6] J. I. Diaz, J.-M. Morel, and L. Oswald, *An elliptic equation with singular nonlinearity*, Comm. Partial Differential Equations **12** (1987), no. 12, 1333–1344.

[7] A. L. Edelson, *Entire solutions of singular elliptic equations*, J. Math. Anal. Appl. **139** (1989), no. 2, 523–532.
[8] M. Guedda and L. Véron, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal. **13** (1989), no. 8, 879–902.
[9] Z. Guo and Z. Zhang, *$W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl. **286** (2003), no. 1, 32–50.
[10] T. Kusano and C. A. Swanson, *Entire positive solutions of singular semilinear elliptic equations*, Japan. J. Math. (N.S.) **11** (1985), no. 1, 145–155.
[11] A. V. Lair and A. W. Shaker, *Classical and weak solutions of a singular semilinear elliptic problem*, J. Math. Anal. Appl. **211** (1997), no. 2, 371–385.
[12] A. C. Lazer and P. J. McKenna, *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc. **111** (1991), no. 3, 721–730.
[13] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), no. 11, 1203–1219.
[14] A. W. Shaker, *On singular semilinear elliptic equations*, J. Math. Anal. Appl. **173** (1993), no. 1, 222–228.
[15] J. Shi and M. Yao, *On a singular nonlinear semilinear elliptic problem*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), no. 6, 1389–1401.
[16] Y. Sun, S. Wu, and Y. Long, *Combined effects of singular and superlinear nonlinearities in some singular boundary value problems*, J. Differential Equations **176** (2001), no. 2, 511–531.
[17] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), no. 3, 191–202.
[18] M. Wiegner, *A degenerate diffusion equation with a nonlinear source term*, Nonlinear Anal. **28** (1997), no. 12, 1977–1995.
[19] Z. Zhang, *Critical points and positive solutions of singular elliptic boundary value problems*, J. Math. Anal. Appl. **302** (2005), no. 2, 476–483.
[20] M. A. del Pino, *A global estimate for the gradient in a singular elliptic boundary value problem*, Proc. Roy. Soc. Edinburgh Sect. A **122** (1992), no. 3-4, 341–352.

Kanishka Perera: Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA
E-mail address: kperera@fit.edu

Zhitao Zhang: Academy of Mathematics and Systems Science, Institute of Mathematics, Chinese Academy of Sciences, Beijing 100080, China
E-mail address: zzt@math.ac.cn