# SECOND-ORDER ESTIMATES FOR BOUNDARY BLOWUP SOLUTIONS OF SPECIAL ELLIPTIC EQUATIONS

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We find a second-order approximation of the boundary blowup solution of the equation $\Delta u = e^{u|u|^{\beta-1}}$, with $\beta > 0$, in a bounded smooth domain $\Omega \subset R^N$. Furthermore, we consider the equation $\Delta u = e^{u+e^u}$. In both cases, we underline the effect of the geometry of the domain in the asymptotic expansion of the solutions near the boundary $\partial\Omega$.



## 1. Introduction

Let $\Omega \subset R^N$ be a bounded smooth domain. In 1916, Bieberbach [10] has investigated the problem

$$\Delta u = e^u \quad \text{in } \Omega, \qquad u(x) \longrightarrow \infty \quad \text{as } x \longrightarrow \partial\Omega, \tag{1.1}$$

and has proved the existence of a classical solution called a boundary blowup (explosive, large) solution. Moreover, if $\delta = \delta(x)$ denotes the distance from $x$ to $\partial\Omega$, we have [10] $u(x) - \log(2/\delta^2(x)) \to 0$ as $x \to \partial\Omega$. Recently, Bandle [4] has improved the previous estimate finding the expansion

$$u(x) = \log\frac{2}{\delta^2(x)} + (N-1)K(\overline{x})\delta(x) + o(\delta(x)), \tag{1.2}$$

where $K(\overline{x})$ denotes the mean curvature of $\partial\Omega$ at the point $\overline{x}$ nearest to $x$, and $o(\delta)$ has the usual meaning. Boundary estimates for various nonlinearities have been discussed in several papers, see for example [1, 3, 5, 8, 13–16].

In Section 2 of the present paper we investigate boundary blowup solutions of the equation $\Delta u = e^{u|u|^{\beta-1}}$, with $\beta > 0$, $\beta \neq 1$. We prove the estimate

$$u(x) = \Phi(\delta) + \beta^{-1}(N-1)K(x)\delta\big(\Phi(\delta)\big)^{1-\beta} + O(1)\delta\big(\Phi(\delta)\big)^{1-2\beta}, \tag{1.3}$$

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where $\Phi(\delta)$ is defined by the equation

$$\int_{\Phi(s)}^{\infty} (2F(t))^{-1/2} = s, \quad F(t) = \int_{-\infty}^{t} e^{\tau|\tau|^{\beta-1}} d\tau, \tag{1.4}$$

$K(x)$ is the mean curvature of the surface $\{x \in \Omega : \delta(x) = \text{constant}\}$, and $O(1)$ denotes a bounded quantity.

In Section 3 we consider boundary blowup solutions of the equation $\Delta u = e^{u+e^u}$. We find the estimate

$$u(x) = \Psi(\delta) + (N-1)K(x)e^{-\Psi(\delta)}\delta + O(1)e^{-2\Psi(\delta)}\delta, \tag{1.5}$$

where $\Psi$ is defined by the equation

$$\int_{\Psi(s)}^{\infty} (2e^{e^t} - 2)^{-1/2} dt = s. \tag{1.6}$$

In this paper, the distance function $\delta = \delta(x)$ plays an important role. Recall that if $\Omega$ is smooth then also $\delta(x)$ is smooth for $x$ near to $\partial\Omega$, and [12]

$$\sum_{i=1}^{N} \delta_{x_i}\delta_{x_i} = 1, \qquad -\sum_{i=1}^{N} \delta_{x_i x_i} = (N-1)K = H, \tag{1.7}$$

where $K = K(x)$ is the mean curvature of the surface $\{x \in \Omega : \delta(x) = \text{constant}\}$.

The effect of the geometry of the domain in the behaviour of boundary blowup solutions for special equations has been observed in various papers, see for example, [2, 7, 9, 11].

## 2. The equation $\Delta u = e^{u|u|^{\beta-1}}$

In what follows we denote with $O(1)$ a bounded quantity.

Lemma 2.1. *Let $\beta > 0$, $f(s) = e^{s|s|^{\beta-1}}$, $F(s) = \int_{-\infty}^{s} f(t)dt$. Then*

$$F(s)f'(s)(f(s))^{-2} = 1 + O(1)s^{-\beta}. \tag{2.1}$$

*Proof.* For $s > 0$ we have

$$\begin{aligned} F(s)f'(s)(f(s))^{-2} &= f'(s)(f(s))^{-2}F(0) + f'(s)(f(s))^{-2}\int_0^s f(t)dt \\ &= \beta e^{-s^\beta}s^{\beta-1}F(0) + e^{-s^\beta}\int_0^s e^{t^\beta}\beta t^{\beta-1}dt + \beta e^{-s^\beta}\int_0^s e^{t^\beta}(s^{\beta-1} - t^{\beta-1})dt \\ &= \beta e^{-s^\beta}s^{\beta-1}F(0) + 1 - e^{-s^\beta} + \beta e^{-s^\beta}\int_0^s e^{t^\beta}(s^{\beta-1} - t^{\beta-1})dt. \end{aligned} \tag{2.2}$$

We have

$$\begin{aligned} \lim_{s\to\infty} s^\beta \beta e^{-s^\beta}s^{\beta-1}F(0) &= 0, \\ \lim_{s\to\infty} s^\beta e^{-s^\beta} &= 0. \end{aligned} \tag{2.3}$$

Moreover, using de l'Hôpital's rule we find

$$\begin{aligned}\lim_{s\to\infty}\frac{\beta\int_0^s e^{t^\beta}(s^{2\beta-1}-s^\beta t^{\beta-1})dt}{e^{s^\beta}} &= \lim_{s\to\infty}\frac{\int_0^s e^{t^\beta}((2\beta-1)s^{\beta-1}-\beta t^{\beta-1})dt}{e^{s^\beta}}\\ &= \lim_{s\to\infty}\frac{(\beta-1)e^{s^\beta}s^{\beta-1}+\int_0^s e^{t^\beta}(2\beta-1)(\beta-1)s^{\beta-2}dt}{\beta e^{s^\beta}s^{\beta-1}}\\ &= \frac{\beta-1}{\beta}+(2\beta-1)(\beta-1)\lim_{s\to\infty}\frac{\int_0^s e^{t^\beta}dt}{\beta e^{s^\beta}s}\\ &= \frac{\beta-1}{\beta}+(2\beta-1)(\beta-1)\lim_{s\to\infty}\frac{1}{\beta(1+\beta s^\beta)}=\frac{\beta-1}{\beta}.\end{aligned}\tag{2.4}$$

The lemma follows. □

*Remark 2.2.* If $\beta = 1$, we have $F(s)f'(s)(f(s))^{-2} = 1$. We do not care of this special case because it has been discussed in [2].

Lemma 2.3. *Let $\Phi = \Phi(\delta)$ be defined by*

$$\int_{\Phi(\delta)}^{\infty}(2F(t))^{-1/2}dt=\delta,\quad F(t)=\int_{-\infty}^{t}f(\tau)d\tau,\ f(\tau)=e^{\tau|\tau|^{\beta-1}}.\tag{2.5}$$

*Then*

$$-\Phi'(\delta)=\left[1+O(1)(\Phi(\delta))^{-\beta}\right]\delta f(\Phi(\delta)).\tag{2.6}$$

*Proof.* By the (trivial) relation

$$-1+2(1+O(1)s^{-\beta})=1+O(1)s^{-\beta},\tag{2.7}$$

using (2.1) we have

$$-1+2F(s)f'(s)(f(s))^{-2}=1+O(1)s^{-\beta}.\tag{2.8}$$

Multiplying by $(2F(s))^{-1/2}$ we find

$$\begin{aligned}-(2F(s))^{-1/2}+(2F(s))^{1/2}f'(s)(f(s))^{-2}&=(2F(s))^{-1/2}+O(1)(2F(s))^{-1/2}s^{-\beta},\\ -\left((2F(s))^{1/2}(f(s))^{-1}\right)'&=(2F(s))^{-1/2}+O(1)(2F(s))^{-1/2}s^{-\beta}.\end{aligned}\tag{2.9}$$

Integrating on $(s,\infty)$ we get

$$(2F(s))^{1/2}(f(s))^{-1}=\int_s^\infty(2F(t))^{-1/2}dt+O(1)\int_s^\infty(2F(t))^{-1/2}t^{-\beta}dt.\tag{2.10}$$

Using de l'Hôpital's rule we find

$$\begin{aligned}\lim_{s\to\infty}\frac{s^{-\beta}\int_s^\infty (2F(t))^{-1/2}dt}{\int_s^\infty (2F(t))^{-1/2}t^{-\beta}dt} &= \lim_{s\to\infty}\frac{(2F(s))^{-1/2}s^{-\beta}+\beta s^{-\beta-1}\int_s^\infty (2F(t))^{-1/2}dt}{(2F(s))^{-1/2}s^{-\beta}}\\ &= 1+\lim_{s\to\infty}\frac{\beta\int_s^\infty (2F(t))^{-1/2}dt}{s(2F(s))^{-1/2}}\\ &= 1+\lim_{s\to\infty}\frac{-\beta}{1-s(2F(s))^{-1}f(s)}=1.\end{aligned}\tag{2.11}$$

In the last step we have used the limit

$$\lim_{s\to\infty}\frac{sf(s)}{F(s)}=\infty,\tag{2.12}$$

which can be proved easily with de l'Hôpital's rule. Using (2.11), (2.10) can be rewritten as

$$(2F(s))^{1/2}(f(s))^{-1}=\int_s^\infty (2F(t))^{-1/2}dt+O(1)s^{-\beta}\int_s^\infty (2F(t))^{-1/2}dt.\tag{2.13}$$

Putting $s=\Phi(\delta)$ and using the equation $-\Phi'(\delta)=(2F(\Phi(\delta)))^{1/2}$, the lemma follows. □

Theorem 2.4. *Let $\Omega$ be a bounded smooth domain in $R^N$, $N\geq 2$, and let $\beta>0$, $\beta\neq 1$. If $u(x)$ is a boundary blowup solution of $\Delta u=e^{u|u|^{\beta-1}}$ in $\Omega$, then*

$$u(x)=\Phi(\delta)+\beta^{-1}H\delta(\Phi(\delta))^{1-\beta}+O(1)\delta(\Phi(\delta))^{1-2\beta},\tag{2.14}$$

*where $\Phi(\delta)$ is defined as in (2.5), $\delta=\delta(x)$ is the distance from $x$ to $\partial\Omega$ and $H$ is defined by (1.7).*

*Proof.* We look for a super-solution of the form

$$w(x)=\Phi(\delta)+\beta^{-1}H\delta(\Phi(\delta))^{1-\beta}+\alpha\delta(\Phi(\delta))^{1-2\beta},\tag{2.15}$$

where $\alpha$ is a positive constant to be determined. Denoting by $'$ differentiation with respect to $\delta$, we have

$$w_{x_i}=\Phi'(\delta)\delta_{x_i}+\beta^{-1}H_{x_i}\delta(\Phi(\delta))^{1-\beta}+\beta^{-1}H\Big(\delta(\Phi(\delta))^{1-\beta}\Big)'\delta_{x_i}+\alpha\Big(\delta(\Phi(\delta))^{1-2\beta}\Big)'\delta_{x_i}.\tag{2.16}$$

Using (1.7) we find

$$\begin{aligned}\Delta w=\Phi''(\delta)-\Phi'(\delta)H+\beta^{-1}\Delta H\delta(\Phi(\delta))^{1-\beta}+2\beta^{-1}\nabla H\cdot\nabla\delta\Big(\delta(\Phi(\delta))^{1-\beta}\Big)'\\ +\beta^{-1}H\Big(\delta(\Phi(\delta))^{1-\beta}\Big)''-\beta^{-1}H^2\Big(\delta(\Phi(\delta))^{1-\beta}\Big)'\\ +\alpha\Big(\delta(\Phi(\delta))^{1-2\beta}\Big)''-\alpha\Big(\delta(\Phi(\delta))^{1-2\beta}\Big)'H.\end{aligned}\tag{2.17}$$

With $f(\tau)=e^{\tau|\tau|^{\beta-1}}$, by (2.5) we have $\Phi''(\delta)=f(\Phi)$. Often we write $\Phi$ instead of $\Phi(\delta)$ and $\Phi'$ instead of $\Phi'(\delta)$. Lemma 2.3 yields

$$-\Phi'=\left[1+O(1)\Phi^{-\beta}\right]\delta f(\Phi). \tag{2.18}$$

Using (2.18) and the equation $\Phi'=-(2F(\Phi))^{1/2}$ we find

$$\begin{aligned}\lim_{\delta\to 0}\frac{(\Phi(\delta))^{1-\beta}}{\delta(\Phi(\delta))^{-\beta}f(\Phi)}&=\lim_{\delta\to 0}\frac{\Phi}{-\Phi'}=\lim_{\delta\to 0}\frac{\Phi}{(2F(\Phi))^{1/2}}\\&=\lim_{s\to\infty}\left(\frac{s^2}{2F(s)}\right)^{1/2}=\lim_{s\to\infty}\left(\frac{s}{f(s)}\right)^{1/2}=0.\end{aligned} \tag{2.19}$$

Let us write the last result as

$$(\Phi(\delta))^{1-\beta}=o(1)\delta(\Phi(\delta))^{-\beta}f(\Phi), \tag{2.20}$$

where $o(1)$ denotes a quantity which tends to zero as $\delta\to 0$. Using (2.18) again we find

$$\lim_{\delta\to 0}\frac{(\Phi(\delta))^{-\beta}\Phi'}{\delta(\Phi(\delta))^{-\beta}f(\Phi)}=-1. \tag{2.21}$$

Therefore,

$$\begin{aligned}\left(\delta(\Phi(\delta))^{1-\beta}\right)'&=(\Phi(\delta))^{1-\beta}+(1-\beta)\delta(\Phi(\delta))^{-\beta}\Phi'\\&=o(1)\delta(\Phi(\delta))^{-\beta}f(\Phi).\end{aligned} \tag{2.22}$$

Further differentiation yields

$$\begin{aligned}\left(\delta(\Phi(\delta))^{1-\beta}\right)''=&\,2(1-\beta)(\Phi(\delta))^{-\beta}\Phi'-\beta(1-\beta)\delta(\Phi(\delta))^{-\beta-1}(\Phi')^2\\&+(1-\beta)\delta(\Phi(\delta))^{-\beta}f(\Phi).\end{aligned} \tag{2.23}$$

Moreover, recalling (2.12) we find

$$\lim_{\delta\to 0}\frac{\delta(\Phi(\delta))^{-\beta-1}(\Phi')^2}{\delta(\Phi(\delta))^{-\beta}f(\Phi)}=\lim_{\delta\to 0}\frac{2F(\Phi)}{\Phi f(\Phi)}=\lim_{s\to\infty}\frac{2F(s)}{sf(s)}=0. \tag{2.24}$$

Using the last result and (2.21), from (2.23) we find

$$\left(\delta(\Phi(\delta))^{1-\beta}\right)''=O(1)\delta(\Phi(\delta))^{-\beta}f(\Phi). \tag{2.25}$$

Similarly, we find

$$\begin{aligned}\left(\delta(\Phi(\delta))^{1-2\beta}\right)'&=o(1)\delta(\Phi(\delta))^{-2\beta}f(\Phi),\\\left(\delta(\Phi(\delta))^{1-2\beta}\right)''&=O(1)\delta(\Phi(\delta))^{-2\beta}f(\Phi).\end{aligned} \tag{2.26}$$

Denoting by $M_1$ a nonnegative constant independent of $\alpha$ and using (2.18), (2.20), (2.22), (2.25), (2.26), by (2.17) we get

$$\Delta w < f(\Phi)\left[1 + H\delta + M_1\delta\Phi^{-\beta} + \alpha M_1\delta\Phi^{-2\beta}\right]. \tag{2.27}$$

On the other side, we have

$$\begin{aligned} f(w) &= e^{(\Phi+\beta^{-1}H\delta\Phi^{1-\beta}+\alpha\delta\Phi^{1-2\beta})^\beta} \\ &= e^{\Phi^\beta(1+\beta^{-1}H\delta\Phi^{-\beta}+\alpha\delta\Phi^{-2\beta})^\beta}. \end{aligned} \tag{2.28}$$

Let us take $\delta_0 > 0$ and $\alpha$ such that for $\{x \in \Omega : \delta(x) < \delta_0\}$ we have

$$-\frac{1}{2} < \beta^{-1}H\delta\big(\Phi(\delta)\big)^{-\beta} + \alpha\delta\big(\Phi(\delta)\big)^{-2\beta} < 1. \tag{2.29}$$

Then, denoting by $M_2$ a nonnegative constant independent of $\alpha$ we find

$$\begin{aligned} f(w) &> e^{\Phi^\beta(1+H\delta\Phi^{-\beta}+\alpha\beta\delta\Phi^{-2\beta}-M_2(\delta\Phi^{-\beta})^2-M_2(\alpha\delta\Phi^{-2\beta})^2)} \\ &= f(\Phi)e^{H\delta+\alpha\beta\delta\Phi^{-\beta}-M_2\delta^2\Phi^{-\beta}-M_2(\alpha\delta)^2\Phi^{-3\beta}} \\ &> f(\Phi)\left[1 + H\delta + \alpha\beta\delta\Phi^{-\beta} - M_2\delta^2\Phi^{-\beta} - M_2(\alpha\delta)^2\Phi^{-3\beta}\right]. \end{aligned} \tag{2.30}$$

By (2.27) and (2.30) we find that

$$\Delta w < f(w) \tag{2.31}$$

when

$$1 + H\delta + M_1\delta\Phi^{-\beta} + \alpha M_1\delta\Phi^{-2\beta} < 1 + H\delta + \alpha\beta\delta\Phi^{-\beta} - M_2\delta^2\Phi^{-\beta} - M_2(\alpha\delta)^2\Phi^{-3\beta}. \tag{2.32}$$

Rearranging we find

$$M_1 + M_2\delta < \alpha\left[\beta - M_2\alpha\delta\Phi^{-2\beta} - M_1\Phi^{-\beta}\right]. \tag{2.33}$$

We can take $\delta_0$ small and $\alpha$ large so that (2.33) and (2.29) hold for $\delta(x) < \delta_0$.

Our function $f(t) = e^{t|t|^{\beta-1}}$ is positive and increasing for all $t$, and $F(t)t^{-2}$ is increasing for large $t$. Moreover, if $G(t) = \int_0^t \sqrt{F(s)}ds$, for $a$ and $b$ such that $1 < a < 2 < b$, we have

$$a\frac{F(t)}{f(t)} \le \frac{G(t)}{G'(t)} \le b\frac{F(t)}{f(t)} \quad \text{for large } t. \tag{2.34}$$

Therefore, by [7, Theorem 4(ii)] we have, for some constant $C > 0$,

$$C\delta^2\Phi'(\delta) + \Phi(\delta) \le u(x) \le \Phi(\delta) + C\delta\Phi(\delta). \tag{2.35}$$

Using the right-hand side of (2.35) we find

$$w(x) - u(x) \ge \Phi(\delta)\Big[\beta^{-1}H\delta\big(\Phi(\delta)\big)^{-\beta} + \alpha\delta\big(\Phi(\delta)\big)^{-2\beta} - C\delta\Big]. \tag{2.36}$$

Take $\alpha$ and $\delta_0$ such that (2.33) holds and put $\alpha\delta_0(\Phi(\delta_0))^{-2\beta} = q$. Decrease $\delta_0$ and increase $\alpha$ so that $\alpha\delta_0(\Phi(\delta_0))^{-\beta} = q$ and

$$\beta^{-1}H\delta(\Phi(\delta))^{-\beta} + q - C\delta > 0 \tag{2.37}$$

for $\delta(x) = \delta_0$. Then, $w(x) \geq u(x)$ on $\{x \in \Omega : \delta(x) = \delta_0\}$. When $\alpha$ is fixed, by (2.36) we get $\liminf_{x\to\partial\Omega}[w(x) - u(x)] \geq 0$. Hence, using (2.31) we find $w(x) \geq u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$.

We look for a subsolution of the form

$$v(x) = \Phi(\delta) + \beta^{-1}H\delta(\Phi(\delta))^{1-\beta} - \alpha\delta(\Phi(\delta))^{1-2\beta}, \tag{2.38}$$

where $\alpha$ is a positive constant to be determined. Instead of (2.27), now we find

$$\Delta v > f(\Phi)\left[1 + H\delta - M_1\delta\Phi^{-\beta} - \alpha M_1\delta\Phi^{-2\beta}\right]. \tag{2.39}$$

Of course, the constant $M_1$ in (2.39) and the constants $M_i$ in what follows are not necessarily the same as in the previous case.

Now we have

$$f(v) = e^{\Phi^\beta(1+\beta^{-1}H\delta\Phi^{-\beta}-\alpha\delta\Phi^{-2\beta})^\beta}. \tag{2.40}$$

Let us take $\delta_0 > 0$ and $\alpha$ such that, for $\{x \in \Omega : \delta(x) < \delta_0\}$ we have

$$-\frac{1}{2} < \beta^{-1}H\delta(\Phi(\delta))^{-\beta} - \alpha\delta(\Phi(\delta))^{-2\beta} < 1. \tag{2.41}$$

Then,

$$\begin{aligned} f(v) &< e^{\Phi^\beta(1+H\delta\Phi^{-\beta}-\alpha\beta\delta\Phi^{-2\beta}+M_2(\delta\Phi^{-\beta})^2+M_2(\alpha\delta\Phi^{-2\beta})^2)} \\ &= f(\Phi)e^{H\delta-\alpha\beta\delta\Phi^{-\beta}+M_2\delta^2\Phi^{-\beta}+M_2(\alpha\delta)^2\Phi^{-3\beta}}. \end{aligned} \tag{2.42}$$

In our next step, we take $\delta$ and $\alpha$ such that

$$\alpha\delta\Phi^{-\beta} < 1, \qquad H\delta - \alpha\beta\delta\Phi^{-\beta} + M_2\delta^2\Phi^{-\beta} + M_2(\alpha\delta)^2\Phi^{-3\beta} < 1. \tag{2.43}$$

Then we find

$$f(v) < f(\Phi)\left[1 + H\delta - \alpha\beta\delta\Phi^{-\beta} + M_3\delta^2 + M_3(\alpha\delta)^2\Phi^{-2\beta}\right]. \tag{2.44}$$

By (2.39) and (2.44) we find that $\Delta v > f(v)$ provided

$$1 + H\delta - M_1\delta\Phi^{-\beta} - \alpha M_1\delta\Phi^{-2\beta} > 1 + H\delta - \alpha\beta\delta\Phi^{-\beta} + M_3\delta^2 + M_3(\alpha\delta)^2\Phi^{-2\beta}. \tag{2.45}$$

Rearranging we have

$$\alpha\left[\beta - M_1\Phi^{-\beta} - M_3\alpha\delta\Phi^{-\beta}\right] > M_1 + M_3\delta\Phi^\beta. \tag{2.46}$$

Since $\delta\Phi^\beta \to 0$ as $\delta \to 0$, inequality (2.46) (in addition to (2.41) and (2.43)) holds for $\delta(x) < \delta_0$ with suitable $\delta_0$ and $\alpha$.

Using the left-hand side of (2.35) we find

$$\begin{aligned} v(x)-u(x) &\le \beta^{-1}H\delta(\Phi(\delta))^{1-\beta}-\alpha\delta(\Phi(\delta))^{1-2\beta}-C\delta^2\Phi'(\delta) \\ &= (\Phi(\delta))^{1-\beta}\left[\beta^{-1}H\delta-\alpha\delta(\Phi(\delta))^{-\beta}-C\delta^2\Phi'(\delta)(\Phi(\delta))^{\beta-1}\right]. \end{aligned} \tag{2.47}$$

Take $\alpha$ and $\delta_0$ such that (2.46) holds, and put $\alpha\delta_0(\Phi(\delta_0))^{-\beta}=q$. Decrease $\delta_0$ and increase $\alpha$ so that $\alpha\delta_0(\Phi(\delta_0))^{-\beta}=q$ and

$$\beta^{-1}H\delta-q-C\delta^2\Phi'(\delta)(\Phi(\delta))^{\beta-1}<0 \tag{2.48}$$

for $\delta(x)=\delta_0$. Note that the previous inequality holds for $\delta$ small because

$$\lim_{\delta\to 0}\frac{\delta^2\Phi'(\delta)}{(\Phi(\delta))^{1-\beta}}=0, \tag{2.49}$$

as one can prove using Lemma 2.3 and de l'Hôpital's rule. It follows from (2.47) that $v(x)\le u(x)$ on $\{x\in\Omega:\delta(x)=\delta_0\}$. By (2.47) we also find that $v(x)-u(x)\le 0$ on $\partial\Omega$. Hence $v(x)\le u(x)$ on $\{x\in\Omega:\delta(x)<\delta_0\}$. The theorem follows. □

## 3. The equation $\Delta u = e^{u+e^u}$

Lemma 3.1. *Let $f(t)=e^{t+e^t}$, $F(s)=\int_{-\infty}^s f(t)dt$. Then*

$$F(s)f'(s)(f(s))^{-2}=1+O(1)e^{-s}, \tag{3.1}$$

*where $O(1)$ is a bounded quantity.*

*Proof.* By computation we find

$$F(s)f'(s)(f(s))^{-2}=1+e^{-s}-e^{-e^s}-e^{-s-e^s}. \tag{3.2}$$

The lemma follows. □

Lemma 3.2. *Let $f(t)$ and $F(s)$ be as in Lemma 3.1. If*

$$\int_{\Psi(\delta)}^{\infty}(2F(s))^{-1/2}ds=\delta \tag{3.3}$$

*we have*

$$-\Psi'(\delta)=[1+O(1)e^{-\Psi(\delta)}]\delta f(\Psi(\delta)). \tag{3.4}$$

*Proof.* By the (trivial) relation

$$-1+2(1+O(1)e^{-s})=1+O(1)e^{-s}, \tag{3.5}$$

using (3.1) we have

$$-1+2F(s)f'(s)(f(s))^{-2}=1+O(1)e^{-s}. \tag{3.6}$$

Multiplying by $(2F(s))^{-1/2}$ we find

$$\begin{gathered}-(2F(s))^{-1/2}+(2F(s))^{1/2}f'(s)(f(s))^{-2}=(2F(s))^{-1/2}+O(1)(2F(s))^{-1/2}e^{-s},\\ -\left((2F(s))^{1/2}(f(s))^{-1}\right)'=(2F(s))^{-1/2}+O(1)(2F(s))^{-1/2}e^{-s}.\end{gathered}\tag{3.7}$$

Integrating on $(s,\infty)$ we get

$$(2F(s))^{1/2}(f(s))^{-1}=\int_s^\infty(2F(t))^{-1/2}dt+O(1)\int_s^\infty(2F(t))^{-1/2}e^{-t}dt.\tag{3.8}$$

Using de l'Hôpital's rule we find

$$\lim_{s\to\infty}\frac{e^{-s}\int_s^\infty(2F(t))^{-1/2}dt}{\int_s^\infty(2F(t))^{-1/2}e^{-t}dt}=1+\lim_{s\to\infty}\frac{\int_s^\infty(2F(t))^{-1/2}dt}{(2F(s))^{-1/2}}=1.\tag{3.9}$$

Using (3.9), (3.8) can be rewritten as

$$(2F(s))^{1/2}(f(s))^{-1}=\int_s^\infty(2F(t))^{-1/2}dt+O(1)e^{-s}\int_s^\infty(2F(t))^{-1/2}dt.\tag{3.10}$$

Putting $s=\Psi(\delta)$ and recalling that $-\Psi'(\delta)=(2F(\Psi(\delta)))^{1/2}$, the lemma follows. □

Theorem 3.3. *Let $\Omega$ be a bounded smooth domain in $R^N$, $N\geq 2$, and let $f(t)=e^{t+e^t}$. If $u(x)$ is a boundary blowup solution of $\Delta u=f(u)$ in $\Omega$, then we have*

$$u(x)=\Psi+He^{-\Psi}\delta+O(1)e^{-2\Psi}\delta,\tag{3.11}$$

*where $\Psi=\Psi(\delta)$ is defined as in Lemma 3.2 and $H=H(x)$ is defined by (1.7).*

*Proof.* We look for a super-solution of the form

$$w(x)=\Psi+He^{-\Psi}\delta+\alpha e^{-2\Psi}\delta,\tag{3.12}$$

where $\alpha$ is a positive constant to be determined. Denoting by $'$ differentiation with respect to $\delta$, we have

$$w_{x_i}=\Psi'\delta_{x_i}+H_{x_i}e^{-\Psi}\delta+H(e^{-\Psi}\delta)'\delta_{x_i}+\alpha(e^{-2\Psi}\delta)'\delta_{x_i}.\tag{3.13}$$

Using (1.7) we find

$$\begin{aligned}\Delta w=\Psi''-\Psi'H+\Delta He^{-\Psi}\delta+(2\nabla H\cdot\nabla\delta-H^2)(e^{-\Psi}\delta)'+H(e^{-\Psi}\delta)''\\ -\alpha H(e^{-2\Psi}\delta)'+\alpha(e^{-2\Psi}\delta)''.\end{aligned}\tag{3.14}$$

By Lemma 3.2 we have $-\Psi'=[1+O(1)e^{-\Psi}]\delta f(\Psi)$, and $\Psi''=f(\Psi)$. Moreover, since $\Psi'\delta\to 0$ as $\delta\to 0$, for $\delta$ small we also find

$$0<(e^{-\Psi}\delta)'=e^{-\Psi}-e^{-\Psi}\Psi'\delta<C_1e^{-\Psi}.\tag{3.15}$$

We denote with $C_i$ positive constants (independent of $\alpha$). Since $f(\Psi)\delta^2 \to 0$ and $f(\Psi)\delta \to \infty$ as $\delta \to 0$, we get

$$0 < (e^{-\Psi}\delta)'' = -2e^{-\Psi}\Psi' - e^{-\Psi}f(\Psi)\delta + e^{-\Psi}(\Psi')^2\delta < C_2 e^{-\Psi}f(\Psi)\delta. \tag{3.16}$$

Similarly, we find

$$\begin{gathered} 0 < (e^{-2\Psi}\delta)' < C_3 e^{-2\Psi}, \\ 0 < (e^{-2\Psi}\delta)'' < C_4 e^{-2\Psi}f(\Psi)\delta. \end{gathered} \tag{3.17}$$

Therefore, by (3.14) we infer

$$\Delta w < f(\Psi)[1 + H\delta + M_1 e^{-\Psi}\delta + \alpha M_2 e^{-2\Psi}\delta]. \tag{3.18}$$

On the other side, since

$$e^w = e^{\Psi + He^{-\Psi}\delta + \alpha e^{-2\Psi}\delta} > e^{\Psi}[1 + He^{-\Psi}\delta + \alpha e^{-2\Psi}\delta], \tag{3.19}$$

we find

$$\begin{aligned} f(w) = e^{w+e^w} &> e^{\Psi + He^{-\Psi}\delta + \alpha e^{-2\Psi}\delta + e^{\Psi}[1 + He^{-\Psi}\delta + \alpha e^{-2\Psi}\delta]} \\ &= e^{\Psi + e^{\Psi}} e^{[He^{-\Psi}\delta + \alpha e^{-2\Psi}\delta + H\delta + \alpha e^{-\Psi}\delta]} \\ &> f(\Psi)[1 - M_3 e^{-\Psi}\delta + H\delta + \alpha e^{-\Psi}\delta]. \end{aligned} \tag{3.20}$$

By (3.18) and (3.20) we have

$$\Delta w < f(w) \tag{3.21}$$

provided

$$1 + H\delta + M_1 e^{-\Psi}\delta + \alpha M_2 e^{-2\Psi}\delta < 1 - M_3 e^{-\Psi}\delta + H\delta + \alpha e^{-\Psi}\delta. \tag{3.22}$$

Rearranging we find

$$M_1 + M_3 < \alpha[1 - M_2 e^{-\Psi(\delta)}]. \tag{3.23}$$

Inequality (3.23) holds provided $\delta$ is small and $\alpha$ is large enough.

The function $f(t) = e^{t+e^t}$ is positive and increasing for all $t$. If $F(t)$ is defined as in Lemma 3.1, the function $F(t)t^{-2}$ is increasing for large $t$. Moreover, if $G(t) = \int_0^t \sqrt{F(s)}ds$, for $1 < a < 2 < b$ we have

$$a\frac{F(t)}{f(t)} \le \frac{G(t)}{G'(t)} \le b\frac{F(t)}{f(t)} \quad \text{for large } t. \tag{3.24}$$

Therefore, by [7, Theorem 4(ii)] we have, for some constant $C > 0$,

$$C\delta^2\Psi'(\delta) + \Psi(\delta) \le u(x) \le \Psi(\delta) + C\delta\Psi(\delta). \tag{3.25}$$

Using the right-hand side of (3.25) we find

$$w(x) - u(x) \geq He^{-\Psi}\delta + \alpha e^{-2\Psi}\delta - C\delta\Psi(\delta). \tag{3.26}$$

Take $\alpha$ and $\delta_0$ so that (3.23) holds for $\delta(x) = \delta_0$ and put $q = \alpha e^{-2\Psi(\delta_0)}\delta_0$. Decrease $\delta_0$ and increase $\alpha$ so that $\alpha e^{-2\Psi(\delta_0)}\delta_0 = q$ and $He^{-\Psi}\delta + q - C\delta\Psi(\delta) > 0$ for $\delta(x) = \delta_0$. Recall that $\delta\Psi(\delta) \to 0$ as $\delta \to 0$. Then, $w(x) \geq u(x)$ on $\{x \in \Omega : \delta(x) = \delta_0\}$. Moreover, by (3.26) we have $w(x) - u(x) \geq 0$ on $\partial\Omega$. Hence, using (3.21) we find $w(x) \geq u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$.

Let us prove that

$$v = \Psi + He^{-\Psi}\delta - \alpha e^{-2\Psi}\delta \tag{3.27}$$

is a subsolution provided $\alpha$ is a suitable positive constant. By computation, instead of (3.18), now we find

$$\Delta v > f(\Psi)\left[1 + H\delta - M_4 e^{-\Psi}\delta - \alpha M_5 e^{-2\Psi}\delta\right]. \tag{3.28}$$

The next step is slightly delicate. Take $\alpha$ and $\delta$ such that

$$e\alpha e^{-\Psi}\delta < 1, \qquad He^{-\Psi}\delta - \alpha e^{-2\Psi}\delta < 1. \tag{3.29}$$

Then, using the second inequality in (3.29), we find

$$e^v = e^{\Psi + He^{-\Psi}\delta - \alpha e^{-2\Psi}\delta} < e^{\Psi}\left[1 + He^{-\Psi}\delta - \alpha e^{-2\Psi}\delta + e\left(He^{-\Psi}\delta\right)^2 + e\left(\alpha e^{-2\Psi}\delta\right)^2\right]. \tag{3.30}$$

Hence, using the first inequality in (3.29), we get

$$\begin{aligned} f(v) = e^{v+e^v} &< e^{\Psi + He^{-\Psi}\delta - \alpha e^{-2\Psi}\delta + e^{\Psi} + H\delta - \alpha e^{-\Psi}\delta + eH^2 e^{-\Psi}\delta^2 + e\alpha^2 e^{-3\Psi}\delta^2} \\ &< f(\Psi)e^{H\delta + M_6 e^{-\Psi}\delta - \alpha e^{-\Psi}\delta} < f(\Psi)\left[1 + H\delta + M_7 e^{-\Psi}\delta - \alpha e^{-\Psi}\delta + \left(\alpha e^{-\Psi}\delta\right)^2\right]. \end{aligned} \tag{3.31}$$

Comparing the last estimate with (3.28) we have

$$\Delta v > f(v) \tag{3.32}$$

provided

$$1 + H\delta - M_4 e^{-\Psi}\delta - \alpha M_5 e^{-2\Psi}\delta > 1 + H\delta + M_7 e^{-\Psi}\delta - \alpha e^{-\Psi}\delta + \left(\alpha e^{-\Psi}\delta\right)^2. \tag{3.33}$$

Rearranging, this inequality reads as

$$\alpha\left[1 - \alpha e^{-\Psi}\delta - M_5 e^{-\Psi}\right] > M_4 + M_7. \tag{3.34}$$

Of course, (3.34) and (3.29) hold provided $\alpha$ is large and $\delta$ is small enough. Using the left-hand side of (3.25), decreasing $\delta_0$ and increasing $\alpha$ if necessary, one proves that $v(x) - u(x) \leq 0$ at all points in $\Omega$ with $\delta(x) = \delta_0$. Moreover, using (3.25) again we observe that $v(x) - u(x) \leq 0$ on $\partial\Omega$. Therefore, by (3.32) it follows that $v(x)$ is a subsolution on $\{x \in \Omega : \delta(x) < \delta_0\}$. The theorem is proved. □

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