THE EXACT ASYMPTOTIC BEHAVIOUR OF THE UNIQUE SOLUTION TO A SINGULAR DIRICHLET PROBLEM

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By Karamata regular variation theory, we show the existence and exact asymptotic behaviour of the unique classical solution $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ near the boundary to a singular Dirichlet problem $-\Delta u = g(u) - k(x)$, $u > 0$, $x \in \Omega$, $u|_{\partial\Omega} = 0$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $g \in C^1((0,\infty),(0,\infty))$, $\lim_{t\to 0^+}(g(\xi t)/g(t)) = \xi^{-\gamma}$, for each $\xi > 0$ and some $\gamma > 1$; and $k \in C^{\alpha}_{\text{loc}}(\Omega)$ for some $\alpha \in (0,1)$, which is nonnegative on Ω and may be unbounded or singular on the boundary.

1. Introduction and the main results

The purpose of this paper is to investigate the existence and exact asymptotic behaviour of the unique classical solution near the boundary to the following model problem:

$$-\triangle u = g(u) - k(x), \quad u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0, \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geq 1)$, $k \in C^{\alpha}_{\text{loc}}(\Omega)$ for some $\alpha \in (0,1)$, which is nonnegative on Ω, and g satisfies

(g_1) $g \in C^1((0,\infty),(0,\infty))$, $g'(s) \leq 0$ for all $s > 0$, $\lim_{s\to 0^+} g(s) = +\infty$.

The problem arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrical conductive materials (see [4, 7, 12, 14]).

The main feature of this paper is the presence of the two terms, the singular term $g(u)$ which is regular varying at zero of index $-\gamma$ with $\gamma > 1$ and includes a large class of singular functions, and the nonhomogeneous term $k(x)$, which may be singular on the boundary.

This type of nonlinear terms arises in the papers of Díaz and Letelier [6], Lasry and Lions [10] for boundary blow-up elliptic problems.

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For $k \equiv 0$ on Ω, problem (1.1) is the following one:

$$-\Delta u = g(u), \quad u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0. \tag{1.2}$$

The problem was discussed and extended to the more general problems in a number of works, see, for instance, [4, 5, 7, 8, 11, 14–17]. Fulks and Maybee [7], Stuart [14], Crandall et al. [4] showed that if g satisfies (g_1), then problem (1.2) has a unique solution $u_0 \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$. Moreover, Crandall et al. [4, Theorems 2.2 and 2.7] showed that there exist positive constants C_1 and C_2 such that

(I) $C_1\psi(d(x)) \le u_0(x) \le C_2\psi(d(x))$ near $\partial\Omega$, where $d(x) = \operatorname{dist}(x, \partial\Omega)$, $\psi \in C[0,a] \cap C^2(0,a]$ is the local solution to the problem

$$-\psi''(s) = g(\psi(s)), \quad \psi(s) > 0, \quad 0 < s < a, \quad \psi(0) = 0. \tag{1.3}$$

Then, for $g(u) = u^{-\gamma}$, $\gamma > 0$, Lazer and McKenna [11], by construction of the global subsolution and supersolution, showed that u_0 has the following properties:

(I_1) if $\gamma > 1$, then $C_1[\phi_1(x)]^{2/(1+\gamma)} \le u_0(x) \le C_2[\phi_1(x)]^{2/(1+\gamma)}$ on $\overline{\Omega}$;

(I_2) if $\gamma > 1$, then $u_0 \notin C^1(\overline{\Omega})$;

(I_3) $u_0 \in H_0^1(\Omega)$ if and only if $\gamma < 3$, this is a basic character to problem (1.2) in the case,

where ϕ_1 is the eigenfunction corresponding to the first eigenvalue of problem $-\Delta u = \lambda u$ in Ω, and $u|_{\partial\Omega} = 0$.

Most recently, when $\int_1^\infty g(s)ds < \infty$, in [16], we showed that

(II) $C_1\psi(d(x)) \le u_0(x) \le C_2\psi(d(x))$, on $\overline{\Omega}$,

where $\psi \in C[0,\infty) \cap C^2(0,\infty)$ is the unique global solution to the problem

$$-\psi''(s) = g(\psi(s)), \quad \psi(s) > 0, \quad s > 0, \quad \psi(0) = 0, \quad \lim_{s\to\infty}\psi(s) = \beta \ge 0. \tag{1.4}$$

Moreover, assume g satisfies (g_1) and

(g_2) there exist positive constants C_0, η_0 and $\gamma \in (0,1)$ such that $g(s) \le C_0 s^{-\gamma}$, for all $s \in (0,\eta_0)$;

(g_3) there exist $\theta > 0$ and $t_0 \ge 1$ such that $g(\xi t) \ge \xi^{-\theta} g(t)$ for all $\xi \in (0,1)$ and $0 < t \le t_0\xi$;

(g_4) the mapping $\xi \in (0,\infty) \to T(\xi) = \lim_{t\to 0^+}(g(\xi t)/\xi g(t))$ is a continuous function.

Ghergu and Rădulescu [8] showed that problem (1.2) has a unique solution $u_0 \in C^{1,1-\alpha}(\overline{\Omega}) \cap C^2(\Omega)$ satisfying

$$\lim_{d(x)\to 0} \frac{u_0(x)}{\psi(d(x))} = \xi_0, \tag{1.5}$$

where $T(\xi_0) = 1$, and $\psi \in C^1[0,a] \cap C^2(0,a]$ $(a \in (0,\eta_0))$ is the local solution to problem (1.3).

For $k \le 0$ on Ω, $k \in L^p(\Omega)$ with $p > N/2$, and $g(u) = u^{-\gamma}$, $\gamma > 0$, Aranda and Godoy [1] showed that problem (1.1) has a unique solution $u \in W^{2,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$.

Most recently, applying Karamata regular variation theory, Cîrstea and Rădulescu [3] and Cîrstea and Du [2] studied the exact asymptotic behaviour of solutions which blow up on the boundary for semilinear elliptic problems.

In this paper, also applying Karamata regular variation theory, and constructing comparison functions, we show the existence and exact asymptotic behaviour of the unique solution near the boundary to problem (1.1).

First we recall a basic definition and a basic property to Karamata regular variation theory [13].

Definition 1.1. A positive measurable function g defined on some neighborhood $(0,b)$ for some $b>0$, is called regularly varying at zero with index β, written $g \in \mathrm{RVZ}_\beta$ if for each $\xi > 0$ and some $\beta \in \mathbb{R}$,

$$\lim_{t\to 0^+} \frac{g(\xi t)}{g(t)} = \xi^\beta. \tag{1.6}$$

When $\beta = 0$, we have the following definition.

Definition 1.2. A positive measurable function L defined on some neighborhood $(0,b)$ for some $b>0$, is called slowly varying at zero, written $L \in \mathrm{RVZ}_0$ if for each $\xi > 0$,

$$\lim_{t\to 0^+} \frac{L(\xi t)}{L(t)} = 1. \tag{1.7}$$

It follows by Definitions 1.1 and 1.2 that if $g \in \mathrm{RVZ}_\beta$, it can be represented in the form

$$g(t) = t^\beta L(t). \tag{1.8}$$

Lemma 1.3 (representation theorem). *The function L is slowly varying at zero if and only if it may be written in the form*

$$L(t) = c(t)\exp\left(\int_t^b \frac{y(s)}{s}\,ds\right), \quad 0<t<b, \tag{1.9}$$

for some $b>0$, where $c(t)$ is a bounded measurable function, $y(t)$ is a continuous function on $[0,b]$, and for $t \to 0^+$, $y(t) \to 0$ and $c(t) \to C_0$, with $C_0 > 0$.

If $c(t)$ is replaced by its limit at zero C_0, a slowly varying function $L_0 \in C^1(0,b]$ of the form

$$L_0(t) = C_0\exp\left(\int_t^b \frac{y(s)}{s}\,ds\right), \quad 0<t<b, \tag{1.10}$$

where $y \in C[0,b]$ with $y(0)=0$, is obtained.

Such a function L_0 is called a normalised slowly varying at zero.

As an important subclass of RVZ_β, *it is defined as*

$$\mathrm{NRVZ}_\beta = \{g \in \mathrm{RVZ}_\beta : g(t)/t^\beta \text{ is a normalised slowly varying at zero}\}. \tag{1.11}$$

Our main results are as follows.

Theorem 1.4. *Let $k \in C^{\alpha}(\Omega)$ be nonnegative, g satisfy* (g_1) *and* $g \in \mathrm{NRVZ}_{-\gamma}$ *with* $\gamma > 1$. *Suppose that there exists a nonnegative constant c_0 such that*

(k) $\lim_{d(x)\to 0}(k(x)/g(\psi(d(x)))) = c_0$;

then problem (1.1) has a unique solution $u \in C(\overline{\Omega}) \cap C^{2+\alpha}(\Omega)$ satisfying

$$\lim_{d(x)\to 0} \frac{u(x)}{\psi(d(x))} = \xi_0, \tag{1.12}$$

where ξ_0 is the unique positive solution to the following equation:

$$\xi^{-1-\gamma} = 1 + \frac{c_0}{\xi}, \tag{1.13}$$

and $\psi \in C[0,a] \cap C^2(0,a]$ is uniquely determined by

$$\int_0^{\psi(t)} \frac{ds}{\sqrt{2G(s)}} = t, \quad G(t) = \int_t^a g(s)ds, \quad a > 0,\ t \in (0,a]. \tag{1.14}$$

Moreover, $\psi \in \mathrm{NRVZ}_{2/(1+\gamma)}$, and there exists $L_0 \in \mathrm{NRVZ}_0$ such that

$$\lim_{d(x)\to 0} \frac{u(x)}{L_0(d(x))\,(d(x))^{2/(1+\gamma)}} = \xi_0. \tag{1.15}$$

In particular, if $g(u) = u^{-\gamma}$, $\gamma > 1$, then $\psi(s) = cs^{2/(1+\gamma)}$, $c = [(1+\gamma)^2/2(\gamma-1)]^{1/(1+\gamma)}$, the unique solution u to problem (1.1) satisfies

$$\lim_{d(x)\to 0} \frac{u(x)}{[d(x)]^{2/(1+\gamma)}} = \left[\frac{(1+\gamma)^2}{2(\gamma-1)}\right]^{1/(1+\gamma)} \xi_0. \tag{1.16}$$

Remark 1.5. In Section 2, we will see that $g \in \mathrm{NRVZ}_{-\gamma}$ with $\gamma > 1$ implies $\lim_{s\to 0^+} g(s) = \infty$ and $G(t) < \infty$, $t > 0$.

Remark 1.6. By the maximum principle [9], one easily sees that problem (1.1) has at most one solution in $C^2(\Omega) \cap C(\overline{\Omega})$.

Remark 1.7. Related to the above result, we raise the following open problem: when $k \le 0$ on Ω and $c_0 < 0$, what is the exact asymptotic behaviour of the unique solution near the boundary to problem (1.1)?

The outline of this article is as follows. In Section 2, we continue to recall some basic properties to Karamata regular variation theory. In Section 3, we prove the asymptotic behaviour of the unique solution u in Theorem 1.4. Finally we show existence of solutions to problem (1.1).

2. Some basic properties of Karamata regular variation theory

Let us continue to recall some basic properties of Karamata regular variation theory (see [13]).

Lemma 2.1. *If L is slowly varying at zero, then*
(i) *for every $\theta > 0$ and $t \to 0^+$,*

$$t^{-\theta}L(t) \longrightarrow \infty, \qquad t^{\theta}L(t) \longrightarrow 0; \tag{2.1}$$

(ii) *for $a > 0$ and $t \to 0^+$,*

$$\int_t^a s^{\beta}L(s)ds \cong (-\beta-1)^{-1}t^{1+\beta}L(t), \quad \text{for } \beta < -1. \tag{2.2}$$

Let Ψ be nondecreasing on $\mathbb{R}$; define (as in [13]) the inverse of Ψ by

$$\Psi^{\leftarrow}(t) = \inf\{s : \Psi(s) \geq t\}. \tag{2.3}$$

Lemma 2.2 [13, Proposition 0.8]. *The following hold:*
(i) *if $f_1 \in \mathrm{RVZ}_{\rho_1}$, $f_2 \in \mathrm{RVZ}_{\rho_2}$ with $\lim_{t\to 0^+} f_2(t) = 0$, then $f_1 \circ f_2 \in \mathrm{RVZ}_{\rho_1\rho_2}$;*
(ii) *if Ψ is nondecreasing on $(0,a)$, $\lim_{t\to 0^+}\Psi(t) = 0$, and $\Psi \in \mathrm{RVZ}_{\rho}$ with $\rho \neq 0$, then $\Psi^{\leftarrow} \in \mathrm{RVZ}_{\rho^{-1}}$.*

By the above lemmas, we can directly obtain the following results.

Corollary 2.3. *If g satisfies (g_1) and $g \in \mathrm{NRVZ}_{-\gamma}$ with $\gamma > 1$, then*

$$\begin{gathered} g(t) = t^{-\gamma}L_0(t), \qquad \int_0^1 g(t)dt = \infty, \\ \lim_{t\to 0^+}\frac{G(t)}{g(t)} = 0, \qquad \lim_{t\to 0^+}\frac{tg(t)}{G(t)} = \gamma - 1, \end{gathered} \tag{2.4}$$

where L_0 is a normalised slowly varying function at zero.

Corollary 2.4. *Under the assumptions in Theorem 1.4, $\psi \in \mathrm{NRVZ}_{2/(1+\gamma)}$.*

Proof. Let $f_1(t) = \int_0^t (ds/\sqrt{2G(s)})$. By the l'Hospital rule and Corollary 2.3, we can easily see that

$$\lim_{t\to 0^+}\frac{tf_1'(t)}{f_1(t)} = 1 + \lim_{t\to 0^+}\frac{tg(t)}{2G(t)} = \frac{1+\gamma}{2}. \tag{2.5}$$

It follows by Lemma 2.2 and [2] that $f_1 \in \mathrm{NRVZ}_{(1+\gamma)/2}$ and $\psi = f_1^{-1} \in \mathrm{NRVZ}_{2/(1+\gamma)}$. □

3. The exact asymptotic behaviour

First we give some preliminary considerations.

Lemma 3.1. *Let g, k, and ψ be in Theorem 1.4. The following hold:*
(i) $\lim_{t\to 0^+}\psi'(t)=\psi'(0)=+\infty$;
(ii) $\lim_{t\to 0^+}(\sqrt{2G(\psi(t))}/g(\psi(t)))=0$.

Proof. By (1.14), we see by a direct calculation that

$$\psi'(t)=\sqrt{2G(\psi(t))},\quad -\psi''(t)=g(\psi(t)),\quad 0<t<a. \tag{3.1}$$

(i) By Corollary 2.4, Lemma 2.1 and $\gamma>1$, we see that there exists $L_0\in \mathrm{NRVZ}_0$ such that

$$\psi(t)=t^{2/(\gamma+1)}L_0(t),\qquad \psi'(t)=t^{(1-\gamma)/(\gamma+1)}L_0(t)\left(\frac{2}{\gamma+1}-y(t)\right). \tag{3.2}$$

So $\lim_{t\to 0^+}\psi'(t)=+\infty$.
(ii) By (g_1) and Corollary 2.3, we see that

$$\lim_{t\to 0^+}\frac{\sqrt{2G(\psi(t))}}{g(\psi(t))}=\lim_{u\to 0^+}\frac{\sqrt{2G(u)}}{g(u)}=\lim_{u\to 0^+}\left(\frac{2G(u)}{g(u)}\right)^{1/2}\lim_{u\to 0^+}\left(\frac{1}{g(u)}\right)^{1/2}=0. \tag{3.3}$$

The exact asymptotic behaviour. Let ξ_0 be the unique positive solution to problem (1.13). For $\varepsilon\in(0,\xi_0^{-1-\gamma}/4)$, denote

$$a_0=\xi_0^{-1-\gamma}=1+\frac{c_0}{\xi_0},\qquad \xi_{1\varepsilon}^{-1-\gamma}=a_0-2\varepsilon,\qquad \xi_{2\varepsilon}^{-1-\gamma}=a_0+2\varepsilon. \tag{3.4}$$

Obviously, $a_0\ge 1$, $c_0/a_0\xi_0=(a_0-1)/a_0\in[0,1)$, and $\xi_0/2<\xi_{2\varepsilon}<\xi_0<\xi_{1\varepsilon}<2\xi_0$. Moreover, it follows by Taylor's formula that

$$c_0\left|\left(\frac{1}{\xi_0}-\frac{1}{\xi_{i\varepsilon}}\right)\right|=\frac{2c_0\varepsilon}{a_0\xi_0(1+\gamma)}+o(\varepsilon)=\frac{2\varepsilon(a_0-1)}{a_0(1+\gamma)}+o(\varepsilon),\quad i=1,2. \tag{3.5}$$

Thus there exist $\varepsilon_1>0$ and $\rho_0\in(2(a_0-1)/a_0(1+\gamma),1)$ such that

$$c_0\left|\left(\frac{1}{\xi_0}-\frac{1}{\xi_{i\varepsilon}}\right)\right|<\rho_0\varepsilon\quad\text{for } \varepsilon\in(0,\varepsilon_1). \tag{3.6}$$

For $\delta>0$, we define $\Omega_\delta=\{x\in\Omega: d(x)\le\delta\}$. By the regularity of $\partial\Omega$ and Lemma 3.1, we can choose δ sufficiently small such that
(i) $d(x)\in C^2(\Omega_\delta)$;
(ii) $|c_0(1/\xi_{i\varepsilon}-1/\xi_0)-(\sqrt{2G(\psi(s))}/g(\psi(s)))\triangle d(x)+(1/\xi_{i\varepsilon})(k(x)/g(\psi(d(x)))-c_0)|<\varepsilon$, for all $(s,x)\in(0,\delta)\times\Omega_\delta$, $i=1,2$;
(iii) $(\xi_{2\varepsilon}g(\psi(d(x)))/g(\xi_{2\varepsilon}\psi(d(x))))(\xi_{2\varepsilon}^{-1-\gamma}-\varepsilon)<1<(\xi_{1\varepsilon}g(\psi(d(x)))/g(\xi_{1\varepsilon}\psi(d(x))))(\xi_{1\varepsilon}^{-1-\gamma}+\varepsilon)$ in Ω_δ.

For any $x \in \Omega_\delta$, define $\overline{u} = \xi_{1\varepsilon}\psi(d(x))$, and $\underline{u} = \xi_{2\varepsilon}\psi(d(x))$. It follows by $|\nabla d(x)| = 1$ that

$$\begin{aligned}
&\triangle\overline{u}(x) + g(\overline{u}(x)) - k(x)\\
&\quad= g(\xi_{1\varepsilon}\psi(d(x))) + \xi_{1\varepsilon}\psi''(d(x)) + \xi_{1\varepsilon}\psi'(d(x))\triangle d(x) - k(x)\\
&\quad= \xi_{1\varepsilon}g(\psi(d(x)))\left[\frac{g(\xi_{1\varepsilon}\psi(d(x)))}{\xi_{1\varepsilon}g(\psi(d(x)))} - \left(1+\frac{c_0}{\xi_0}\right) - c_0\left(\frac{1}{\xi_{1\varepsilon}} - \frac{1}{\xi_0}\right)\right.\\
&\qquad\qquad\left. + \frac{\sqrt{2G(\psi(d(x)))}}{g(\psi(d(x)))}\triangle d(x) - \frac{1}{\xi_{1\varepsilon}}\left(\frac{k(x)}{g(\psi(d(x)))} - c_0\right)\right]\\
&\quad\le \xi_{1\varepsilon}g(\psi(d(x)))\left[\left(1+\frac{\lambda c_0}{\xi_0} - \varepsilon\right) - \left(1+\frac{c_0}{\xi_0}\right) - c_0\left(\frac{1}{\xi_{1\varepsilon}} - \frac{1}{\xi_0}\right)\right.\\
&\qquad\qquad\left. + \frac{\sqrt{2G(\psi(d(x)))}}{g(\psi(d(x)))}\triangle d(x) - \frac{1}{\xi_{1\varepsilon}}\left(\frac{k(x)}{g(\psi(d(x)))} - c_0\right)\right] \le 0;\\
&\triangle\underline{u}(x) + g(\underline{u}(x)) - k(x)\\
&\quad= g(\xi_{2\varepsilon}\psi(d(x))) + \xi_{2\varepsilon}\psi''(d(x)) + \xi_{2\varepsilon}\psi'(d(x))\triangle d(x) - k(x)\\
&\quad= \xi_{2\varepsilon}g(\psi(d(x)))\left[\frac{g(\xi_{2\varepsilon}\psi(d(x)))}{\xi_{2\varepsilon}g(\psi(d(x)))} - \left(1+\frac{c_0}{\xi_0}\right) - c_0\left(\frac{1}{\xi_{2\varepsilon}} - \frac{1}{\xi_0}\right)\right.\\
&\qquad\qquad\left. + \frac{\sqrt{2G(\psi(d(x)))}}{g(\psi(d(x)))}\triangle d(x) - \frac{1}{\xi_{2\varepsilon}}\left(\frac{k(x)}{g(\psi(d(x)))} - c_0\right)\right]\\
&\quad\ge \xi_{2\varepsilon}g(\psi(d(x)))\left[\left(1+\frac{c_0}{\xi_0} + \varepsilon\right) - \left(1+\frac{c_0}{\xi_0}\right) - c_0\left(\frac{1}{\xi_{2\varepsilon}} - \frac{1}{\xi_0}\right)\right.\\
&\qquad\qquad\left. + \frac{\sqrt{2G(\psi(d(x)))}}{g(\psi(d(x)))}\triangle d(x) - \frac{1}{\xi_{2\varepsilon}}\left(\frac{k(x)}{g(\psi(d(x)))} - c_0\right)\right] \ge 0.
\end{aligned} \tag{3.7}$$

Let $u \in C(\overline{\Omega}) \cap C^{2+\alpha}(\Omega)$ be the unique solution to problem (1.1). We assert

$$\xi_{2\varepsilon}\psi(d(x)) = \underline{u}(x) \le u(x) \le \overline{u}(x) = \xi_{1\varepsilon}\psi(d(x)) \quad \forall x \in \Omega_\delta. \tag{3.8}$$

In fact, denote $\Omega_\delta = \Omega_{\delta+} \cup \Omega_{\delta-}$, where $\Omega_{\delta+} = \{x \in \Omega_\delta : u(x) \ge \underline{u}(x)\}$ and $\Omega_{\delta-} = \{x \in \Omega_\delta : u(x) < \underline{u}(x)\}$. We see by ($g_1$) that

$$-\Delta(u - \underline{u})(x) \ge g(u(x)) - g(\underline{u}(x)) > 0, \quad x \in \Omega_{\delta-}. \tag{3.9}$$

Since $(u - \underline{u})(x) = 0$, $x \in \partial\Omega_{\delta-}$, we see by the maximum principle [9, Theorem 2.3] that $u(x) \ge \underline{u}(x)$, $x \in \Omega_{\delta-}$, that is, $\Omega_{\delta-} = \emptyset$. Thus $\xi_{2\varepsilon}\psi(d(x)) \le u(x)$, for all $x \in \Omega_\delta$. In the same way, we can see that $u(x) \le \xi_{1\varepsilon}\psi(d(x))$, for all $x \in \Omega_\delta$. Let $\varepsilon \to 0$, we see that $\lim_{d(x)\to 0}(u(x)/\psi(d(x))) = \xi_0$. By Corollary 2.4, the proof is finished. □

4. Existence of solutions

First we introduce a sub-supersolution method with the boundary restriction (see [5]).

We consider the more general following problem:

$$-\Delta u = f(x,u), \quad u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0. \tag{4.1}$$

Definition 4.1. A function $\underline{u} \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ is called a subsolution to problem (4.1) if

$$-\Delta \underline{u} \le f(x,\underline{u}), \quad \underline{u} > 0,\ x \in \Omega,\ \underline{u}|_{\partial\Omega} = 0. \tag{4.2}$$

Definition 4.2. A function $\overline{u} \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ is called a supersolution to problem (4.1) if

$$-\Delta \overline{u} \ge f(x,\overline{u}), \quad \overline{u} > 0,\ x \in \Omega,\ \overline{u}|_{\partial\Omega} = 0. \tag{4.3}$$

Lemma 4.3 [5, Lemma 3]. *Let $f(x,s)$ be locally Hölder continuous in $\Omega \times (0,\infty)$ and continuously differentiable with respect to the variable s. Suppose problem (4.1) has a supersolution $\overline{u}$ and a subsolution $\underline{u}$ such that $\underline{u} \le \overline{u}$ on Ω, then problem (4.1) has at least one solution $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ in the ordered interval $[\underline{u},\overline{u}]$.*

Denote

$$|u|_\infty = \max_{x\in\overline{\Omega}} |u(x)|, \quad u \in C(\overline{\Omega}). \tag{4.4}$$

Now we apply Lemma 4.3 to consider existence of solutions to problem (1.1).

Let $u_0 \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ be the unique solution to problem (1.2). Obviously, $\overline{u} = u_0$ is a supersolution to problem (1.1). To construct a subsolution to problem (1.1), let $w \in C^{2+\alpha}(\Omega) \cap C^1(\overline{\Omega})$ be the unique solution to the following problem:

$$-\Delta w = 1, \quad w > 0, \quad x \in \Omega, \quad w|_{\partial\Omega} = 0. \tag{4.5}$$

It follows by the Höpf maximum principle that there exist positive constants c_1 and c_2 such that

$$c_1 d(x) \le w(x) \le c_2 d(x) \quad \forall x \in \Omega, \qquad \nabla w(x) \ne 0 \quad \forall x \in \partial\Omega. \tag{4.6}$$

Let $a > |w|_\infty$ in (1.14) and

$$M_0 = \sup_{x\in\overline{\Omega}} \left(|\nabla w(x)|^2 + \frac{\sqrt{2G(\psi(w(x)))}}{g(\psi(w(x)))} \right), \qquad M_1 = \sup_{x\in\overline{\Omega}} \left(\frac{k(x)}{g(\psi(d(x)))} \frac{g(\psi(c_2^{-1}w(x)))}{g(\psi(w(x)))} \right). \tag{4.7}$$

By Corollary 2.4 and Lemma 2.2, we see that $g \circ \psi \in \mathrm{NRVZ}_{-2\gamma/(1+\gamma)}$. It follows by the assumption (k) and Lemma 3.1 that $M_0, M_1 \in (0,\infty)$.

Define

$$\underline{u} = m\psi(w(x)), \tag{4.8}$$

where m is a positive constant to be chosen.

It follows that

$$\begin{aligned} &-\Delta\underline{u}(x) + k(x) \\ &\quad = g(\psi(w(x)))\left[m\left(|\nabla w(x)|^2 + \frac{\sqrt{2G(\psi(w(x)))}}{g(\psi(w(x)))}\right) + \frac{k(x)}{g(\psi(d(x)))}\frac{g(\psi(d(x)))}{g(\psi(w(x)))}\right] \\ &\quad \le (mM_0 + M_1)g(\psi(w(x))), \quad x \in \Omega. \end{aligned} \tag{4.9}$$

Let us analyze the function

$$F_m(x) = \frac{g(m\psi(w(x)))}{g(\psi(w(x)))}, \quad x \in \Omega. \tag{4.10}$$

By $\lim_{x\to\partial\Omega} F_m(x) = m^{-\gamma}$, we see that there exist positive constants δ_0 and m_0 such that for $m \in (0, m_0)$,

$$F_m(x) \ge mM_0 + M_1 \quad \forall x \in \overline{\Omega}_{\delta_0}, \tag{4.11}$$

where $\Omega_{\delta_0} = \{x \in \Omega : d(x) < \delta_0\}$ and m_0 is the unique positive root of the equation

$$m^{-\gamma} = 2(mM_0 + M_1). \tag{4.12}$$

Let

$$A_0 = \max_{x\in\overline{\Omega}/\Omega_{\delta_0}} \psi(w(x)), \qquad a_0 = \min_{x\in\overline{\Omega}/\Omega_{\delta_0}} \psi(w(x)). \tag{4.13}$$

It follows by (g_1) that there exists $m_1 > 0$ such that

$$F_m(x) \ge \frac{g(mA_0)}{g(a_0)} \ge mM_0 + M_1 \quad \forall m \in (0, m_1). \tag{4.14}$$

Thus $-\Delta\underline{u}(x) \le g(\underline{u}(x)) - k(x)$, $x \in \Omega$, that is, $\underline{u} = m\psi(w(x))$ is a subsolution to problem (1.1) for $0 < m < \min\{m_1, m_0\}$. Moreover, we see by the maximum principle [9, Theorem 2.3] that $\underline{u} \le u_0$ on $\overline{\Omega}$ and by Lemma 4.3 that problem (1.1) has at least one solution $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ in ordered interval $[\underline{u}, u_0]$.

The proof is complete.

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