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Infinitely many solutions to superlinear second order m -point boundary value problems

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Abstract

We consider the boundary value problem

$$u''(x) + g(u(x)) + p(x, u(x), u'(x)) = 0, \quad x \in (0, 1),$$
$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i),$$

where:

- (1) $m \geq 3$, $\eta_i \in (0, 1)$ and $\alpha_i > 0$ with $A := \sum_{i=1}^{m-2} \alpha_i < 1$;
- (2) $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies

$$g(s)s > 0, \quad s \neq 0,$$

and

$$\lim_{s \rightarrow \infty} \frac{g(s)}{s} = \infty;$$

- (3) $p : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous and satisfies

$$|p(x, u, v)| \leq C + \beta|u|, \quad x \in [0, 1], (u, v) \in \mathbb{R}^2$$

for some $C > 0$ and $\beta \in (0, 1/2)$.

We obtain infinitely many solutions having specified nodal properties by the bifurcation techniques.

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1 Introduction

We consider the nonlinear boundary value problem

$$u''(x) + g(u(x)) + p(x, u(x), u'(x)) = 0, \quad x \in (0, 1), \tag{1.1}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{1.2}$$

where

(H1) $m \geq 3$, $\eta_i \in (0, 1)$ and $\alpha_i > 0$ with

$$A := \sum_{i=1}^{m-2} \alpha_i < 1;$$

(H2) $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies

$$g(s)s > 0, \quad s \neq 0, \tag{1.3}$$

and

$$\lim_{s \rightarrow \infty} \frac{g(s)}{s} = \infty; \tag{1.4}$$

(H3) $p : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous and satisfies

$$|p(x, u, v)| \leq C + \beta|u|, \quad x \in [0, 1], (u, v) \in \mathbb{R}^2 \tag{1.5}$$

for some $C > 0$ and $\beta \in (0, 1/2)$.

In order to state our results, we first recall some standard notations to describe the nodal properties of solutions. For any integer, $n \geq 0$, $C^n[0, 1]$ will denote the usual Banach space of n -times continuously differentiable functions on $[0, 1]$, with the usual sup-type norm, denoted by $\|\cdot\|_n$. Let $X := \{u \in C^2[0, 1] : u \text{ satisfies (1.2)}\}$, $Y := C^0[0, 1]$, with the norms $\|\cdot\|_2$ and $\|\cdot\|_0$, respectively. Let

$$E := \{u \in C^1[0, 1] : u \text{ satisfies (1.2)}\},$$

with the norms $\|\cdot\|_E$.

We define a linear operator $L : X \rightarrow Y$ by

$$Lu := -u'', \quad u \in X. \tag{1.6}$$

In addition, for any continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ and any $u \in Y$, let $g(u) \in Y$ denote the function $g(u(x))$, $x \in [0, 1]$.

Next, we state some notations to describe the nodal properties of solutions of (1.1), see [1] for the details. For any C^1 function u , if $u(x_0) = 0$, then x_0 is a simple zero of u , if $u'(x_0) \neq 0$. Now, for any integer $k \geq 1$ and any $v \in \{+, -\}$, we define sets $S_k^v, \Gamma_k^v \subset C^2[0, 1]$ consisting of the set of functions $u \in C^2[0, 1]$ satisfying the following conditions:

$$\underline{S}_k^v$$

(i) $u(0) = 0$, $vu'(0) > 0$; (ii) u has only simple zeros in $[0, 1]$ and has exactly $k - 1$ zeros in $(0, 1)$.

$$\underline{\Gamma}_k^v$$

(i) $u(0) = 0$, $vu'(0) > 0$; (ii) u' has only simple zeros in $(0, 1)$ and has exactly k such zeros; (iii) u has a zero strictly between each two consecutive zeros of u' .

Remark 1.1 If we add the restriction $u'(1) \neq 0$ on the functions in Γ_k^v then Γ_k^v becomes the set T_k^v , which used in [1]. The reason we use Γ_k^v rather than T_k^v is that the Equation (1.1) is not autonomous anymore.

In [1, Remarks 2.1 and 2.2], Rynne pointed out that

- a. If $u \in T_k^v$, then u has exactly one zero between each two consecutive zeros of u' , and all zeros of u are simple. Thus, u has at least $k - 1$ zeros in $(0, 1)$, and at most k zeros in $(0, 1]$;
- b. The sets T_k^v are open in X and disjoint;
- c. When considering the multi-point boundary condition (1.2), the sets T_k^v are in fact more appropriate than the sets S_k^v .

The main result of this paper is the following

Theorem 1.1 Let (H1)-(H3) hold. Then there exists an integer $k_0 \geq 1$ such that for all integers $k \geq k_0$ and each $v \in \{+, -\}$ the problem (1.1), (1.2) has at least one solution $u_k^v \in \Gamma_k^v$.

Superlinear problems with classical boundary value conditions have been considered in many papers, particularly in the second and fourth order cases, with either periodic or separated boundary conditions, see for example [2-11] and the references therein. Specifically, the second order periodic problem is considered in [2,3], while [4-7] consider problems with separated boundary conditions, and results similar to Theorem 1.1 were obtained in each of these papers. The fourth order periodic problem is considered in [8-10]. Rynne [11] and De Coster [12] consider some general higher order problems with separated boundary conditions also.

Calvert and Gupta [13] studied the superlinear three-point boundary value problem

$$u''(x) + g(u(x)) + p(x, u(x), u'(x)) = 0, \quad x \in (0, 1), \tag{1.7}$$

$$u(0) = 0, \quad u(1) = \beta u(\eta), \tag{1.8}$$

(which is a nonlocal boundary value problem), under the assumptions:

(A0) $\beta \in (0, 1) \cup (1, \infty)$;

(A1) $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies $g(s)s > 0, s \neq 0, \frac{g(s)}{s}$ is increasing and

$$\lim_{|s| \rightarrow \infty} \frac{g(s)}{s} = \infty;$$

(A2) $p : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function satisfying the Carathéodory conditions and satisfies

$$|p(x, u, v)| \leq M_1(t, \max(|u|, |v|)), \quad x \in [0, 1], (u, v) \in \mathbb{R}^2,$$

where $M_1 : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ satisfies the condition: for each $s \in [0, \infty)$, $M_1(\cdot, s)$ is integrable on $[0, 1]$ and for each $t \in [0, 1]$, $M_1(t, \cdot)$ is increasing on $[0, \infty)$ with $s^{-1} \int_0^1 M_1(t, s) ds \rightarrow 0$ as $s \rightarrow \infty$.

Calvert and Gupta used Leray-Schauder degree and some ideas from Henrard [14] and Cappieto et al. [5] to prove the existence of infinity many solutions for (1.7), (1.8). Their results extend the main results in [14].

It is the purpose of this paper to use the global bifurcation theorem, see [15] and [1], to obtain infinity many nodal solutions to m -point boundary value problems (1.1), (1.2) under the assumptions (H1)-(H3). Obviously, our conditions (H2) and (H3) are much weaker than the corresponding restrictions imposed in [13]. Our paper uses some of ideas of Rynne [10], which deals with fourth order two-point boundary value problems. By the way, the proof [10, Lemma 2.8] contains a small error (since $\|u''\|_0 \geq \zeta_4(0) \boxtimes \|u''\|_0 \geq \zeta_4(R)$ there). So, we introduce a new function χ (see (3.7)) with

$$\chi(0) \geq \zeta_2(R)$$

which are required in applying Lemma 3.4.

2 Eigenvalues of the linear problem

First, we state some preliminary results related to the linear eigenvalue problem

$$Lu = \lambda u, \quad u \in X. \tag{2.1}$$

Denote the spectrum of L by $\sigma(L)$. The following spectrum results on (2.1) were established by Rynne [1], which extend the main result of Ma and O'Regan [16].

Lemma 2.1. [1, Theorem 3.1] The spectrum $\sigma(L)$ consists of a strictly increasing sequence of eigenvalues $\lambda_k > 0$, $k = 1, 2, \dots$, with corresponding eigenfunctions $\phi_k(x) = \sin(\lambda_k^{1/2}x)$. In addition,

- (i) $\lim_{k \rightarrow \infty} \lambda_k = \infty$;
- (ii) $\phi \in T_k^p$, for each $k \geq 1$, and ϕ_1 is strictly positive on $(0, 1)$.

Lemma 2.2 [1, Theorem 3.8] For each $k \geq 1$, the algebraic multiplicity of the characteristic value λ_k of $L^{-1} : Y \rightarrow Y$ is equal to 1.

3 Proof of the main results

For any $u \in X$, we define $e(u)(\cdot) : [0, 1] \rightarrow \mathbb{R}$ by

$$e(u)(x) = p(x, u(x), u'(x)), \quad x \in [0, 1].$$

It follows from (1.5) that

$$|e(u)(x)| \leq C + \beta|u(x)|, \quad x \in [0, 1]. \tag{3.1}$$

For any $s \in \mathbb{R}$, let

$$G(s) = \int_0^s g(\tau) d\tau \geq 0,$$

and for any $s \geq 0$, let

$$\gamma(s) = \max\{|g(r)| : |r| \leq s\}, \quad \Gamma(s) = \max\{G(r) : |r| \leq s\}.$$

We now consider the boundary value problem

$$u'' + \lambda u + \alpha(g(u) + e(u)) = 0, \quad u \in X, \tag{3.2}$$

where $\alpha \in [0, 1]$ is an arbitrary fixed number and $\lambda \in \mathbb{R}$. In the following lemma $(\lambda, u) \in \mathbb{R} \times X$ will be an arbitrary solution of (3.2).

By (H2), we can choose $b_1 \geq 1$ such that

$$|s| \geq b_1 \Rightarrow |g(s)| \geq C + \beta|s|. \tag{3.3}$$

By (1.2), we have the following

Lemma 3.1. Let (H1) hold and let $u \in X$. Then

$$|u|_0 \leq |u'|_0 \leq |u''|_0. \tag{3.4}$$

Lemma 3.2. Let u be a solution of (3.2). Then for any $x_0, x_1 \in [0, 1]$,

$$u'(x_1)^2 + \lambda u(x_1)^2 + 2\alpha G(u(x_1)) = u'(x_0)^2 + \lambda u(x_0)^2 + 2\alpha G(u(x_0)) - 2\alpha \int_{x_0}^{x_1} e(u)(s)u'(s)ds.$$

Proof. Multiply (3.2) by u' and integrate from x_0 to x_1 , then we get the desired result.

■

In the following, let us fix $R \in (0, \infty)$ so large that $R \geq b_1$ and

$$\begin{aligned} g(r) + p(t, r, v) &> 0, \quad t \in [0, 1], v \in \mathbb{R}, r > R, \\ g(r) + p(t, r, v) &< 0, \quad t \in [0, 1], v \in \mathbb{R}, r < -R. \end{aligned} \tag{3.5}$$

Lemma 3.3. There exists an increasing function $\zeta_1 : [0, \infty) \rightarrow [0, \infty)$, such that for any solution u of (3.2) with $0 \leq \lambda \leq R$ and $|u(x_0)| + |u'(x_0)| \leq R$ for some $x_0 \in [0, 1]$, we have

$$|u'|_0 \leq \zeta_1(R).$$

Proof. Choose $x_1 \in [0, 1]$ such that $|u'|_0 = |u'(x_1)|$. We obtain from Lemma 3.2 that

$$\begin{aligned} |u'|_0^2 &= u'(x_1)^2 \\ &\leq u'(x_1)^2 + \lambda u(x_1)^2 + 2\alpha G(u(x_1)) \\ &= u'(x_0)^2 + \lambda u(x_0)^2 + 2\alpha G(u(x_0)) - 2\alpha \int_{x_0}^{x_1} e(u)(\xi)u'(\xi)d\xi. \end{aligned}$$

Combining this with (3.1), (3.4), it concludes that

$$|u'|_0^2 \leq R^2 + R^3 + 2\Gamma(R) + 2(C + \beta|u|_0)|u'|_0 \leq K(R) + 2C|u'|_0 + 2\beta|u'|_0^2,$$

with

$$K(R) = R^2 + R^3 + 2\Gamma(R).$$

This implies

$$|u'|_0 \leq \zeta_1(R) := \frac{2C + \sqrt{4C^2 + 4(1 - 2\beta)K(R)}}{2(1 - 2\beta)}.$$

■

Define

$$\zeta_2(s) = \zeta_1(s + s^2) + 1, \quad s > 0. \tag{3.6}$$

Clearly, the function is nondecreasing.

Lemma 3.4 Let u be a solution of (3.2) with $0 \leq \lambda \leq R$ and $|u'|_0 \geq \zeta_2(R)$ for some $R > 0$. Then, for any $x \in [0, 1]$ with $|u(x)| \leq R$, we have $|u'(x)| \geq R^2$.

Proof. Suppose, on the contrary that there exists $x_0 \in (0, 1)$ such that $|u(x_0)| \leq R$ and $|u'(x_0)| < R^2$. Then

$$|u(x_0)| + |u'(x_0)| < R + R^2.$$

Combining this with $\lambda \leq R < R + R^2$ and using Lemma 3.3, it concludes that

$$|u'|_0 \leq \zeta_1(R + R^2).$$

However, this is impossible if $|u'|_0 \geq \zeta_2(R)$. ■

For fixed $R > b_1$, let us define

$$\chi(s) := \zeta_2(R) + \zeta_2(s), \quad s \geq 0. \tag{3.7}$$

Let us now consider the problem

$$u'' + \lambda u + \theta(|u'|_0/\chi(\lambda))(g(u) + e(u)) = 0, \quad u \in X, \tag{3.8}$$

where $\theta : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing, C^∞ -function with $\theta(s) = 0, s \leq 1$ and $\theta(s) = 1, s \geq 2$. The nonlinear term in (3.8) is a continuous function of $(\lambda, u) \in \mathbb{R} \times X$ and is zero for $\lambda \in \mathbb{R}, |u'|_0 \leq \chi(\lambda)$, so (3.8) becomes a linear eigenvalue problem in this region, and overall the problem can be regarded as a bifurcation (from $u = 0$) problem.

The next lemma now follows immediately.

Lemma 3.5 The set of solutions (λ, u) of (3.8) with $|u'|_0 \leq \chi(\lambda)$ is

$$\{(\lambda, 0) : \lambda \in \mathbb{R}\} \cup \{(\lambda_k, t\phi_k) : k \geq 1, |t| \leq \chi(\lambda_k)/|\phi'_k|_0\}.$$

We also have the following global bifurcation result for (3.8).

Lemma 3.6 For each $k \geq 1$ and $v \in \{+, -\}$, there exists a connected set $C_k^v \subset \mathbb{R} \times E$ of nontrivial solutions of (3.8) such that $C_k^v \cup (\lambda_k, 0)$ is closed and connected and:

- (i) there exists a neighborhood N_k of $(\lambda_k, 0)$ in $\mathbb{R} \times E$ such that $N_k \cap C_k^v \subset \mathbb{R} \times \Gamma_k^v$,
- (ii) C_k^v meets infinity in $\mathbb{R} \times E$ (that is, there exists a sequence $(\lambda_n, u_n) \in C_k^v, n = 1, 2, \dots$, such that $|\lambda_n| + |u_n|_E \rightarrow \infty$).

Proof. Since $L^{-1} : Y \rightarrow X$ exists and is bounded, (3.8) can be rewritten in the form

$$u = \lambda L^{-1}u + \theta(|u'|_0/\chi(\lambda))L^{-1}(g(u) + e(u)), \tag{3.9}$$

and since L^{-1} can be regarded as a compact operator from Y to E , it is clear that finding a solution (λ, u) of (3.8) in $\mathbb{R} \times E$ is equivalent to finding a solution of (3.9) in $\mathbb{R} \times E$. Now, by the similar method used in the proof of [1, Theorem 4.2]), we may deduce the desired result.

■

Since $e(u)(t) \sigma 0$ in (3.8), nodal properties need not be preserved. However, we will rely on preservation of nodal properties for “large” solutions, encapsulated in the following result.

Lemma 3.7 If (λ, u) is a solution of (3.8) with $\lambda \geq 0$ and $|u'|_0 > \chi(\lambda)$, then $u \in \Gamma_k^v$ for some $k \geq 1$ and $v \in \{+, -\}$.

Proof. If $u \notin \Gamma_k^v$ for any $k \geq 1$ and v , then one of the following cases must occur:

Case 1. $u'(0) = 0$;

Case 2. $u'(\tau) = u''(\tau) = 0$ for some $\tau \in (0, 1]$.

In the Case 1, $u(t) \equiv 0$ on $[0, 1]$. This contradicts the assumption $|u'|_0 > \chi(\lambda) \geq \zeta_2(\lambda)$. So this case cannot occur.

In the Case 2, we have from (3.8) that

$$\lambda u(\tau) + \theta(|u'|_0/\chi(\lambda))(g(u(\tau)) + e(u)(\tau)) = 0. \tag{3.10}$$

Since $|u'|_0 > \chi(\lambda)$, we have from the definition of θ that

$$\theta(|u'|_0/\chi(\lambda)) > 0. \tag{3.11}$$

It follows from Lemma 3.4 that $|u(\tau)| > R \geq b_1$. Combining this with (3.11) and (3.3), it concludes that

$$\lambda u(\tau) + \theta(|u'|_0/\chi(\lambda))(g(u(\tau)) + e(u)(\tau)) \neq 0, \tag{3.12}$$

which contradicts (3.10). So, Case 2 cannot occur.

Therefore, $u \in \Gamma_k^v$ for any $k \geq 1$ and $v \in \{+, -\}$. ■

In view of Lemmas 3.5 and 3.7, in the following lemma, we suppose that (λ, u) is an arbitrary nontrivial solution of (3.8) with $\lambda \geq 0$ and $u \in \Gamma_{k_0}^v$ for some $k \geq 1$ and v .

Lemma 3.8. There exists an integer $k_0 \geq 1$ (depending only on $\chi(0)$) such that for any nontrivial solution u of (3.8) with $\lambda = 0$ and $\chi(0) \leq |u'|_0 \leq 2\chi(0)$, we have

$$k < k_0. \tag{3.13}$$

Proof. Let x_1, x_2 be consecutive zeros of u . Then there exists $x_3 \in (x_1, x_2)$ such that $u'(x_3) = 0$, and hence, Lemma 3.4, (3.3), and (3.7) yield that $|u(x_3)| > 1$. Since

$$\begin{aligned} 2 &< |u(x_2) - u(x_3)| + |u(x_3) - u(x_1)| \\ &= |(x_2 - x_3)u'(\tau_1)| + |(x_3 - x_1)u'(\tau_2)| \\ &\leq (x_2 - x_3)|u'|_0 + (x_3 - x_1)|u'|_0 \\ &= (x_2 - x_1)|u'|_0 \end{aligned}$$

for some $\tau_1 \in (x_3, x_2)$, $\tau_2 \in (x_1, x_3)$, it follows that

$$|x_2 - x_1| > 2/|u'|_0. \tag{3.14}$$

Notice that $|u'|_0 > \chi(0) \geq \zeta_2(R)$ implies that $u \in \Gamma_k^v$ for some $k \in \infty$ and $v \in \{+, -\}$, and subsequently, there exist $0 < r_1 < r_2 < \dots < r_{k-1}$, such that

$$u(r_j) = 0, \quad j = 1, \dots, k - 1.$$

This together with (3.14) imply that

$$1 > (k - 1) \cdot 2/|u'|_0,$$

and accordingly, $k < |u'|_0/2 + 1 \leq \chi(0) + 1$. ■

Now let

$$V_R(u) = \{x \in [0, 1] : |u(x)| \geq R\}, \quad W_R(u) = \{x \in [0, 1] : |u(x)| < R\}.$$

Lemma 3.9. Suppose that $0 \leq \lambda \leq R$ and $|u'|_0 \geq \chi(R)$. Then $W_R(u)$ consists of at least k intervals and at most $k + 1$ intervals, each of length less than $2/R$, and $V_R(u)$ consists of at least k intervals and at most $k + 1$ intervals.

Proof. Lemma 3.4 implies that $|u'(x)| \geq R^2$ for all $x \in W_R(u)$. For any interval $I \subset W_R(u)$, u' does not change sign on I , say,

$$u'(x) \geq R^2, \quad x \in I.$$

We claim that the length of I is less than $2/R$.

In fact, for $x, y \in I$ with $x > y$, say,

$$u(x) - u(y) = \int_y^x u'(s) ds \geq R^2(x - y).$$

Thus,

$$x - y \leq \frac{R - (-R)}{R^2} = \frac{2}{R},$$

which implies

$$|I| \leq \frac{2}{R}.$$

The case

$$u'(x) \leq -R^2, \quad x \in I$$

can be treated by the similar method. Since u is monotonic in any subinterval containing in $W_R(u)$, the desired result is followed. ■

Lemma 3.10. There exists ζ_3 with $\lim_{R \rightarrow \infty} \zeta_3(R) = 0$, and $\eta_1 \geq 0$ such that, for any $R \geq \eta_1$, if either

- (a) $0 \leq \lambda \leq R$ and $|u'|_0 = 2\chi(R)$, or
- (b) $\lambda = R$ and $\chi(R) \leq |u'|_0 \leq 2\chi(R)$,

then the length of each interval of $V_R(u)$ is less than $\zeta_3(R)$.

Proof. Define $H = H(R)$ by

$$H(R)^2 := \min\{R, \min\{g(\xi)/\xi : |\xi| \geq R\} - (C/R + \beta)\},$$

and let $\zeta_3(R) := 2\pi/H(R)$. By (1.4), $\lim_{R \rightarrow \infty} H(R) = \infty$, so $\lim_{R \rightarrow \infty} \zeta_3(R) = 0$, and we may choose $\eta_1 \geq b_1$ sufficiently large that $H(R) > 0$ for all $R \geq \eta_1$.

We firstly show that

$$|u(\tau)| > R, \quad \text{for some } \tau \in (0, 1). \tag{3.15}$$

In fact, if $|u(x)| \leq R$ on $[0, 1]$, then Lemma 3.4 yields that either

$$u'(x) \geq R^2, \quad x \in [0, 1],$$

or

$$u'(x) \leq -R^2, \quad x \in [0, 1].$$

However, these contradict the boundary conditions (1.2), since (H1) implies $u'(s_0) = 0$ for some $s_0 \in (0, 1)$. Therefore, (3.15) is valid.

Now, Let us choose x_0, x_2 such that either

- (1) $u(x_0) = u(x_2) = R$ and $u > R$ on (x_0, x_2) or
- (2) $u(x_0) = R, x_2 = 1$ and $u > R$ on $(x_0, 1]$.

(the case of intervals on which $u < 0$ is similar). Let

$$I = [x_0, x_2].$$

By (3.8) and the construction of $H(R)$, if either (a) or (b) holds then

$$-u''(x) \geq H(R)^2 u(x) > 0, \quad x \in I,$$

and by Lemma 3.4, $u'(x_0) > 0$, and $u'(x_2) < 0$, if $x_2 < 1$.

Suppose, now on the contrary that $x_2 - x_0 > \zeta_3(R)$, that is, $l := 2\pi/(x_2 - x_0) < H(R)$.

Defining $x_1 = (x_0 + x_2)/2$ and

$$v(x) = 1 + \cos l(x - x_1), \quad x \in I,$$

we have

$$\begin{aligned} v(x_0) = v(x_2) = 0, \quad v'(x_0) = v'(x_2) = 0, \\ v''(x) = -l^2(v(x) - 1), \quad x \in I, \end{aligned}$$

and hence

$$\begin{aligned} 0 &= \int_{x_0}^{x_2} \frac{d}{dx}(u'v - uv') dx \\ &= \int_{x_0}^{x_2} (u''v - uv'') dx \\ &\leq \int_{x_0}^{x_2} (-H^2 uv + l^2(v - 1)u) dx \\ &= -l^2 \int_{x_0}^{x_2} u dx \\ &\leq -l^2 R < 0, \end{aligned}$$

and this contradiction shows that $x_2 - x_0 \leq \zeta_3(R)$, which proves the lemma.

■

Now, we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1 Now, choose an arbitrary integer $k \geq k_0$ and $v \in \{+, -\}$, and choose $\Lambda > \max\{\eta_1, \mu_k\}$ (Here, we assume $\Lambda > \eta_1$, so that Lemma 3.10 could be applied!) such that

$$(k + 1) \frac{2}{\Lambda} + (k + 1) \zeta_3(\Lambda) < 1. \tag{3.16}$$

Notice that Lemma 3.9 implies that if $|u'|_0 \geq \chi(\Lambda)$, then the length of each interval of $W_\Lambda(u)$ is less than $\frac{2}{\Lambda}$ for $0 \leq \lambda \leq \Lambda$. This together with (3.16) and Lemma 3.10 imply that there exists no solution (λ, u) of (3.8), which satisfies either

- (a) $0 \leq \lambda \leq \Lambda$ and $|u'|_0 = 2\chi(\Lambda)$ or
- (b) $\lambda = \Lambda$ and $\chi(\Lambda) \leq |u'|_0 \leq 2\chi(\Lambda)$.

Now, let us denote

$$\begin{aligned} B &= \{(\lambda, u) : 0 \leq \lambda \leq \Lambda, \chi(\lambda) \leq |u'|_0 \leq 2\chi(\Lambda)\}, \\ D_1 &= \{(\lambda, u) : 0 \leq \lambda \leq \Lambda, |u'|_0 = \chi(\lambda)\}, \\ D_2 &= \{(0, u) : 2\chi(0) \leq |u'|_0 \leq 2\chi(\Lambda)\}. \end{aligned}$$

It follows from Lemma 3.5 that C_k^v “enters” B through the set D_1 , while from Lemma 3.7, $C_k^v \cap B \subset \mathbb{R} \times \Gamma_k^v$. Thus, by Lemma 3.6 and the fact

$$|u|_0 \leq |u'|_0,$$

C_k^v must “leave” B . (Suppose, on the contrary that C_k^v does not “leave” B , then

$$|u|_0 \leq |u'|_0 \leq 2\chi(\Lambda),$$

which contradicts the fact that C_k^v joins $(\mu_k, 0)$ to infinity in $\mathbb{R} \times E$.) Since C_k^v is connected, it must intersect ∂B . However, Lemmas 3.8-3.10 (together with (3.16)) show that the only portion of ∂B (other than D_1), which C_k^v can intersect is D_2 . Thus, there exists a point $(0, u_k^v) \in C_k^v \cap D_2$, and clearly u_k^v provides the desired solution of (1.1)-(1.2).

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Authors' contributions

The authors declare that the work was realized in collaboration with same responsibility. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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