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Spatial estimates for a class of hyperbolic equations with nonlinear dissipative boundary conditions

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Abstract

This paper is concerned with investigating the spatial behavior of solutions for a class of hyperbolic equations in semi-infinite cylindrical domains, where nonlinear dissipative boundary conditions imposed on the lateral surface of the cylinder. The main tool used is the weighted energy method.

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1 Introduction

The aim of this paper is to study the spatial asymptotic behavior of solutions of the problem determined by the equation

$$u_{tt} = \Delta u_t - au_t - \Delta^2 u, \quad (x, t) \in \Omega \times (0, \infty), \tag{1.1}$$

where a is a positive constant and

$$\Omega = \{x \in R^n : x_n \in R^+,\ x' = (x_1, \ldots, x_{n-1}) \in \Gamma_{x_n} \subset R^{n-1}\},$$

where

$$\Gamma_\tau = \{(x', x_n) \in \Omega : x_n = \tau\}.$$

When we consider equation (1.1), we impose the initial and boundary conditions

$$u(x, 0) = u_t(x, 0) = 0, \quad x \in \Omega, \tag{1.2}$$

$$u(x', 0, t) = h_1(x', t),\ \frac{\partial u}{\partial \nu}(x', 0, t) = h_2(x', t), \quad (x', t) \in \Gamma_0 \times (0, \infty), \tag{1.3}$$

$$u = 0,\ \Delta u = -f\left(\frac{\partial u}{\partial \nu}\right), \quad (x, t) \in \Sigma_0 \times (0, \infty), \tag{1.4}$$

where ν is the outward normal to the boundary and

$$\Sigma_\tau = \{x \in R^n : x' \in \partial\Gamma_{x_n}, \tau \le x_n < \infty\},$$

where $\tau \to \bar{\Gamma}_\tau$ is a map from R^+ into family of bounded domains in R^{n-1} with sufficiently smooth boundary $\partial\Gamma_\tau$ such that

$$0 < m_0 \le \inf_\tau |\Gamma_\tau| \le \sup_\tau |\Gamma_\tau| \le m_1 < \infty.$$

In the sequel, we are using

$$\Omega_\tau = \Omega \cap \{x \in R^n : 0 < x_n < \tau\},$$

$$R_\tau = \Omega \cap \{x \in R^n : \tau < x_n < \infty\},$$

and assume f satisfies

$$F(v) = \int_0^v f(\xi)\mathrm{d}\xi \ge \alpha v f(v) > 0, \quad \alpha > 0,\ \forall v \in R, \tag{1.5}$$

$$v f(v) \ge \gamma |v|^{2p}, \quad p > \frac{1}{2}, \gamma > 0,\ \forall v \in R. \tag{1.6}$$

In recent years, much attention has been directed to the study of spatial behavior of solutions of partial differential equations and systems. The history and development of this question is explained in the work of Horgan and Knowles [1]. The interested reader is referred to the papers [2-9] and the reviews by Horgan and Knowles [1,10,11]. The energy method is widely used to study such results.

Spatial growth or decay estimates for nontrivial solutions of initial -boundary value problems in semi-infinite domains with nonlinearities on the boundary have been studied by many authors. Since 1908, when Edvard Phragmén and Ernst Lindelöf published their idea [12], many authors have obtained spatial growth or decay results by Phragmén-Lindelöf theorems. In [13], Horgan and Payne proved some these types of theorems and showed the asymptotic behavior of harmonic functions defined on a three-dimensional semi-infinite cylinder when homogeneous nonlinear boundary conditions are imposed on the lateral surface of the cylinder. Payne and Schaefer [14] proved such results for some classes of heat conduction problems. In [15], Quintanilla investigate the spatial behavior of several nonlinear parabolic equations with nonlinear boundary conditions, (see also [16,17]).

Under nonlinear dissipative feedbacks on the boundary, Nouria [18] proved a polynomial stability for regular initial data and exponential stability for some analytic initial data of a square Euler-Bernoulli plate. For the used methodology, one can see [19,20] where the stabilities are investigated in the cases bounded and unbounded feedbacks for some evolution equations. Recently, Celebi and Kalantarov [21] established a Phragmén-Lindelöf type theorems for a linear wave equation under nonlinear boundary conditions. In our study, we establish Phragmén-Lindelöf type theorems for equation (1.1) with nonlinear dissipative feedback terms on the boundary. Our study is inspired by the results of [21].

For the proof of our results, we will use the following Lemma.

Lemma [22]*Let* ψ *be a monotone increasing function with* $\psi(0) = 0$ *and* $\lim_{z\to\infty}\psi(z) = \infty$. *Then* $\phi(z) > 0$ *satisfying* $\phi(z) < \psi(\phi'(z))$, $z > 0$, *tends to* $+\infty$ *when* $z \to +\infty$.

(i) *If* $\psi(z) \le cz^m$ *for some* c *and* $m > 1$ *for* $z \ge z_1$, *then*

$$\liminf_{z\to+\infty} z^{-\frac{m}{m-1}}\varphi(z) > 0.$$

(ii) *If* $\psi(z) \le cz$ *for some* c *and* $z \ge z_1$, *then*

$$\liminf_{z\to+\infty} \varphi(z)\exp\left(-\frac{z}{c}\right) > 0.$$

2 Spatial estimates

With the solutions of (1.1-1.4) with $h_i(x', t) = 0$, $i = 1, 2$ is naturally associated an energy function

$$E(\tau) = \int_0^T \left[||u_t||^2_{\Omega_\tau} + ||\nabla u_t||^2_{\Omega_\tau} + ||\Delta u||^2_{\Omega_\tau} + \int_0^\tau \int_{\partial\Gamma_\eta} \nabla u f(\nabla u) \mathrm{d}s\mathrm{d}\eta \right] \mathrm{d}t, \tag{2.1}$$

where $||.||_\Omega$ denotes the usual norm in $L^2(\Omega)$.

A multiplication of equation (1.1) by u_t, integrating over Ω_τ and using (1.3-1.5):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}||u_t||^2_{\Omega_\tau} + \frac{1}{2}||\Delta u||^2_{\Omega_\tau} + \int_0^\tau \int_{\partial\Gamma_\eta} F(\nabla u)\mathrm{d}s\mathrm{d}\eta\right] + a||u_t||^2_{\Omega_\tau}$$
$$+||\nabla u_t||^2_{\Omega_\tau} = -(u_t, u_{x_nx_nx_n})_{\Gamma_\tau} + (u_{tx_n}, u_{x_nx_n})_{\Gamma_\tau} + (u_t, u_{tx_n})_{\Gamma_\tau}.$$

Since

$$(u_t, u_{x_nx_nx_n})_{\Gamma_\tau} = -(u_{tx_n}, u_{x_nx_n})_{\Gamma_\tau},$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}||u_t||^2_{\Omega_\tau} + \frac{1}{2}||\Delta u||^2_{\Omega_\tau} + \int_0^\tau \int_{\partial\Gamma_\eta} F(\nabla u)\mathrm{d}s\mathrm{d}\eta\right] + a||u_t||^2_{\Omega_\tau} \\ +||\nabla u_t||^2_{\Omega_\tau} = 2(u_{tx_n}, u_{x_nx_n})_{\Gamma_\tau} + (u_t, u_{tx_n})_{\Gamma_\tau}.\end{aligned} \tag{2.2}$$

Let $\delta > 0$. Multiplying (1.1) by δu, integrating over Ω_τ, and adding to (2.2), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}||u_t||^2_{\Omega_\tau} + \frac{1}{2}||\Delta u||^2_{\Omega_\tau} + \delta(u, u_t)_{\Omega_\tau}\right. \\ &\left. + \frac{a\delta}{2}||u||^2_{\Omega_\tau} + \frac{\delta}{2}||\nabla u||^2_{\Omega_\tau} + \int_0^\tau \int_{\partial\Gamma_\eta} F(\nabla u)\mathrm{d}s\mathrm{d}\eta\right\} \\ &+(a-\delta)||u_t||^2_{\Omega_\tau} + ||\nabla u_t||^2_{\Omega_\tau} + \delta||\Delta u||^2_{\Omega_\tau} + \delta\int_0^\tau \int_{\partial\Gamma_\eta} \nabla u f(\nabla u)\mathrm{d}s\mathrm{d}\eta \\ &= 2(u_{tx_n}, u_{x_nx_n})_{\Gamma_\tau} + (u_t, u_{tx_n})_{\Gamma_\tau} + 2\delta(u_{x_n}, u_{x_nx_n})_{\Gamma_\tau} + \delta(u, u_{tx_n})_{\Gamma_\tau}.\end{aligned} \tag{2.3}$$

Integrating (2.3) with respect to t over (0, T) and using (1.5), one can find

$$\begin{aligned}
&\frac{1}{2}||u_t||^2_{\Omega_\tau} + \frac{1}{2}||\Delta u||^2_{\Omega_\tau} + \frac{\delta}{2}||\nabla u||^2_{\Omega_\tau} + \frac{a\delta}{2}||u||^2_{\Omega_\tau} \\
&+ \delta(u, u_t)_{\Omega_\tau} + \alpha\int_0^\tau\int_{\partial\Gamma\eta} \nabla u f(\nabla u)\mathrm{d}s\mathrm{d}\eta \\
&+(a-\delta)\int_0^T ||u_t||^2_{\Omega_\tau}\mathrm{d}t + \delta\int_0^T ||\Delta u||^2_{\Omega_\tau}\mathrm{d}t + \int_0^T ||\nabla u_t||^2_{\Omega_\tau}\mathrm{d}t \\
&+\delta\int_0^T\int_0^\tau\int_{\partial\Gamma\eta} \nabla u f(\nabla u)\mathrm{d}s\mathrm{d}\eta dt \le \int_0^T [2(u_{tx_n}, u_{x_nx_n})\Gamma_\tau + (u_t, u_{tx_n})\Gamma_\tau]\,\mathrm{d}t \\
&+\int_0^T [2\delta(u_{x_n}, u_{x_nx_n})\Gamma_\tau + \delta(u, u_{tx_n})\Gamma_\tau]\,\mathrm{d}t.
\end{aligned} \tag{2.4}$$

On exploiting (2.1) and the inequality $-\left(\frac{1}{4}\right)\|u_t\|^2_{\Omega_\tau} - \delta^2\|u\|^2_{\Omega_\tau} \le \delta(u, u_t)_{\Omega_\tau}$, the estimate (2.4) takes the form

$$\begin{aligned}
\sigma^{-1}E(\tau) \le &\int_0^T [(u_{tx_n}, u_{x_nx_n})\Gamma_\tau + (u_t, u_{tx_n})\Gamma_\tau]\,\mathrm{d}t \\
&+\int_0^T [(u_{x_n}, u_{x_nx_n})\Gamma_\tau + (u, u_{tx_n})\Gamma_\tau]\,\mathrm{d}t,
\end{aligned} \tag{2.5}$$

by choosing $\delta = \frac{a}{2}$, $\delta_1 = \min\{1, \frac{a}{2}\}$, $\sigma = \max\{\frac{a}{\delta_1}, \frac{2}{\delta_1}\}$. Now we find upper bounds for the right hand side of (2.5). Using the Young's and Schwartz inequalities, we have

$$\int_0^T (u_{tx_n}, u_{x_nx_n})_{\Gamma_\tau}\mathrm{d}t \le \frac{1}{2}\int_0^T ||\nabla u_t||^2_{\Gamma_\tau}\mathrm{d}t + \frac{1}{2}\int_0^T ||\Delta u||^2_{\Gamma_\tau}\mathrm{d}t, \tag{2.6}$$

$$\int_0^T (u_t, u_{tx_n})_{\Gamma_\tau}\mathrm{d}t \le \frac{1}{2}\int_0^T ||u_t||^2_{\Gamma_\tau}\mathrm{d}t + \frac{1}{2}\int_0^T ||\nabla u_t||^2_{\Gamma_\tau}\mathrm{d}t, \tag{2.7}$$

$$\int_0^T (u_{x_n}, u_{x_nx_n})_{\Gamma_\tau}\mathrm{d}t \le \int_0^T ||u_{x_n}||_{\Gamma_\tau}||u_{x_nx_n}||_{\Gamma_\tau}\mathrm{d}t. \tag{2.8}$$

By the Poincaré inequality, it is not difficult to see

$$\|v\|^2_D \le \lambda^{-1}\|\nabla v\|^2_D + |D|^{-1}\left(\int_D v\mathrm{d}A\right)^2. \tag{2.9}$$

Inserting (2.9) into (2.8), we get

$$\int_0^T (u_{x_n}, u_{x_n x_n})_{\Gamma_\tau} \mathrm{d}t \le \int_0^T \left\{ \lambda_\tau^{-\frac{1}{2}} ||\Delta' u||_{\Gamma_\tau} + |\Gamma_\tau|^{-\frac{1}{2}} \left| \int_{\Gamma_\tau} \nabla' u \mathrm{d}A \right| \right\} ||u_{x_n x_n}||_{\Gamma_\tau} \mathrm{d}t, \tag{2.10}$$

where Δ' and ∇' are Laplacian and gradient operators in R^{n-1}, respectively, $|\Gamma_\tau|$ is the area of Γ_τ and λ_τ is the Poincaré constant. Now, we recall the inequality

$$\int_D v \mathrm{d}A \le \frac{r_0}{2} \int_{\partial D} |v| \mathrm{d}s + \frac{I_0^{\frac{1}{2}}}{2} \left(\int_D |\nabla v|^2 \mathrm{d}A \right)^{\frac{1}{2}}, \tag{2.11}$$

from [13] where $r_0^2 = \sup_D |x'|^2$ and $I_0 = \int_D |x'|^2 \mathrm{d}A$. Using (2.11) and the Hölder's inequality to estimate the boundary integral $\left| \int_{\Gamma_\tau} \nabla' u \mathrm{d}A \right|$ in (2.10), we obtain

$$\int_0^T (u_{x_n}, u_{x_n x_n})_{\Gamma_\tau} \mathrm{d}t \le \int_0^T \left\{ M_1 ||\Delta' u||_{\Gamma_\tau} + \gamma^{\frac{1}{2p}} M_2 \left(\int_{\partial\Gamma_\tau} |\nabla' u|^{2p} \mathrm{d}A \right)^{\frac{1}{2p}} \right\} ||u_{x_n x_n}||_{\Gamma_\tau} \mathrm{d}t, \tag{2.12}$$

where $M_1 = \lambda^{-\frac{1}{2}} + \frac{I^{1/2}}{2m^{1/2}}$, $M_2 = \frac{1}{2} r L^{(2p-1)/2p} m^{-1/2} \gamma^{-1/2p}$, such that $r = \sup_\tau r_\tau$, $\lambda = \inf_\tau \lambda_\tau$, $I = \sup_\tau I_\tau$, $L = \sup_\tau L_\tau$ and $m = \inf_\tau |\Gamma_\tau|$ in which L_τ is the area of $\partial\Gamma_\tau$. From (1.6) the inequality (2.12) yields

$$\int_0^T (u_{x_n}, u_{x_n x_n})_{\Gamma_\tau} \mathrm{d}t \le M_1 \int_0^T ||\Delta u||_{\Gamma_\tau}^2 \mathrm{d}t + M_2 \int_0^T \left(\int_{\partial\Gamma_\tau} \nabla' u f(\nabla' u) \mathrm{d}A \right)^{\frac{1}{2p}} ||u_{x_n x_n}||_{\Gamma_\tau} \mathrm{d}t. \tag{2.13}$$

Consequently

$$\left(\int_{\partial\Gamma_\tau} \nabla' u f(\nabla' u)\mathrm{d}s\right)^{\frac{1}{2p}} \left(\int_{\Gamma_\tau} u_{x_n x_n}^2 \mathrm{d}A\right)^{\frac{1}{2}}$$

$$= \left[\left(\int_{\partial\Gamma_\tau} \nabla' u f(\nabla' u)\mathrm{d}s\right)^{\frac{1}{p+1}} \left(\int_{\Gamma_\tau} u_{x_n x_n}^2 \mathrm{d}A\right)^{\frac{p}{p+1}}\right]^{\frac{p+1}{2p}}$$

$$\le \left[\frac{\mu^p}{1+p}\int_{\partial\Gamma_\tau} \nabla' u f(\nabla' u)\mathrm{d}s + \frac{p}{\mu(1+p)}\int_{\Gamma_\tau} u_{x_n x_n}^2 \mathrm{d}A\right]^{\frac{p+1}{2p}},$$

where the Young's inequality

$$\alpha^\varepsilon \beta^{1-\varepsilon} = (\alpha\gamma)^\varepsilon \left[\beta\gamma^{\frac{-\varepsilon}{1-\varepsilon}}\right]^{(1-\varepsilon)} \le \varepsilon\alpha\gamma + (1-\varepsilon)\beta\gamma^{\frac{-\varepsilon}{1-\varepsilon}},$$

for $0 < \varepsilon < 1$, $\mu = p^{\frac{1}{p+1}}$ and $\gamma = \mu^p$ have been used. Therefore,

$$\begin{aligned} &\left(\int_{\partial\Gamma_\tau} \nabla' u f(\nabla' u)\,\mathrm{d}s\right)^{\frac{1}{2p}} \left(\int_{\Gamma_\tau} u_{x_n x_n}^2 \,\mathrm{d}A\right)^{\frac{1}{2}} \\ &\le \left[N(p)\left(\int_{\partial\Gamma_\tau} \nabla' u f(\nabla' u)\,\mathrm{d}s + \int_{\Gamma_\tau} u_{x_n x_n}^2 \,\mathrm{d}A\right)\right]^{\frac{p+1}{2p}}, \end{aligned} \tag{2.14}$$

where

$$N(p) = \frac{p^{\frac{p}{p+1}}}{(1+p)}.$$

By using (2.13) and (2.14), we get

$$\begin{aligned} &\int_0^T (u_{x_n}, u_{x_n x_n})_{\Gamma_\tau}\,\mathrm{d}t \le M_1 \int_0^T ||\Delta u||_{\Gamma_\tau}^2 \,\mathrm{d}t \\ &+ M_2 \tilde{N}(p) \int_0^T \left(\int_{\partial\Gamma_\tau} \nabla' u f(\nabla' u)\,\mathrm{d}s + \int_{\Gamma_\tau} u_{x_n x_n}^2 \,\mathrm{d}A\right)^{\frac{p+1}{2p}} \mathrm{d}t, \end{aligned} \tag{2.15}$$

where $\tilde{N}(p) = \left[N(p)\right]^{\frac{p+1}{2p}}$. From (2.15), it is easy to see

$$\int_0^T (u_{x_n}, u_{x_n x_n})_{\Gamma_\tau}\,\mathrm{d}t \le M_1 \int_0^T ||\Delta u||^2_{\Gamma_\tau}\,\mathrm{d}t$$

$$+M_2 C\tilde{N}(p)\left(\int_0^T\int_{\partial\Gamma_\tau} \nabla u f(\nabla u)\,\mathrm{d}s\mathrm{d}t + \int_0^T ||\Delta u||^2_{\Gamma_\tau}\,\mathrm{d}t\right)^{\frac{p+1}{2p}},$$

where C is a positive constant.

Next, we exploit Poincaré inequality to estimate

$$\int_0^T (u, u_{tx_n})_{\Gamma_\tau}\,\mathrm{d}t \le \frac{\rho^{-1}}{2}\int_0^T ||\Delta u||^2_{\Gamma_\tau}\,\mathrm{d}t + \frac{1}{2}\int_0^T ||\nabla u_t||^2_{\Gamma_\tau}\,\mathrm{d}t, \tag{2.17}$$

where ρ is the Poincaré constant.

Now, from the inequalities (2.5-2.7), (2.16), and (2.17), one can find

$$E(\tau) \le \int_0^T \left[\frac{\sigma}{2}||u_t||^2_{\Gamma_\tau} + \frac{3}{2}\sigma||\nabla u_t||^2_{\Gamma_\tau} + \sigma\left(\frac{1}{2} + M_1 + \frac{\rho^{-1}}{2}\right)||\Delta u||^2_{\Gamma_\tau}\right]\mathrm{d}t$$

$$+\int_0^T\int_{\partial\Gamma_\tau} \nabla u f(\nabla u)\,\mathrm{d}s\mathrm{d}t + \sigma M_2 C\tilde{N}(p)\left\{\int_0^T\int_{\partial\Gamma_\tau} \nabla u f(\nabla u)\,\mathrm{d}s\mathrm{d}t\right.$$

$$\left.+\int_0^T \left[||\nabla u_t||^2_{\Gamma_\tau}\,\mathrm{d}t + ||\Delta u||^2_{\Gamma_\tau} + ||u_t||^2_{\Gamma_\tau}\right]\mathrm{d}t\right\}^{\frac{p+1}{2p}}.$$

Upon inserting (2.1) into the right hand side of (2.18), we may write an inequality in the form

$$E(\tau) \le \sigma\left(\frac{5}{2} + M_1 + \frac{\rho^{-1}}{2}\right)E'(\tau) + \sigma M_2 C\tilde{N}(p)\,\left[E'(\tau)\right]^{\frac{p+1}{2p}}. \tag{2.19}$$

At this point, by the inequality (2.19), the function $\psi(z) = \alpha_1 z + \alpha_2 z^{\frac{p+1}{2p}}$ satisfies in the hypothesis of the **Lemma**. Therefore, we have proved the following theorem.

Theorem 1 *Let* $u(x, t)$ *be a nontrivial solution of* (1.1) - (1.4) *with* $h_i(x', t) = 0$, $i = 1, 2$ *under the conditions* (1.5) *and* (1.6). *Then*

$$\liminf_{\tau\to+\infty} E(\tau)\tau^{-\frac{p+1}{1-p}} > 0, \quad p \in (\frac{1}{2}, 1),$$

and

$$\liminf_{\tau\to+\infty} E(\tau)\exp(-\frac{\tau}{c}) > 0, \quad p \in [1, +\infty),$$

where

$$c = \max\left\{\sigma\left(\frac{5}{2} + M_1 + \frac{\rho^{-1}}{2}\right), \sigma M_2 C\tilde{N}(p)\right\}.$$

Theorem 2 *Consider the equation* (1.1) *subject to the conditions* $u(x', 0, t) = h_1(x', t)$ *and* $\frac{\partial u}{\partial v}(x', 0, t) = h_2(x', t)$*for* $x' \in \Gamma_0$. *If* $E(+\infty)$ *is finite, then*

$$\lim_{\tau\to+\infty}\left(\int_0^T ||u_t||^2_{R_\tau}\,\mathrm{d}t + \int_0^T ||\nabla u_t||^2_{R_\tau}\,\mathrm{d}t + \int_0^T ||\Delta u||^2_{R_\tau}\,\mathrm{d}t\right) = 0. \quad (2.20)$$

proof By the same manner followed in theorem 1, it is easy to find the inequality

$$(a-\delta)\int_0^T ||u_t||^2_{R_\tau}\,\mathrm{d}t + \int_0^T ||\nabla u_t||^2_{R_\tau}\,\mathrm{d}t + \delta\int_0^T ||\Delta u||^2_{R_\tau}\,\mathrm{d}t \le \frac{1}{2}\int_0^T ||u_t||^2_{\Gamma_\tau}\,\mathrm{d}t$$
$$+\left(\frac{3}{2}+\frac{\delta}{2}\right)\int_0^T ||\nabla u_t||^2_{\Gamma_\tau}\,\mathrm{d}t + [1+\delta(1+\lambda_\tau^{-1}+\frac{1}{2}\lambda_\tau^{-2})]\int_0^T ||\Delta u||^2_{\Gamma_\tau}\,\mathrm{d}t,$$

where λ_τ is the Poincaré constant. Choosing $\delta \in (0, a)$, $\eta = \min\{a - \delta, \delta, 1\}$ and

$$\tilde{\gamma} = \eta^{-1}\max\{\frac{3}{2}+\frac{\delta}{2}, 1+\delta(1+\lambda_\tau^{-1}+\frac{1}{2}\lambda_\tau^{-2})\},$$

we obtain

$$\tilde{E}(\tau) \le -\tilde{\gamma}\tilde{E}'(\tau), \quad (2.21)$$

where

$$\tilde{E}(\tau) = \int_0^T ||u_t||^2_{R_\tau}\,\mathrm{d}t + \int_0^T ||\nabla u_t||^2_{R_\tau}\,\mathrm{d}t + \int_0^T ||\Delta u||^2_{R_\tau}\,\mathrm{d}t.$$

Thus, (2.20) follows from (2.21). ■

Authors' contributions

The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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