EXISTENCE OF SOLUTIONS FOR A NONLINEAR ELLIPTIC DIRICHLET BOUNDARY VALUE PROBLEM WITH AN INVERSE SQUARE POTENTIAL

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Via the linking theorem, the existence of nontrivial solutions for a nonlinear elliptic Dirichlet boundary value problem with an inverse square potential is proved.

1. Introduction

This paper is concerned with the existence of nontrivial solutions to the following problem:

$$\begin{aligned} -\triangle u - \frac{\mu}{|x|^2}u &= |u|^{p-2}u + \lambda u \quad \text{in } \Omega \setminus \{0\}, \\ u(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $0 \in \Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $0 \leq \mu < \overline{\mu} = ((N-2)/2)^2$, and $\overline{\mu}$ is the best constant in the Hardy inequality:

$$C\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,dx \leq \int_{\mathbb{R}^N} |\nabla u|^2\,dx \tag{1.2}$$

(cf. [3, Lemma 2.1]), $2 < p < 2^*$, where $2^* = 2N/(N-2)$ is the so-called critical Sobolev exponent and $\lambda > 0$ is a parameter.

Finally, in Theorem 1.3 we prove, for small $\lambda > 0$, the existence of a solution to

$$\begin{aligned} -\triangle u - \frac{\mu}{|x|^2}u &= u^{p-1} + \lambda u \quad \text{in } \Omega \setminus \{0\}, \\ u(x) &> 0 \quad \text{in } \Omega \setminus \{0\}, \\ u(x) &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

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In the case $\mu = 0$, problem (1.1) has been studied extensively. For example, when $p = 2^*$, Capozzi et al. [1] have shown that (1.1) has at least one positive solution for $N \geq 5$. When $2 < p < 2^*$, the existence of positive solutions of (1.1) has been shown in [5, Chapter 1].

Our results are the following.

Theorem 1.1. *Let $0 \in \Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be an open bounded domain. If $0 \leq \mu < \overline{\mu}$, then for any $\lambda > 0$, problem (1.1) possesses a nontrivial solution.*

Remark 1.2. We mention that when $p = 2^*$, the existence of nontrivial solutions of (1.1) has been proved in [2, Theorem 1.3].

Theorem 1.3. *Let $0 \in \Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be an open bounded domain. If $0 \leq \mu < \bar{\mu}$, problem (1.3) has a positive solution for $0 < \lambda < \lambda_1$, where λ_1 denotes the first eigenvalue of the operator $-\triangle - \mu/|x|^2$.*

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proof of Theorem 1.1. The proof of Theorem 1.3 is contained in Section 4.

2. Notations and preliminaries

Throughout this paper, c, c_i will denote various positive constants whose exact values are not important. $H_0^1(\Omega)$ will be denoted as the standard Sobolev space, whose norm $\|\cdot\|$ is deduced by the standard inner product. By $|\cdot|_p$, we denote the norm of $L^p(\Omega)$. All integrals are taken over Ω unless stated otherwise. On $H_0^1(\Omega)$, we use the norm

$$\|u\|_\mu^2 = \int \left(|\nabla u|^2 - \frac{\mu}{|x|^2} u^2 \right) dx. \tag{2.1}$$

It follows from the Hardy inequality that the norm $\|\cdot\|_\mu$ is equivalent to the usual norm $\|\cdot\|$ of $H_0^1(\Omega)$. $H_0^1(\Omega)$ with the norm $\|\cdot\|_\mu$ is simply denoted by H.

By using the critical point theory, we define the action function on H:

$$J_\mu(u) = \frac{1}{2} \int \left(|\nabla u|^2 - \frac{\mu}{|x|^2} u^2 \right) dx - \frac{1}{p} \int |u|^p \, dx - \frac{\lambda}{2} \int |u|^2 \, dx. \tag{2.2}$$

It is well known that a weak solution $u \in H_0^1(\Omega)$ of (1.1) is precisely a critical point of J_μ. That is,

$$\langle J_\mu'(u), \varphi \rangle = \int \left(\nabla u \nabla \varphi - \frac{\mu}{|x|^2} u\varphi \right) dx - \int |u|^{p-2} u\varphi \, dx - \lambda \int u\varphi \, dx = 0 \tag{2.3}$$

holds for any $\varphi \in H_0^1(\Omega)$. The following definition has become standard.

Definition 2.1 (see [6, Definition 1.16]). Let $c \in \mathbb{R}$, let E be a Banach space, and let $I \in C^1(E, \mathbb{R})$. Say that I satisfies $(\mathrm{PS})_c$ condition if any sequence $\{u_n\}$ in E such that $I(u_n) \to c$ and $\|I'(u_n)\|_{E^{-1}} \to 0$ has a convergent subsequence. If this holds for every $c \in \mathbb{R}$, I satisfies (PS) condition.

Now we will prove that J_μ satisfies (PS) condition, which is contained in the following two lemmas.

Lemma 2.2. *If $0 \le \mu < \overline{\mu} = ((N-2)/2)^2$, then any sequence $\{u_n\} \subset H_0^1(\Omega)$ satisfying*

$$J_\mu(u_n) \longrightarrow c, \quad J_\mu'(u_n) \longrightarrow 0, \quad n \longrightarrow \infty, \tag{2.4}$$

is bounded in $H_0^1(\Omega)$.

Proof. Since

$$\begin{aligned} J_\mu(u_n) &= \frac{1}{2}\int \left(|\nabla u_n|^2 - \frac{\mu}{|x|^2}u_n^2\right)dx - \frac{1}{p}\int |u_n|^p\,dx - \frac{\lambda}{2}\int |u_n|^2\,dx, \\ \langle J_\mu'(u_n), \varphi\rangle &= \int \left(\nabla u_n \nabla\varphi - \frac{\mu}{|x|^2}u_n\varphi\right)dx - \int |u_n|^{p-2}u_n\varphi\,dx - \lambda\int u_n\varphi\,dx. \end{aligned} \tag{2.5}$$

Choose $2 < q < p$, and let $\varphi = u_n$ in (2.5). For n large enough,

$$\begin{aligned} &c + 1 + o(1)\|u_n\|_\mu \\ &\quad \ge J_\mu(u_n) - \frac{1}{q}\left\langle J_\mu'(u_n), u_n\right\rangle \\ &\quad = \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|_\mu^2 + \left(\frac{1}{q} - \frac{1}{p}\right)\int |u_n|^p\,dx + \left(\frac{1}{q} - \frac{1}{2}\right)\lambda\int |u_n|^2\,dx \\ &\quad \ge \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|_\mu^2 + \left(\frac{1}{q} - \frac{1}{p}\right)\int |u_n|^p\,dx + \left(\frac{1}{q} - \frac{1}{2}\right)\lambda C\|u_n\|_\mu^2. \end{aligned} \tag{2.6}$$

It follows from $p > 2$ that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. □

Lemma 2.3. *Under the assumption of Lemma 2.2, $\{u_n\}$ possesses a convergent subsequence in H.*

Proof. By Lemma 2.2, going if necessary to a subsequence, we can assume that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } H, \\ &u_n \longrightarrow u \quad \text{in } L^r(\Omega) \text{ for } 1 \le r < 2^*. \end{aligned} \tag{2.7}$$

Let $f(u) = |u|^{p-2}u$, [5, Theorem A.2] implies that $f(u_n) \to f(u)$ in L^s, where $s = r/(r-1)$. Observe that

$$\|u_n - u\|_\mu^2 = \left\langle J_\mu'(u_n) - J_\mu'(u), u_n - u\right\rangle + \int \left[(f(u_n) - f(u))(u_n - u) + \lambda(u_n - u)^2\right]dx. \tag{2.8}$$

It is clear that

$$\left\langle J_\mu'(u_n) - J_\mu'(u), u_n - u\right\rangle \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.9}$$

It follows from the Hölder inequality that

$$\int [(f(u_n) - f(u))(u_n - u)]\,dx \le |f(u_n) - f(u)|_{r/(r-1)}|u_n - u|_r \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.10}$$

Thus we have proved that $\|u_n - u\|_\mu \to 0$, $n \to \infty$. □

3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 via the following linking theorem from Rabinowitz [5, Theorem 5.3] (see also [6]).

Proposition 3.1. *Let E be a Banach space with $E = Y \oplus X$, where $\dim Y < \infty$. Suppose that $I \in C^1(E,\mathbb{R})$ and satisfies that*
 (i) *there exist $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \ge \alpha$;*
 (ii) *there exist $e \in \partial B_1 \cap X$ and $R > \rho$ such that if $Q \equiv (\overline{B_\rho} \cap Y) \oplus \{re;\ 0 < r < R\}$, then $I|_{\partial Q} \le 0$.*

If I satisfies $(PS)_c$ condition with

$$c = \inf_{h\in\Gamma}\max_{u\in Q} I(h(u)), \tag{3.1}$$

where

$$\Gamma = \{h \in C(\overline{Q}, E); h|_{\partial Q} = \mathrm{id}\}, \tag{3.2}$$

then c is a critical value of I and $c \ge \alpha$.

Remark 3.2 (see [5, Remark 5.5(iii)]). Suppose $I|_Y \le 0$ and there are an $e \in \partial B_1 \cap X$ and $\tilde{T} > \rho$ such that $I(u) \le 0$ for $u \in Y \oplus \operatorname{span}\{e\}$ and $\|u\| \ge \tilde{T}$, then for any large T, $Q = (\overline{B_\rho} \cap Y) \oplus \{te; 0 < t < T\}$ satisfies $I|_{\partial Q} \le 0$.

To continue our discussion, we may assume that there is k such that $\lambda_k \le \lambda < \lambda_{k+1}$, where λ_k is the kth eigenvalue of the operator $(-\triangle - \mu/|x|^2)$ with Dirichlet boundary condition (see [2, 4]). Let

$$Y := Y_k = \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_k\}, \tag{3.3}$$

here ϕ_i denotes the eigenfunction corresponding to λ_i. Decompose $H_0^1(\Omega) = Y \oplus X$ (where X is the topological complement of Y in $H_0^1(\Omega)$). For any $y \in Y$, we have that

$$\int \left(|\nabla y|^2 - \frac{\mu}{|x|^2}y^2\right)dx \le \lambda_k \int y^2\,dx, \tag{3.4}$$

$$\int \left(|\nabla u|^2 - \frac{\mu}{|x|^2}u^2\right)dx \ge \lambda_{k+1} \int u^2\,dx \quad \text{for any } u \in X. \tag{3.5}$$

Now we will show that J_μ satisfies (i), (ii) in Proposition 3.1 in our situation.

Proposition 3.3. *There exist $\rho, \alpha > 0$ such that $J_\mu \mid_{\partial B_\rho \cap X} \geq \alpha$.*

Proof. For any $u \in X$, $\lambda_k \leq \lambda < \lambda_{k+1}$, we obtain from (3.5) and Sobolev inequality that

$$\begin{aligned} J_\mu(u) &= \frac{1}{2}\int \left(|\nabla u|^2 - \frac{\mu}{|x|^2}u^2\right) dx - \frac{1}{p}\int |u|^p\, dx - \frac{\lambda}{2}\int |u|^2\, dx \\ &\geq \frac{1}{2}\frac{\lambda_{k+1}-\lambda}{\lambda_{k+1}}\int \left(|\nabla u|^2 - \frac{\mu}{|x|^2}u^2\right) dx - \frac{1}{p}\int |u|^p\, dx \\ &\geq \frac{1}{2}\frac{\lambda_{k+1}-\lambda}{\lambda_{k+1}}\|u\|_\mu^2 - c\|u\|_\mu^p. \end{aligned} \tag{3.6}$$

Then we can choose $\|u\|_\mu = \rho$ sufficiently small and $\alpha > 0$ such that $J_\mu \mid_{\partial B_\rho \cap X} \geq \alpha$. □

Proposition 3.4. *J_μ verifies (ii) of Proposition 3.1.*

Proof. First, for any $y \in Y$, we obtain from (3.4) that

$$\begin{aligned} J_\mu(y) &= \frac{1}{2}\int \left(|\nabla y|^2 - \frac{\mu}{|x|^2}y^2\right) dx - \frac{1}{p}\int |y|^p\, dx - \frac{\lambda}{2}\int |y|^2\, dx \\ &\leq \frac{1}{2}\frac{\lambda_k-\lambda}{\lambda_k}\int \left(|\nabla y|^2 - \frac{\mu}{|x|^2}y^2\right) dx - \frac{1}{p}\int |y|^p\, dx \\ &= \frac{1}{2}\frac{\lambda_k-\lambda}{\lambda_k}\|y\|_\mu^2 - \frac{1}{p}|y|_p^p. \end{aligned} \tag{3.7}$$

Thus $J_\mu(y) \leq 0$ since all norms are equivalent on Y. Let $e := \phi_{k+1}$ be the $(k+1)$th eigenfunction of $(-\triangle - \mu/\mid x\mid^2)$, since for any $y \in Y$,

$$J_\mu(y + t\phi_{k+1}) \longrightarrow -\infty \qquad \text{as } t \longrightarrow \infty. \tag{3.8}$$

It follows from Remark 3.2 that we can take T sufficiently large and define $Q = (\overline{B_T} \cap Y) \oplus \{re; 0 < t < T\}$ such that Proposition 3.4 holds. □

The proof in the case of $c \geq \alpha$ is the same as in the proof of [5, Theorem 5.3], by now we have completed the proof of Theorem 1.1.

4. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3. Here we define the following Euler-Lagrange functional of (1.3) on H:

$$\widetilde{J}_\mu(u) = \frac{1}{2}\int \left(|\nabla u|^2 - \frac{\mu}{|x|^2}u^2\right) dx - \frac{1}{p}\int (u^+)^p\, dx - \frac{\lambda}{2}\int (u^+)^2\, dx, \tag{4.1}$$

where $u^+ = \max\{u, 0\}$, and for any $\varphi \in C_0^\infty(\Omega)$,

$$\left\langle \widetilde{J}_\mu(u), \varphi \right\rangle = \int \left(\nabla u \nabla \varphi - \frac{\mu}{|x|^2}u\varphi\right) dx - \int (u^+)^{p-1}\varphi\, dx - \lambda\int (u^+)\varphi\, dx. \tag{4.2}$$

By using the same method in the proof of Theorem 1.1, we obtain that $\widetilde{J}_\mu$ satisfies (PS) condition. Next, we just use the mountain pass theorem to prove Theorem 1.3.

It is easy to check that $\widetilde{J}_\mu(u) \in C^1(H_0^1(\Omega), \mathbb{R})$, we will verify the assumptions of the mountain pass theorem. By the Sobolev theorem, there exists $c_1 > 0$, such that for $u \in H$, $\|u\|_{L^p(\Omega)} \leq c_1 \|u\|_\mu$. Hence we have

$$\begin{aligned}\widetilde{J}_\mu(u) &= \frac{1}{2}\int \left(|\nabla u|^2 - \frac{\mu}{|x|^2}u^2 \right) dx - \frac{1}{p}\int (u^+)^p\, dx - \frac{\lambda}{2}\int (u^+)^2\, dx \\ &\geq \frac{1}{2}\|u\|_\mu^2 - \frac{c_1}{p}\|u\|_\mu^p - \frac{\lambda}{2\lambda_1}\|u\|_\mu^2 \\ &= \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|_\mu^2 - \frac{c_1}{p}\|u\|_\mu^p.\end{aligned} \tag{4.3}$$

So there is $r > 0$ such that

$$b := \inf_{\|u\|_\mu = r} \widetilde{J}_\mu(u) > 0 = \widetilde{J}_\mu(0). \tag{4.4}$$

Let $u \in H$ with $u > 0$ on Ω, we have, for $t \geq 0$,

$$\widetilde{J}_\mu(tu) = \frac{t^2}{2}\int \left(|\nabla u|^2 - \frac{\mu}{|x|^2}u^2 \right) dx - \frac{t^p}{p}\int (u^+)^p\, dx - \frac{\lambda t^2}{2}\int (u^+)^2\, dx. \tag{4.5}$$

Since $p > 2$, there exists $e := tu$, such that $\|e\|_\mu > r$ and $\widetilde{J}_\mu(e) \leq 0$. By the mountain pass theorem, $\widetilde{J}_\mu$ has a positive critical value, and problem

$$\begin{gathered} -\triangle u - \frac{\mu}{|x|^2}u = (u^+)^{p-1} + \lambda u^+ \quad \text{in } \Omega \setminus \{0\}, \\ u \in H_0^1(\Omega) \end{gathered} \tag{4.6}$$

has a nontrivial solution u. Multiplying the equation by u^- and integrating over Ω, we find

$$0 = \int \left(|\nabla u^-|^2 - \frac{\mu}{|x|^2}(u^-)^2 \right) dx = \|u^-\|_\mu^2. \tag{4.7}$$

Hence $u^- = 0$, that is, $u \geq 0$. A standard elliptic regularity argument implies that $u \in C^2(\Omega \setminus \{0\})$, in which case, by the strong maximum principle, u is positive, thus is the solution of problem (1.3).

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