UNIQUENESS RESULTS FOR ELLIPTIC PROBLEMS WITH SINGULAR DATA

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We obtain some uniqueness results for the Dirichlet problem for second-order elliptic equations in an unbounded open set Ω without the cone property, and with data depending on appropriate weight functions. The leading coefficients of the elliptic operator are VMO functions. The hypotheses on the other coefficients involve the weight function.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 3$. Consider in Ω the uniformly elliptic differential operator with measurable coefficients

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a, \tag{1.1}$$

and the Dirichlet problem

$$Lu = 0, \quad u \in W^{2,p}(\Omega) \cap \overset{o}{W}{}^{1,p}(\Omega), \tag{D}$$

with $p \in]1, +\infty[$.

Suppose that Ω verifies suitable regularity assumptions.

If $p \geq n$, $a_{ij} \in L^\infty(\Omega)$ $(i, j = 1, \dots, n)$, and the coefficients a_i $(i = 1, \dots, n)$, a satisfy certain local summability conditions (with $a > 0$), then it is possible to obtain a uniqueness result for the problem (D) using a classical result of Alexandrov and Pucci (see [17] for the case of bounded open sets and [6, Section 1] for the unbounded case).

If $p < n$, some more assumptions on the a_{ij}'s are necessary to get uniqueness results for the problem (D). If Ω is bounded, problem (D) has been widely studied by several authors under various hypotheses on the leading coefficients. In particular, if the coefficients a_{ij}

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belong to the space $C^o(\overline{\Omega})$, then uniqueness results for problem (D) have been obtained (see [12–15]). On the other hand, when the coefficients a_{ij} are required to be discontinuous, the classical result by Miranda [16] must be quoted, where the author assumed that the a_{ij}'s belong to $W^{1,n}(\Omega)$ (and consider the case $p = 2$). More recently, a relevant contribution has been given in [11, 22], where the coefficients a_{ij} are supposed to be in the class VMO and $p \in]1,\infty[$; observe here that VMO contains both classes $C^o(\overline{\Omega})$ and $W^{1,n}(\Omega)$ (see [10]). If Ω is unbounded, uniqueness results for problem (D), under assumptions similar to those required in [16], have been for istance obtained in [4, 18, 19] with $p = 2$ and in [5] with $p \in]1,\infty[$. Moreover, futher uniqueness results for (D), when the a_{ij}'s are in VMO and $p \in]1,\infty[$, can be found in [6, 9].

Suppose now that Ω has singular boundary. In [8], a problem of type (D) has been investigated, with $(a_{ij})_{x_k}$, a_i and a singular near a nonempty subset S_ρ of $\partial\Omega$, and $p = 2$. In particular, the data are supposed to be depending on an appropriate weight function ρ related to the distance function from S_ρ.

The aim of this paper is to obtain uniqueness results for a Dirichlet problem of type (D) under hypotheses weaker than those of [8] on the a_{ij}'s, and with $p > 1$. More precisely, if there exist extensions a^o_{ij} of the coefficients a_{ij} $(i,j = 1,\dots,n)$ in $\mathrm{VMO}(\Omega_o) \cap L^\infty(\Omega_o)$, where Ω_o is a regular open set containing Ω, and the functions ρa_i $(i = 1,\dots,n)$, $\rho^2 a$ are assumed to be bounded with $\operatorname{ess\,inf}_\Omega \rho^2 a > 0$, we can prove a uniqueness result for the problem

$$Lu = 0, \quad u \in W^{2,p}_{\mathrm{loc}}(\overline{\Omega} \setminus S_\rho) \cap \overset{o}{W}{}^{1,p}_{\mathrm{loc}}(\overline{\Omega} \setminus S_\rho) \cap L^p_t(\Omega), \tag{D$_1$}$$

where $L^p_t(\Omega)$, $t \in \mathbb{R}$, is a weighted Sobolev space.

Observe that if $S_\rho = \partial\Omega$ and Ω has the segment property, we are able to deduce from the above result that the problem

$$u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap L^p(\Omega), \quad Lu = 0, \tag{D$_2$}$$

admits only the trivial solution.

2. Notation and function spaces

Let G be any Lebesgue measurable subset of $\mathbb{R}^n$ and let $\Sigma(G)$ be the collection of all Lebesgue measurable subsets of G. If $F \in \Sigma(G)$, denote by $|F|$ the Lebesgue measure of F and by $\mathcal{D}(F)$ the class of restrictions to F of functions $\zeta \in C^\infty_o(\mathbb{R}^n)$ with $\overline{F} \cap \operatorname{supp}\zeta \subseteq F$. Moreover, for $p \in [1,+\infty]$, let $L^p_{\mathrm{loc}}(F)$ be the class of functions g such that $\zeta g \in L^p(F)$ for all $\zeta \in \mathcal{D}(F)$.

Let Ω be an open subset of $\mathbb{R}^n$. We put

$$\Omega(x,r) = \Omega \cap B(x,r) \quad \forall x \in \mathbb{R}^n,\ \forall r \in \mathbb{R}_+, \tag{2.1}$$

where $B(x,r)$ is the open ball of radius r centered at x.

Denote by $\mathcal{A}(\Omega)$ the class of all measurable functions $\rho : \Omega \to \mathbb{R}_+$ such that

$$\gamma^{-1}\rho(y) \le \rho(x) \le \gamma\rho(y) \quad \forall y \in \Omega,\ \forall x \in \Omega(y,\rho(y)), \tag{2.2}$$

where $\gamma \in \mathbb{R}_+$ is independent of x and y. For $\rho \in \mathcal{A}(\Omega)$, we put

$$S_\rho = \left\{ z \in \partial\Omega : \lim_{x \to z} \rho(x) = 0 \right\}. \tag{2.3}$$

It is known that

$$\rho \in L^\infty_{\mathrm{loc}}(\overline{\Omega}), \qquad \rho^{-1} \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \setminus S_\rho), \tag{2.4}$$

and, if $S_\rho \neq \emptyset$,

$$\rho(x) \leq \operatorname{dist}(x, S_\rho) \quad \forall x \in \Omega \tag{2.5}$$

(see [7, 20]).

If $r \in \mathbb{N}$, $1 \leq p \leq +\infty$, $s \in \mathbb{R}$, and $\rho \in \mathcal{A}(\Omega)$, we consider the space $W^{r,p}_s(\Omega)$ of distributions u on Ω such that $\rho^{s+|\alpha|-r}\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq r$, equipped with the norm

$$\|u\|_{W^{r,p}_s(\Omega)} = \sum_{|\alpha| \leq r} \|\rho^{s+|\alpha|-r}\partial^\alpha u\|_{L^p(\Omega)}. \tag{2.6}$$

Moreover, we denote by $\overset{o}{W}{}^{r,p}_s(\Omega)$ the closure of $C^\infty_o(\Omega)$ in $W^{r,p}_s(\Omega)$ and put $W^{0,p}_s(\Omega) = L^p_s(\Omega)$. A detailed account of properties of the above-defined function spaces can be found in [21].

If Ω has the property

$$|\Omega(x,r)| \geq A r^n \quad \forall x \in \Omega,\ \forall r \in]0,1], \tag{2.7}$$

where A is a positive constant independent of x and r, it is possible to consider the space $\mathrm{BMO}(\Omega,t)$ ($t \in \mathbb{R}_+$) of functions $g \in L^1_{\mathrm{loc}}(\overline{\Omega})$ such that

$$[g]_{\mathrm{BMO}(\Omega,t)} = \sup_{\substack{x\in\Omega \\ r\in]0,t]}} \fint_{\Omega(x,r)} \left| g - \fint_{\Omega(x,r)} g \right| dy < +\infty, \tag{2.8}$$

where $\fint_{\Omega(x,r)} g\,dy = (1/|\Omega(x,r)|)\int_{\Omega(x,r)} g\,dy$. We will say that $g \in \mathrm{VMO}(\Omega)$ if $g \in \mathrm{BMO}(\Omega) = \mathrm{BMO}(\Omega, t_A)$, where

$$t_A = \sup_{t \in \mathbb{R}_+} \left(\sup_{\substack{x\in\Omega \\ r\in]0,t]}} \frac{r^n}{|\Omega(x,r)|} \leq \frac{1}{A} \right), \tag{2.9}$$

and $[g]_{\mathrm{BMO}(\Omega,t)} \to 0$ for $t \to 0^+$.

3. Some density results

Let $\rho \in \mathcal{A}(\Omega)$. We consider the following conditions on ρ.

(i_1) There exists an open subset Ω_o of $\mathbb{R}^n$ with the segment property such that

$$\Omega \subset \Omega_o, \quad \partial\Omega \setminus S_\rho \subset \partial\Omega_o. \tag{3.1}$$

(i_2) $H = \inf_\Omega \rho^{-n}(x)|\Omega(x,\rho(x))| \in \mathbb{R}_+$.

Remark 3.1. If condition (i_2) holds, then it is possible to find a function $\sigma \in \mathcal{A}(\Omega) \cap C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ which is equivalent to ρ and such that

$$\left|\partial^\alpha \sigma(x)\right| \le c_\alpha \sigma^{1-|\alpha|}(x) \quad \forall x \in \Omega,\ \forall \alpha \in \mathbb{N}_o^n, \tag{3.2}$$

where c_α is independent of x (see [20]).

Fix $r \in \mathbb{N}$ and $p \in [1,+\infty[$. We denote by $\overset{o}{W}{}^{r,p}(\overline{\Omega} \setminus S_\rho)$ the space of distributions u on Ω such that

$$u \in W^{r,p}(\Omega), \quad \operatorname{supp} u \subset \overline{\Omega} \setminus S_\rho. \tag{3.3}$$

Lemma 3.2. *Assume that condition (i_1) holds. Then $\mathcal{D}(\overline{\Omega} \setminus S_\rho)$ is dense in $\overset{o}{W}{}^{r,p}(\overline{\Omega} \setminus S_\rho)$.*

Proof. Fix $u \in \overset{o}{W}{}^{r,p}(\overline{\Omega} \setminus S_\rho)$ and denote by u_o the zero extension of u to Ω_o. It is easy to prove that u_o belongs to $W^{r,p}(\Omega_o)$. It follows from (i_1) that there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\overline{\Omega}_o)$ such that

$$u_k \longrightarrow u_o \quad \text{in } W^{r,p}(\Omega_o) \tag{3.4}$$

(see [1, Theorem 3.18]).

Let $\psi \in \mathcal{D}(\overline{\Omega} \setminus S_\rho)$ such that $\psi = 1$ on $\operatorname{supp} u$. Observe that $\{\psi u_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\overline{\Omega} \setminus S_\rho)$ and

$$\|\psi u_k - u\|_{W^{r,p}(\Omega)} \le \|\psi(u_k - u_o)\|_{W^{r,p}(\Omega_o)} \le c_1 \|u_k - u_o\|_{W^{r,p}(\Omega_o)}, \tag{3.5}$$

where c_1 depends on n, ψ. Thus the statement is a consequence of (3.4). □

Lemma 3.3. *Assume that conditions (i_1) and (i_2) hold. Then $\mathcal{D}(\overline{\Omega} \setminus S_\rho)$ is dense in $W_s^{r,p}(\Omega)$.*

Proof. It follows from (i_1), (i_2), and [20, Theorem 4.1] that there exists a sequence $\{\delta_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\overline{\Omega} \setminus S_\rho)$ such that

$$\lim_{k\to+\infty} \partial^\alpha(1 - \delta_k(x)) = 0 \quad \forall x \in \Omega,\ \forall \alpha \in \mathbb{N}_o^n, \tag{3.6}$$

$$\sup_{k\in\mathbb{N}} \left|\partial^\alpha \delta_k(x)\right| \le c_\alpha \rho^{-|\alpha|}(x) \quad \forall x \in \Omega,\ \forall \alpha \in \mathbb{N}_o^n, \tag{3.7}$$

where c_α is independent of x.

Fix $u \in W_s^{r,p}(\Omega)$. Observe that condition (3.7) implies that $\delta_k u \in W_s^{r,p}(\Omega)$ for all $k \in \mathbb{N}$. Moreover, by (3.6) we have that

$$\delta_k u \longrightarrow u \quad \text{in } W_s^{r,p}(\Omega). \tag{3.8}$$

On the other hand, using (2.4), it is easy to show that $\delta_k u \in W^{r,p}(\Omega)$, and so $\delta_k u \in \overset{o}{W}{}^{r,p}(\overline{\Omega} \setminus S_\rho)$. For each $k \in \mathbb{N}$, Lemma 3.2 yields that there exists a sequence $\{u_h^k\}_{h\in\mathbb{N}} \subset \mathfrak{D}(\overline{\Omega} \setminus S_\rho)$ such that

$$u_h^k \longrightarrow \delta_k u \quad \text{in } W^{r,p}(\Omega). \tag{3.9}$$

Moreover, let $\psi_k \in C_o^\infty(\mathbb{R}^n)$ such that $\psi_k = 1$ on $\operatorname{supp}(\delta_k u)$. Thus by (2.4), we have

$$||\psi_k u_h^k - \delta_k u||_{W_s^{r,p}(\Omega)} \le c_1 ||u_h^k - \delta_k u||_{W^{r,p}(\Omega)}, \tag{3.10}$$

where $c_1 \in \mathbb{R}_+$ depends on ρ, r, s, k. It follows from (3.9) that there exists $h_k \in \mathbb{N}$ such that

$$||\psi_k u_{h_k}^k - \delta_k u||_{W_s^{r,p}(\Omega)} \le \frac{1}{k}. \tag{3.11}$$

If $\varphi_k = \psi_k u_{h_k}^k$, $k \in \mathbb{N}$, we obtain from (3.8) and (3.11) that

$$\varphi_k \longrightarrow u \quad \text{in } W_s^{r,p}(\Omega), \tag{3.12}$$

and the lemma is proved. □

If $r \in \mathbb{N}$, $1 \le p < +\infty$, we will denote by $\overset{o}{W}{}^{r,p}_{\text{loc}}(\overline{\Omega} \setminus S_\rho)$ the set of distributions u on Ω such that $\zeta u \in \overset{o}{W}{}^{r,p}(\Omega)$ for any $\zeta \in \mathfrak{D}(\overline{\Omega} \setminus S_\rho)$.

Lemma 3.4. *Assume that conditions (i_1) and (i_2) hold. Then*

$$\overset{o}{W}{}^{r,p}_{\text{loc}}(\overline{\Omega} \setminus S_\rho) \cap W_s^{r,p}(\Omega) = \overset{o}{W}{}_s^{r,p}(\Omega). \tag{3.13}$$

Proof. It is clearly enough to show that

$$\overset{o}{W}{}^{r,p}_{\text{loc}}(\overline{\Omega} \setminus S_\rho) \cap W_s^{r,p}(\Omega) \subseteq \overset{o}{W}{}_s^{r,p}(\Omega). \tag{3.14}$$

Let $u \in \overset{o}{W}{}^{r,p}_{\text{loc}}(\overline{\Omega} \setminus S_\rho) \cap W_s^{r,p}(\Omega)$ and consider a sequence $\{\delta_k\}_{k\in\mathbb{N}} \subset \mathfrak{D}(\overline{\Omega} \setminus S_\rho)$ satisfying (3.6) and (3.7). Since each $\delta_k u$ belongs to $\overset{o}{W}{}^{r,p}(\Omega)$, for any $k \in \mathbb{N}$, there exists a sequence $\{u_h^k\}_{h\in\mathbb{N}} \subset C_o^\infty(\Omega)$ such that

$$u_h^k \longrightarrow \delta_k u \quad \text{in } W^{r,p}(\Omega). \tag{3.15}$$

Let $\psi_k \in C_o^\infty(\mathbb{R}^n)$ such that $\psi_k = 1$ on $\operatorname{supp}(\delta_k u)$. Since $\psi_k u_h^k \in C_o^\infty(\Omega)$, the same argument used in Lemma 3.3 allows to deduce from (3.15) that for every $k \in \mathbb{N}$, there exists $h_k \in \mathbb{N}$ such that

$$||\psi_k u_{h_k}^k - \delta_k u||_{W_s^{r,p}(\Omega)} \le \frac{1}{k}. \tag{3.16}$$

We put $\varphi_k = \psi_k u_{h_k}^k$ for each k. Therefore it follows from (3.16) that

$$||\varphi_k - u||_{W_s^{r,p}(\Omega)} \le \frac{1}{k} + ||\delta_k u - u||_{W_s^{r,p}(\Omega)}. \tag{3.17}$$

As the sequence $\{\delta_k\}_{k\in\mathbb{N}}$ satisfies (3.8), (3.17) yields that the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ converges to u in $W_s^{r,p}(\Omega)$, and hence (3.14) holds. □

4. Main results

Let Ω be an open subset of $\mathbb{R}^n$, $n\geq 3$, with the segment property. Fix $\rho\in\mathcal{A}(\Omega)\cap L^\infty(\Omega)$ and consider the following condition on Ω.

(h_1) There exists an open subset Ω_o of $\mathbb{R}^n$ with the uniform $C^{1,1}$-regularity property, such that

$$\Omega\subset\Omega_o,\quad \partial\Omega\setminus S_\rho\subset\partial\Omega_o. \tag{4.1}$$

Remark 4.1. If condition (h_1) holds and $\rho\in\mathcal{A}(\Omega)\cap L^\infty(\Omega)$, then Ω satisfies (i_2) (see [20]).

Let $p\in]1,+\infty[$, and let L be the differential operator in Ω defined by

$$L=-\sum_{i,j=1}^n a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^n a_i\frac{\partial}{\partial x_i}+a. \tag{4.2}$$

Consider the following conditions on the coefficients of L:

(h_2) there exist extensions a_{ij}^o of a_{ij} to Ω_o such that

$$\begin{gathered}a_{ij}^o=a_{ji}^o\in L^\infty(\Omega_o)\cap \mathrm{VMO}(\Omega_o),\quad i,j=1,\dots,n,\\ \exists\nu\in\mathbb{R}_+:\sum_{i,j=1}^n a_{ij}^o\xi_i\xi_j\geq\nu|\xi|^2\quad\text{a.e. in }\Omega_o,\ \forall\xi\in\mathbb{R}^n,\end{gathered} \tag{4.3}$$

(h_3)

$$\begin{gathered}a_i\in L_1^\infty(\Omega),\quad i=1,\dots,n,\ a\in L_2^\infty(\Omega),\\ a_o=\operatorname*{ess\,inf}_{\Omega}\left(\sigma^2(x)a(x)\right)>0,\end{gathered} \tag{4.4}$$

where σ is the function defined in Remark 3.1.

Moreover, we suppose that the following hypothesis on ρ holds:

(h_4)

$$\lim_{k\to+\infty}\left(\sup_{\Omega\setminus\Omega_k}\left((\sigma(x))_x+\sigma(x)(\sigma(x))_{xx}\right)\right)=0, \tag{4.5}$$

where

$$\Omega_k=\left\{x\in\Omega:\sigma(x)>\frac{1}{k}\right\},\quad k\in\mathbb{N}. \tag{4.6}$$

In the proof of our main theorem, we need the following uniqueness result.

Lemma 4.2. *Assume that conditions (h_1)–(h_4) hold and also that $p > n/2$. Then the problem*

$$\begin{gathered} Lu = 0, \quad u \in W^{2,p}_{\mathrm{loc}}(\Omega), \\ \lim_{x \to x_o} (\sigma^s u)(x) = 0, \quad \forall x_o \in \partial\Omega, \\ \lim_{|x| \to +\infty} (\sigma^s u)(x) = 0, \quad \text{if } \Omega \text{ is unbounded}, \end{gathered} \tag{4.7}$$

admits only the zero solution.

Proof. The statement can be proved as [2, Corollary 5.4]. In fact, the proof of that result also works if the condition $S_p = \partial\Omega$ is replaced by the assumption (h_1). □

Theorem 4.3. *Suppose that conditions (h_1)–(h_4) are satisfied. Then for any $t \in \mathbb{R}$, the problem*

$$u \in W^{2,p}_{\mathrm{loc}}(\overline{\Omega} \setminus S_p) \cap \overset{o}{W}{}^{1,p}_{\mathrm{loc}}(\overline{\Omega} \setminus S_p) \cap L^p_t(\Omega), \quad Lu = 0, \tag{4.8}$$

admits only the zero solution.

Proof. Let u be a solution of the problem (4.8). It follows from [3, Theorem 5.2] that $u \in W^{2,p}_{t+2}(\Omega)$. Moreover, u belongs to $W^{1,p}_{t+1}(\Omega)$, and hence Lemma 3.4 yields that $u \in W^{2,p}_{t+2}(\Omega) \cap \overset{o}{W}{}^{1,p}_{t+1}(\Omega)$. Using Remark 3.1, it is easy to prove that

$$\sigma^{t+2} u \in W^{2,p}(\Omega) \cap \overset{o}{W}{}^{1,p}(\Omega). \tag{4.9}$$

Put $v = \sigma^{t+2} u$ and denote by v_o the zero extension of v to Ω_o. Then

$$v_o \in W^{2,p}(\Omega_o) \cap \overset{o}{W}{}^{1,p}(\Omega_o) \tag{4.10}$$

by Lemma 3.3. Suppose first that $p > n/2$. By the Sobolev embedding theorem, v_o belongs to $C^0(\overline{\Omega}_o) \cap \overset{o}{W}{}^{1,p}(\Omega_o)$, and hence $v_{o|\partial\Omega_o} = 0$. On the other hand, $v_o \in W^{2,p}(\Omega_o)$, so that another application of the Sobolev embedding theorem gives that $\lim_{|x| \to +\infty} v_o(x) = 0$. Thus by ($h_1$), we have that

$$\lim_{|x| \to +\infty} (\sigma^{t+2} u)(x) = 0, \qquad (\sigma^{t+2} u)(x)_{|\partial\Omega} = 0. \tag{4.11}$$

In this case the statement follows now from Lemma 4.2.

Assume now that $p \in]1, n/2]$. Then by the Sobolev embedding theorem, we have that $v_o \in L^q(\Omega_o)$, where $1/q \geq 1/p - 2/n$. It follows from [3, Theorem 5.2] that $v_o \in W^{2,q}_2(\Omega_o)$, and hence v_o belongs to $W^{2,q}(\Omega_o)$ by (2.4). If $q > n/2$, the previous case can be used to complete the proof. If finally $q \leq n/2$, an iterated application of [3, Theorem 5.2] yields that $v_o \in W^{2,q'}(\Omega_o)$ with $q' > n/2$. Thus the first case applies again to complete the proof. □

As an application of Theorem 4.3, we consider the case $S_p = \partial\Omega$ (examples of such situation can for instance be found in [20]). The condition (h_1) is obviously satisfied by each $\Omega_o \supset \Omega$ with the uniform $C^{1,1}$-regularity property; in this case, condition (h_2) means that the coefficients a_{ij} admit extensions outside Ω in the class $L^\infty(\Omega_o) \cap \mathrm{VMO}(\Omega_o)$.

COROLLARY 4.4. *Assume that (h_2), (h_3), (h_4) hold and that $S_\rho = \partial\Omega$. Then the problem*

$$u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap L^p(\Omega), \quad Lu = 0 \tag{4.12}$$

admits only the zero solution.

Proof. The statement follows from Theorem 4.3 observing that, in this case, u belongs to $\overset{o}{W}{}^{1,p}_{\mathrm{loc}}(\Omega)$. □

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