Liu et al. Boundary Value Problems 2011, 2011:30
http://www.boundaryvalueproblems.com/content/2011/1/30

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Regularity of large solutions for the compressible magnetohydrodynamic equations

Xin Liu[1*], Yuming Qin[2] and Xiaozhen Peng[3]

* Correspondence: xinliu120@126.com
[1]College of Information Sciences and Technology, Donghua University, Songjiang Shanghai, 201620, People's Republic of China
Full list of author information is available at the end of the article

Abstract

In this paper, we consider the initial-boundary value problem of one-dimensional compressible magnetohydrodynamics flows. The existence and continuous dependence of global solutions in H^1 have been established in Chen and Wang (Z Angew Math Phys 54, 608-632, 2003). We will obtain the regularity of global solutions under certain assumptions on the initial data by deriving some new a priori estimates.

Keywords: magnetohydrodynamics (MHD), global solutions, regularity, initial-boundary value problem

1 Introduction

Magnetohydrodynamics (MHD) is concerned with the flow of electrically conducting fluids in the presence of magnetic fields, either externally applied or generated within the fluid by inductive action. The application of magnetohydrodynamics covers a very wide range of physical areas from liquid metals to cosmic plasmas, for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, high-speed aerodynamics, and plasma physics. There is a complex interaction between the magnetic and fluid dynamic phenomena, and both hydrodynamic and electrodynamic effects have to be considered. For convenience, we consider the following plane magnetohydrodynamic equations in the Lagrangian coordinate system:

$$v_t - u_y = 0, \tag{1.1}$$

$$u_t + (p + \frac{1}{2}|\boldsymbol{b}|^2)_y = \left(\frac{\lambda u_y}{v}\right)_y, \tag{1.2}$$

$$\boldsymbol{w}_t - \boldsymbol{b}_y = \left(\frac{\mu \boldsymbol{w}_y}{v}\right)_y, \tag{1.3}$$

$$(v\boldsymbol{b})_t - \boldsymbol{w}_y = \left(\frac{\nu \boldsymbol{b}_y}{v}\right)_y, \tag{1.4}$$

$$E_t + \left(u(p + \frac{1}{2}|\boldsymbol{b}|^2) - \boldsymbol{w}\cdot\boldsymbol{b}\right)_y = \left(\frac{\lambda u u_y + \mu \boldsymbol{w}\cdot\boldsymbol{w}_y + \nu \boldsymbol{b}\cdot\boldsymbol{b}_y + \kappa\theta_y}{v}\right)_y. \tag{1.5}$$

Here, v, u, $\boldsymbol{w}$, $\boldsymbol{b}$, θ, and p are the specific volume, the longitudinal velocity, the transverse velocity, the transverse magnetic filed, the absolute temperature, and the pressure, respectively; λ, μ, ν, and κ are the bulk viscosity coefficient, the shear viscosity coefficient, the magnetic diffusivity, and the heat conductivity, respectively.

We consider problem (1.1)-(1.5) in the region $\{y \in \Omega: = (0, 1), t \geq 0\}$ under the initial-boundary conditions

$$(v,\ u,\ \boldsymbol{w},\ \boldsymbol{b},\ \theta)|_{t=0} = (v_0,\ u_0,\ \boldsymbol{w}_0,\ \boldsymbol{b}_0,\ \theta)(y), \quad y \in \Omega, \tag{1.6}$$

$$(u,\ \boldsymbol{w},\ \boldsymbol{b},\ \theta_y)|_{\partial\Omega} = 0. \tag{1.7}$$

In this paper, we focus on an initial-boundary problem for the magnetohydrodynamic flows of a perfect gas with following equations of state:

$$p = \frac{R\theta}{v}, \quad e = c_v\theta,$$

where R is the gas constant and c_v is the heat capacity of the gas at constant volume. For concreteness, we assume that λ, μ, and ν are constants, and κ depends on the temperature θ with $C_1 \leq \kappa(\theta)/(1 + \theta^r) \leq C_2$ for some positive constants C_1, C_2 and, $r \geq 2$. The growth condition assumed on κ is motivated by the physical fact: $\kappa \propto \theta^{5/2}$ for important physical regimes (see [1,2]). The total energy of the magnetohydrodynamics flows is

$$E = e + \frac{1}{2}(u^2 + |\boldsymbol{w}|^2) + \frac{1}{2}v\,|\boldsymbol{b}|^2.$$

Before showing our main results, let us first recall the related results in the literature. For the one-dimensional ideal gas, i.e.,

$$e = c_v\theta, \quad \sigma = \frac{R\theta}{v} + \mu\frac{u_y}{v}, \quad Q = -\kappa\frac{\theta_y}{v}, \tag{1.8}$$

with suitable positive constants c_v, R. Kazhikhov and Shelukhin [3-5], Kawashima and Nishida [6] established the existence of global smooth solutions. Zheng and Qin [7] proved the existence of maximal attractors in $H^i (i = 1, 2)$. However, under very high temperatures and densities, constitutive relations (1.8) become inadequate. Thus, a more realistic model would be a linearly viscous gas (or Newtonian fluid)

$$\sigma(v,\ \theta,\ u_y) = -p(v,\ \theta) + \frac{\mu(v,\theta)}{v}u_y \tag{1.9}$$

satisfying Fourier's law of heat flux

$$Q(v,\ \theta,\ \theta_y) = -\frac{\kappa(v,\theta)}{v}\theta_y \tag{1.10}$$

whose internal energy e and pressure p are coupled by the standard thermodynamical relation (1.8). In this case, Kawohl [8] obtained the existence of global solutions with the exponents $r \in [0, 1]$, $q \geq 2r + 2$. Jiang [9] also established the global existence with basically same constitutive relations as those in [8] but with the exponents $r \in [0, 1]$, $q \geq r + 1$. When the exponents q, r satisfy the more general constitutive relations than those in [8,9], Qin [10] established the regularity and asymptotic behavior of global solutions with arbitrary initial data for a one-dimensional viscous heat-conductive real gas.

For the radiative and reactive gas, Ducomet [11] established the global existence and exponential decay in H^1 of smooth solutions, and Umehara and Tani [12] proved the global existence of smooth solutions for a self-gravitating radiative and reactive gas.

For the radiative magnetohydrodynamic equations with self-gravitation, Ducomet and Feireisl [13] proved the existence of global-in-time solutions of this problem with arbitrarily large initial data and conservative boundary conditions on a bounded spatial domain in $\mathbb{R}^3$. Recently, under the technical condition that $\kappa(\rho, \theta)$ satisfies

$$k_1(1+\theta^q) \le \kappa(\rho, \theta) \le k_2(1+\theta^q), \quad k_1(1+\theta^q) \le |\kappa_\rho(\rho, \theta)| \le k_2(1+\theta^q),$$

for some $q > \frac{5}{2}$, Zhang and Xie [14] investigated the existence of global smooth solutions.

For the non-radiative and non self-gravitation magnetohydrodynamic flows, there have been a number of studies under various conditions by several authors (see, e.g., [2,15-22]). The existence and uniqueness of local smooth solutions were first obtained in [21]; moreover, the existence of global smooth solutions with small smooth initial data was shown in [20]. Chen and Wang [15] investigated a free boundary problem with general large initial data with exponents $r \in [0, 1]$, $q \ge 2r + 2$. Under the technical condition that $\kappa(\rho, \theta)$ satisfies

$$C^{-1}(1+\theta^q) \le \kappa(\rho, \theta) \le C(1+\theta^q)$$

for $q \ge 2$, Chen and Wang [16] also proved the existence and continuous dependence of global strong solutions with large initial data. Wang [22] established large solutions to the initial-boundary value problem for planar magnetohydrodynamics. Under the technical condition upon

$$\kappa(\rho, \theta) \equiv \kappa(\rho) > \frac{C}{\rho},$$

Fan et al. [18] investigated the uniqueness of the weak solutions of MHD with Lebesgue initial data. Fan et al. [19] also considered a one-dimensional plane compressible MHD flows and proved that as the shear viscosity goes to zero, global weak solutions converge to a solution of the original equations with zero shear viscosity. The uniqueness and continuous dependence of weak solutions for the Cauchy problem have been proved by Hoff and Tsyganov [17].

As mentioned above, the global existence in $H_+^i (i = 2, 4)$ of global solutions has never been studied for Equations (1.1)-(1.5) of the nonlinear one-dimensional compressible magnetohydrodynamics flows with initial-boundary conditions (1.6)-(1.7). The main aim of this paper is to prove the regularity of solutions in the subspace H_+^i of $(H^i[0, 1])^7 (i = 2, 4)$ for systems (1.1)-(1.7). In order to obtain higher regularity of global solutions, there are many complicated estimates on higher derivations of solutions to be involved, this is our main difficulty. To overcome this difficulty, we should use some proper embedding theorems, the interpolation techniques as well as many delicate estimates. This is the novelty of the paper.

We define three spaces as follows:

$$\begin{aligned}H_+^1 &= \Big\{(v,\ u,\ \boldsymbol{w},\ \boldsymbol{b},\ \theta) \in (H^1(\Omega))^7\ :\ v(x) > 0,\ \theta(x) > 0, \quad x \in \Omega,\\ &\quad u(0) = u(1) = 0,\ \boldsymbol{w}(0) = \boldsymbol{w}(1) = \boldsymbol{b}(0) = \boldsymbol{b}(1) = \boldsymbol{0}\Big\},\\ H_+^i &= \Big\{(v,\ u,\ \boldsymbol{w},\ \boldsymbol{b},\ \theta) \in (H^i(\Omega))^7\ :\ v(x) > 0,\ \theta(x) > 0,\ x \in \Omega,\\ &\quad u(0) = u(1) = 0,\ \boldsymbol{w}(0) = \boldsymbol{w}(1) = \boldsymbol{b}(0) = \boldsymbol{b}(1) = \boldsymbol{0},\\ &\quad \theta'(0) = \theta'(1) = 0\Big\}, \quad i = 2, 4.\end{aligned}$$

The notation in this paper will be stated as follows:

L^p, $1 \le p \le +\infty$, $W^{m,\ p}$, $m \in N$, $H^1 = W^{1,2}$, $H_0^1 = W_0^{1,2}$ denote the usual (Sobolev) spaces on Ω. In addition, $||\cdot||_B$ denotes the norm in the space B, we also put $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. Constants $C_i (i = 1, 2, 3, 4)$ depend on the H_+^i norm of the initial data $(v_0, u_0, \boldsymbol{w_0}, \boldsymbol{b}_0, \theta_0)$ and $T > 0$.

Now we are in a position to state our main results.

Theorem 1.1 *Assume that the initial data* $(v_0,\ u_0,\ \boldsymbol{w}_0,\ \boldsymbol{b}_0,\ \theta_0) \in H_+^2$*and e, p,* and κ *are C^3 functions. Then, the problem (1.1) -(1.7) admits a unique global solution* $(v(t),\ u(t),\ \boldsymbol{w}(t),\ \boldsymbol{b}(t),\ \theta(t)) \in H_+^2$*such that for any* $T > 0$,

$$\begin{aligned}&||v(t) - \bar{v}||_{H^2}^2 + ||u(t)||_{H^2}^2 + ||\boldsymbol{w}(t)||_{H^2}^2 + ||\boldsymbol{b}(t)||_{H^2}^2 + ||\theta(t) - \bar{\theta}||_{H^2}^2 + ||u_t(t)||^2 + ||\boldsymbol{w}_t(t)||^2\\ &+||\boldsymbol{b}_t(t)||^2 + ||\theta_t(t)||^2 + \int_0^t (||v - \bar{v}||_{H^2}^2 + ||u||_{H^3}^2 + ||\boldsymbol{w}||_{H^3}^2 + ||\boldsymbol{b}||_{H^3}^2 + ||\theta - \bar{\theta}||_{H^3}^2\\ &+||u_{ty}||^2 + ||\boldsymbol{w}_{ty}||^2 + ||\boldsymbol{b}_{ty}||^2 + ||\theta_{ty}||^2)(s)ds \le C_2,\ \forall t \in [0, T],\end{aligned} \tag{1.11}$$

where $\bar{v} = \int_0^1 v\mathrm{d}y = \int_0^1 v_0\mathrm{d}y$, constant $\bar{\theta} > 0$ is determined by

$$e(\bar{v}, \bar{\theta}) = \int_0^1 \left(\frac{1}{2}(u_0^2 + |\mathbf{w}_0|^2 + \ v_0|\mathbf{b}_0|^2) + e(v_0,\ \theta_0)\right)(y)\mathrm{d}y.$$

Theorem 1.2 *Assume that the initial data* $(v_0,\ u_0,\ \boldsymbol{w}_0,\ \boldsymbol{b}_0,\ \theta_0) \in H_+^4$*and e, p,* and κ *are C^5 functions on $0 < v < +\infty$ and $0 \le \theta < +\infty$. Then, the problem (1.1)-(1.7) admits a unique global solution* $(v(t),\ u(t),\ \boldsymbol{w}(t),\ \boldsymbol{b}(t),\ \theta(t)) \in H_+^4$*such that for any* $T > 0$,

$$\begin{aligned}&\|v(t) - \bar{v}\|_{H^4}^2 + \|u(t)\|_{H^4}^2 + \|\boldsymbol{w}(t)\|_{H^4}^2 + \|\boldsymbol{b}(t)\|_{H^4}^2 + \|\theta(t) - \bar{\theta}\|_{H^4}^2 + \|u_{tt}(t)\|^2 + \|\boldsymbol{w}_{tt}(t)\|^2\\ &+ \|\boldsymbol{b}_{tt}(t)\|^2 + \|u_t(t)\|_{H^2}^2 + \|\boldsymbol{w}_t(t)\|_{H^2}^2 + \|\boldsymbol{b}_t(t)\|_{H^2}^2 + \|\theta_t(t)\|_{H^2}^2 + \|\theta_{tt}(t)\|^2\\ &+ \int_0^t (\|v - \bar{v}\|_{H^4}^2 + \|u\|_{H^5}^2 + \|\boldsymbol{w}\|_{H^5}^2 + \|\boldsymbol{b}\|_{H^5}^2 + \|\theta - \bar{\theta}\|_{H^5}^2 + \|u_t\|_{H^3}^2 + \|\boldsymbol{w}_t\|_{H^3}^2 + \|\boldsymbol{b}_t\|_{H^3}^2\\ &+ \|\theta_t\|_{H^3}^2 + \|u_{tt}\|_{H^1}^2 + \|\boldsymbol{w}_{tt}\|_{H^1}^2 + \|\boldsymbol{b}_{tt}\|_{H^1}^2 + \|\theta_{tt}\|_{H^1}^2)(s)ds \le C_4,\ \forall t \in [0,\ T].\end{aligned} \tag{1.12}$$

2 Proof of Theorem 1.1

In this section, we study the global existence of problem (1.1)-(1.7) in H_+^2 by establishing a series of priori estimates. Without loss of generality, we take $c_v = R = 1$. We begin with the following lemma.

Lemma 2.1 *Assume that the initial data* $(v_0,\ u_0,\ \boldsymbol{w}_0,\ \boldsymbol{b}_0,\ \theta_0) \in H_+^1$*and e, p,* and κ *are C^2 functions on $0 < v < +\infty$ and $0 \le \theta < +\infty$ and there exists a positive constant C_0 such that*

$$0 < C_0^{-1} \le v_0(y) \le C_0, \quad 0 < C_0^{-1} \le \theta_0(y) \le C_0.$$

Then, for the initial data $(v_0,\ u_0,\ \boldsymbol{w}_0,\ \boldsymbol{b}_0,\ \theta_0) \in H_+^1$, *the problem (1.1) -(1.7) admits a unique global solution* $(v(t),\ u(t),\ w(t),\ b(t),\ \theta(t)) \in H_+^1$ *such that for any* $T > 0$

$$0 < C_1^{-1} \le v(y,\ t) \le C_1, \quad 0 < C_1^{-1} \le \theta(y,\ t) \le C_1, \quad \forall (y,\ t) \in [0, 1] \times [0,\ T] \tag{2.1}$$

and for any $t \in\ [0,\ T]$,

$$\begin{aligned} &\| v(t) - \bar{v} \|_{H^1}^2 + \| u(t) \|_{H^1}^2 + \| \boldsymbol{w}(t) \|_{H^1}^2 + \| \boldsymbol{b}(t) \|_{H^1}^2 + \| \theta(t) - \bar{\theta} \|_{H^1}^2 + \int_0^t (\| v - \bar{v} \|_{H^1}^2 + \| u \|_{H^2}^2 \\ &+ \| \boldsymbol{w} \|_{H^2}^2 + \| \boldsymbol{b} \|_{H^2}^2 + \| \theta - \bar{\theta} \|_{H^2}^2 + \| u_t \|^2 + \| \boldsymbol{w}_t \|^2 + \| \boldsymbol{b}_t \|^2 + \| \theta_t \|^2)\ (s) ds \le C_1. \end{aligned} \tag{2.2}$$

Proof. See, e.g., [16].

Lemma 2.2 *Under the assumptions in Theorem 1.1, the following estimate holds:*

$$\begin{aligned} &\| u_t(t) \|^2 + \| \boldsymbol{w}_t(t) \|^2 + \| \boldsymbol{b}_t(t) \|^2 + \| \theta_t(t) \|^2 \\ &+ \int_0^t (\| u_{ty} \|^2 + \| \boldsymbol{w}_{ty} \|^2 + \| \boldsymbol{b}_{ty} \|^2 + \| \theta_{ty} \|^2)\ (s) ds \le C_2,\ \forall t \in [0,\ T]. \end{aligned} \tag{2.3}$$

Proof. Differentiating (1.2) with respect to t, multiplying the resultant by u_t, and then integrating the resulting equation over $Q_t := \Omega \times [0,\ t]$, we infer

$$\begin{aligned} &\| u_t(t) \|^2 + \int_0^t \| u_{ty}(s) \|^2 \mathrm{d}s \\ &\le \varepsilon \int_0^t \| u_{ty}(s) \|^2 \mathrm{d}s + C_2 \int_0^t (\| \theta_t(s) \|^2\ \| + \| \mathbf{b} \cdot \mathbf{b}_t(s) \|^2 + \| u_y(s) \|_{L^4}^4)\ \mathrm{d}s \\ &\le \varepsilon \int_0^t \| u_{ty}(s) \|^2 \mathrm{d}s + C_2 \int_0^t (\| u_y(s) \|_{L^\infty}^2 + \| \theta_t(s) \|^2 + \| \mathbf{b}_t(s) \|^2)\ \mathrm{d}s \\ &\le C_2 + \varepsilon \int_0^t \| u_{ty}(s) \|^2 \mathrm{d}s, \end{aligned}$$

which implies

$$\| u_t(t) \|^2 + \int_0^t \| u_{ty}(s) \|^2 \mathrm{d}s \le C_2. \tag{2.4}$$

Analogously, we have

$$\| \boldsymbol{w}_t(t) \|^2 + \ \boldsymbol{b}_t(t) \|^2 + \| \theta_t(t) \|^2 + \int_0^t (\| \boldsymbol{w}_{ty} \|^2 + \| \boldsymbol{b}_{ty} \|^2 + \| \theta_{ty} \|^2)\ (s) \mathrm{d}s \le C_2. \tag{2.5}$$

Thus, (2.3) follows from (2.4)-(2.5).

Lemma 2.3 *Under the assumptions in Theorem 1.1, the following estimate holds:*

$$\| u_{yy}(t) \|^2 + \| \boldsymbol{w}_{yy}(t) \|^2 + \| \boldsymbol{b}_{yy}(t) \|^2 + \| \theta_{yy}(t) \|^2 + \int_0^t (\| u_{yyy}\|^2 + \| \boldsymbol{w}_{yyy}\|^2 + \| \boldsymbol{b}_{yyy}\|^2 + \| \theta_{yyy}\|^2)(s)ds \le C_2, \ \forall t \in [0, T]. \tag{2.6}$$

Proof. Equation (1.2) can be rewritten as

$$u_t = -\frac{\lambda\theta_y}{v} + \frac{\lambda\theta v_y}{v} - \frac{\lambda u_y v_y}{v^2} + \frac{\lambda u_{yy}}{v} - \mathbf{b}\cdot\mathbf{b}_y. \tag{2.7}$$

Using equation (2.7), Lemmas 2.1-2.2, Sobolev's embedding theorem and Young's inequality, we have

$$\begin{aligned}\|u_{yy}(t)\| &\le C_2(||u_t(t)|| + ||\theta_y(t)|| + ||\mathbf{b}\cdot\mathbf{b}_y(t)|| + ||\theta v_y(t)|| + ||v_y u_y(t)||)\\ &\le C_2(||u_t(t)|| + ||\theta_y(t)|| + \| \theta(t)\|_{L^\infty}||v_y(t)|| + ||\mathbf{b}_y(t)||^2 + ||u_y(t)||_{L^\infty}||v_y(t)||)\\ &\le \varepsilon||u_{yy}(t)|| + C_2(||u_t(t)|| + 1),\end{aligned}$$

which leads to

$$\|u_{yy}(t)\| \le C_2, \int_0^t \| u_{yyy}(s) \|^2 \mathrm{d}s \le C_2 \int_0^t \| u_{ty}(s) \|^2 \mathrm{d}s \le C_2. \tag{2.8}$$

Similarly, we derive

$$\| \boldsymbol{w}_{yy}(t) \| + \| \boldsymbol{b}_{yy}(t) \| + \| \theta_{yy}(t)\| \le C_2(\| \boldsymbol{w}_t(t) \| + \| \boldsymbol{b}_t(t) \| + \| \theta_t(t) \| + 1) \le C_2, \tag{2.9}$$

$$\int_0^t (\| \boldsymbol{w}_{yyy}\|^2 + \| \boldsymbol{b}_{yyy}\|^2 + \| \theta_{yyy}\|^2)(s)\mathrm{d}s \le C_2. \tag{2.10}$$

Thus, (2.6) follows from (2.8)-(2.10).

Lemma 2.4 *Under the assumptions in Theorem 1.1, the following estimate holds:*

$$\| v_{yy}(t)||^2 + \int_0^t \| v_{yy}(s) \|^2 ds \le C_2, \quad \forall t \in [0, T]. \tag{2.11}$$

Proof. Differentiating (1.2) with respect to y, we obtain

$$\lambda \frac{d}{\mathrm{d}t}\left(\frac{v_{yy}}{v}\right) + \frac{\theta}{v^2}v_{yy} = u_{ty} + E(y, t), \tag{2.12}$$

where

$$E(y, t) = \frac{\theta_{yy}}{v} + \frac{2v_y(\lambda u_{yy} - \theta_y)}{v^2} + \frac{2v_y^2(\theta - \lambda u_y)}{v^3} + \boldsymbol{b}\cdot\boldsymbol{b}_{yy} + |\boldsymbol{b}_y|^2.$$

Multiplying (2.12) by $\frac{v_{yy}}{v}$, integrating the resulting equation over Q_t, and then using the Young inequality and interpolation theorem, we can conclude

$$
\begin{aligned}
&\left|\frac{v_{yy}}{v}(t)\right|^2 + C_1^{-1}\int_0^t \left|\frac{v_{yy}}{v}(s)\right|^2 ds \le \frac{1}{4C_1}\int_0^t \left|\frac{v_{yy}}{v}(s)\right|^2 ds + C_2\int_0^t (\|u_{ty}\|^2 + \|\theta_{yy}\|^2 \\
&+ \|v_y u_{yy}\|^2 + \|v_y\|_{L^4}^4 + \|u_y v_y^2\|^2 + \|\boldsymbol{b}_y\|_{L^4}^4 + \|\boldsymbol{b}\|_{L^\infty}^2 \|\boldsymbol{b}_{yy}\|^2)(s)ds \\
&\le \frac{1}{2C_1}\int_0^t \left|\frac{v_{yy}}{v}(s)\right|^2 ds + C_2\int_0^t (\|v_y\|^2 + \|u_{ty}\|^2 + \|u_{yy}\|_{L^\infty}^2 + \|\theta_{yy}\|^2 + \|\boldsymbol{b}_{yy}\|^2)(s)ds,
\end{aligned}
$$

which, together with Lemmas 2.1-2.3, yields (2.11).

Proof of Theorem 1.1. By Lemmas 2.1-2.4, we complete the proof of Theorem 1.1.

3 Proof of Theorem 1.2

In this section, we study the global existence of problem (1.1)-(1.7) in H_+^4 by establishing a series of priori estimates. We begin with the following lemmas.

Lemma 3.1 *Under the assumptions in Theorem 1.2, the following estimates hold:*

$$\|u_{ty}(y,0)\| + \|\boldsymbol{w}_{ty}(y,0)\| + \|\boldsymbol{w}_{ty}(y,0)\| + \|\theta_{ty}(y,0)\| \le C_3, \tag{3.1}$$

$$
\begin{aligned}
&\|u_{tt}(y,0)\| + \|\boldsymbol{w}_{tt}(y,0)\| + \|\boldsymbol{b}_{tt}(y,0)\| + \|\theta_{tt}(y,0)\| \\
&+ \|u_{tyy}(y,0)\| + \|\boldsymbol{w}_{tyy}(y,0)\| + \|\boldsymbol{b}_{tyy}(y,0)\| + \|\theta_{tyy}(y,0)\| \le C_3.
\end{aligned} \tag{3.2}
$$

Proof. We easily infer from (1.2), Lemma 2.1 and Theorems 1.1 that

$$
\begin{aligned}
\|u_t(t)\| &\le C_3(\|v_y(t)\| + \|\theta_y(t)\| + \|u_{yy}(t)\| + \|u_y(t)\|_{L^\infty}\|v_y(t)\| + \|\boldsymbol{b}(t)\|_{L^\infty}\|\boldsymbol{b}_y(t)\|) \\
&\le C_3(\|v_y(t)\| + \|\theta_y(t)\| + \|u_{yy}(t)\| + \|b_y(t)\|).
\end{aligned}
$$

Differentiating (1.2) with respect to y, and using Theorem 1.1, we get

$$\|u_{ty}(t)\| \le C_3(\|v_y(t)\|_{H^1} + \|\theta_y(t)\|_{H^1} + \|u_y(t)\|_{H^2} + \|\boldsymbol{b}_y(t)\|_{H^1}), \tag{3.3}$$

or

$$\|u_{yyy}(t)\| \le C_3(\|v_y(t)\|_{H^1} + \|\theta_y(t)\|_{H^1} + \|\boldsymbol{b}_y(t)\|_{H^1} + \|u_{ty}(t)\|). \tag{3.4}$$

Differentiating (1.2) with respect to y twice, using the embedding theorem and Theorem 1.1, we conclude

$$\|u_{tyy}(t)\| \le C_3(\|v_y(t)\|_{H^2} + \|\theta_y(t)\|_{H^2} + \|u_y(t)\|_{H^3} + \|\boldsymbol{b}_y(t)\|_{H^2}), \tag{3.5}$$

or

$$\|u_{yyyy}(t)\| \le C_3(\|v_y(t)\|_{H^2} + \|\theta_y(t)\|_{H^2} + \|\boldsymbol{b}_y(t)\|_{H^2} + \|u_{tyy}(t)\|). \tag{3.6}$$

Similarly, we have

$$
\begin{aligned}
&\|\boldsymbol{w}_t(t)\| \le C_3(\|\boldsymbol{w}_y(t)\|_{H^1} + \|\boldsymbol{b}_y(t)\| + \|v_y(t)\|), \\
&\|\boldsymbol{w}_{ty}(t)\| \le C_3(\|\boldsymbol{w}_y(t)\|_{H^2} + \|\boldsymbol{b}_y(t)\|_{H^1} + \|v_y(t)\|_{H^1}),
\end{aligned} \tag{3.7}
$$

or

$$\|\boldsymbol{w}_{yyy}(t)\| \le C_3(\|\boldsymbol{b}_y(t)\|_{H^1} + \|v_y(t)\|_{H^1} + \|\boldsymbol{b}_{ty}(t)\|), \tag{3.8}$$

$$\|\boldsymbol{w}_{tyy}(t)\| \le C_3(\|\boldsymbol{w}_y(t)\|_{H^3} + \|\boldsymbol{b}_y(t)\|_{H^2} + \|v_y(t)\|_{H^2}), \tag{3.9}$$

or

$$\| \boldsymbol{w}_{yyyy}(t) \| \le C_3(\| \boldsymbol{b}_y(t) \|_{H^2} + \| v_y(t) \|_{H^2} + \| \boldsymbol{w}_{tyy}(t) \|), \tag{3.10}$$

$$\begin{aligned} &\| \boldsymbol{b}_t \| \le C_3(\| \boldsymbol{b}_y\|_{H^1} + \| \boldsymbol{w}_y \| + \| v_y \|), \\ &\| \boldsymbol{b}_{ty}(t) \| \le C_3(\| \boldsymbol{b}_y(t) \|_{H^2} + \| \boldsymbol{w}_y(t) \|_{H^1} + \| v_y(t) \|_{H^1}), \end{aligned} \tag{3.11}$$

or

$$\| \boldsymbol{b}_{yyy}(t) \| \le C_3(\| \boldsymbol{w}_y(t) \|_{H^1} + \| v_y(t) \|_{H^1} + \| \boldsymbol{b}_{ty}(t) \|), \tag{3.12}$$

$$\| \boldsymbol{b}_{tyy}(t) \| \le C_3(\| \boldsymbol{b}_y(t) \|_{H^3} + \| \boldsymbol{w}_y(t) \|_{H^2} + \| v_y(t) \|_{H^2}), \tag{3.13}$$

or

$$\| \boldsymbol{b}_{yyyy}(t) \| \le C_3(\| \boldsymbol{w}_y(t) \|_{H^2} + \| v_y(t) \|_{H^2} + \| \boldsymbol{b}_{tyy}(t) \|), \tag{3.14}$$

$$\begin{aligned} \| \theta_t(t) \| \le\, & C_3(\| u_y(t) \| + \| v_y(t) \| + \| \theta_{yy}(t) \| + \| u_y(t) \|_{L^\infty} \| u_y(t) \| \\ & + \| \boldsymbol{w}_y(t) \|_{L^\infty} \| \boldsymbol{w}_y(t) \| + \| \boldsymbol{b}_y(t) \|_{L^\infty} \| \boldsymbol{b}_y(t) \| + \| \theta_y(t) \|_{L^\infty} \| \theta_y(t) \|), \\ & \le C_3(\| \theta_{yy}(t) \| + \| u_{yy}(t) \| + \| \boldsymbol{w}_{yy}(t) \| + \| \boldsymbol{b}_{yy}(t) \|), \end{aligned} \tag{3.15}$$

$$\begin{aligned} \| \theta_{ty}(t) \| \le\, & C_3(\| \theta_t(t) \| + \| \theta_y(t) \|_{H^2} + \| v_y(t) \|_{H^1} + \| u_y(t) \|_{H^1} + \| \boldsymbol{w}_y(t) \|_{H^1} \\ & + \| \boldsymbol{b}_y(t) \|_{H^1}), \end{aligned} \tag{3.16}$$

or

$$\begin{aligned} \| \theta_{yyy}(t) \| \le\, & C_3(\| v_y(t) \|_{H^1} + \| u_y(t) \|_{H^1} + \| \boldsymbol{w}_y(t) \|_{H^1} + \| \boldsymbol{b}_y(t) \|_{H^1} \\ & + \| \theta_{ty}(t) \|), \end{aligned} \tag{3.17}$$

$$\| \theta_{tyy}(t) \| \le C_3(\| \theta_y(t) \|_{H^3} + \| v_y(t) \|_{H^2} + \| u_y(t) \|_{H^2} + \| \boldsymbol{w}_y(t) \|_{H^2} + \| \boldsymbol{b}_y(t) \|_{H^2}), \tag{3.18}$$

or

$$\begin{aligned} \| \theta_{yyyy}(t) \| \le\, & C_3(\| v_y(t) \|_{H^2} + \| u_y(t) \|_{H^2} + \| \boldsymbol{w}_y(t) \|_{H^2} + \| \boldsymbol{b}_y(t) \|_{H^2} \\ & + \| \theta_{tyy}(t) \|). \end{aligned} \tag{3.19}$$

Differentiating (1.2) with respect to t, and using Theorem 1.1, (3.3), (3.5), (3.11)-(3.12) and (3.16), we derive

$$\| u_{tt}(t) \| \le C_3(\| v_y(t) \|_{H^2} + \| u_y(t) \|_{H^3} + \| \boldsymbol{b}_y(t) \|_{H^2} + \| \theta_y(t) \|_{H^2}). \tag{3.20}$$

Similarly, we can conclude

$$\| \boldsymbol{w}_{tt}(t) \| \le C_3(\| v_y(t) \|_{H^2} + \| \boldsymbol{b}_y(t) \|_{H^2} + \| \boldsymbol{w}_y(t) \|_{H^3}), \tag{3.21}$$

$$\| \boldsymbol{b}_{tt}(t) \| \le C_3(\| v_y(t) \|_{H^2} + \| \boldsymbol{b}_y(t) \|_{H^3} + \| \boldsymbol{w}_y(t) \|_{H^2}), \tag{3.22}$$

$$\| \theta_{tt}(t) \| \le C_3(\| v_y(t) \|_{H^2} + \| u_y(t) \|_{H^2} + \| \boldsymbol{b}_y(t) \|_{H^2} + \| \boldsymbol{w}_y(t) \|_{H^2} + \| \theta_y(t) \|_{H^3}). \tag{3.23}$$

Thus, (3.1) follows from (3.3), (3.7), (3.11) and (3.16), and (3.2) from (3.5), (3.9), (3.13), (3.18) and (3.20)-(3.23).

Lemma 3.2 *Under the assumptions in Theorem 1.2, the following estimates hold, for any* $t \in [0, T]$,

$$\| u_{tt}(t) \|^2 + \int_0^t \| u_{tty}(s) \|^2 ds \le C_3 + C_3 \int_0^t (\| \boldsymbol{b}_{tyy}\|^2 + \| \theta_{tyy}\|^2)(s) ds, \tag{3.24}$$

$$\begin{aligned} &\| \boldsymbol{w}_{tt}(t) \|^2 + \| \boldsymbol{b}_{tt}(t) \|^2 + \int_0^t (\| \boldsymbol{w}_{tty}(s) \|^2 + \| \boldsymbol{b}_{tty}(s) \|^2) ds \\ &\le C_3 + C_3 \int_0^t (\| \boldsymbol{b}_{tyy}(s) \|^2 + \| \boldsymbol{w}_{tyy}(s) \|^2) ds, \end{aligned} \tag{3.25}$$

$$\begin{aligned} &\| \theta_{tt}(t) \|^2 + \int_0^t \| \theta_{tty}(s) \|^2 ds \le C_3 + C_2 \varepsilon^{-1} \int_0^t \| \theta_{tyy}(s) \|^2 ds \\ &+ C_1 \varepsilon \int_0^t (\| u_{tyy}\|^2 + \| u_{tty}\|^2 + \| \boldsymbol{w}_{tyy}\|^2 + \| \boldsymbol{w}_{tty}\|^2 + \| \boldsymbol{b}_{tyy}\|^2 + \| \boldsymbol{b}_{tty}\|^2)(s) ds. \end{aligned} \tag{3.26}$$

Proof. Differentiating (1.2) with respect to t twice, multiplying the resulting equation by u_{tt}, performing an integration by parts, and using Lemma 2.1, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 u_{tt}^2(y, t) dy &\le -\lambda \| u_{tty}(t) \|^2 + C_4(\| \theta_{tt}(t) \| + \| u_{yy}(t) \| + \| \theta_t u_y(t) \| \\ &+ \| \boldsymbol{b} \cdot \boldsymbol{b}_{tt}(t) \| + \| \boldsymbol{b}_t^2(t) \|^2 + \| u_{ty}(t) \|) \| u_{tty}(t) \| \\ &\le -C_1^{-1} \| u_{tty}(t) \|^2 + C_4(\| \theta_{tt}(t) \|^2 + \| \boldsymbol{b}_{tt}(t) \|^2 + \| u_{ty}(t) \|^2 + \| u_y(t)\|^2 + \| \theta_t(t)\|^2) \end{aligned} \tag{3.27}$$

Thus, using Theorem 1.1 and Lemma 3.1, we get

$$\| u_{tt}(t) \|^2 + \int_0^t \| u_{tty}(s) \|^2 \mathrm{d}s \le C_3 + C_3 \int_0^t (\| \boldsymbol{b}_{tyy}\|^2 + \| \theta_{tyy}\|^2)(s) \mathrm{d}s.$$

Analogously, we obtain

$$\begin{aligned} &\| \boldsymbol{w}_{tt}(t) \|^2 + \| \boldsymbol{b}_{tt}(t) \|^2 + \int_0^t (\| \boldsymbol{w}_{tty}\|^2 + \| \boldsymbol{b}_{tty}\|^2)(s) \mathrm{d}s \\ &\le C_3 + C_3 \int_0^t (\| \boldsymbol{w}_{tyy}\|^2 + \| \boldsymbol{b}_{tyy}\|^2)(s) \mathrm{d}s. \end{aligned}$$

Equation (1.5) can be rewritten as

$$(c_v \theta)_t + p u_y = \left(\frac{\kappa \theta_y}{v} \right)_y + \frac{\lambda u_y^2 + \mu |\mathbf{w}_y|^2 + \nu |\mathbf{b}_y|^2}{v}. \tag{3.28}$$

Differentiating (3.28) with respect to t twice, multiplying the resulting equation by θ_{tt} in L^2 0[1] and integrating by parts, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 c_v\theta_{tt}(y,\ t)dy = -\int_0^1\left(\frac{\kappa\theta_y}{v}\right)_{tt}\theta_{tty}(y,\ t)dy\\
&-\int_0^1\left(p-\frac{\lambda u_y}{v}\right)u_{tty}\theta_{tt}(y,\ t)dy+\int_0^1\left(\frac{\mu \boldsymbol{w}_y}{v}\boldsymbol{w}_{tty}+\frac{v\boldsymbol{b}_y}{v}\boldsymbol{b}_{tty}\right)\theta_{tt}(y,\ t)dy\\
&-2\int_0^1\left(p_t-(\frac{\lambda u_y}{v})_t\right)u_{ty}\theta_{tt}(y,\ t)dy+2\int_0^1\left[(\frac{\mu \boldsymbol{w}_y}{v})_t\boldsymbol{w}_{ty}+(\frac{v\boldsymbol{b}_y}{v})_t\boldsymbol{b}_{ty}\right]\theta_{tt}(y,\ t)dy\\
&+\int_0^1\left(-p+\frac{\lambda u_y}{v}\right)_{tt}u_y\theta_{tt}(y,\ t)dy+\int_0^1\left[(\frac{\mu \boldsymbol{w}_y}{v})_{tt}\boldsymbol{w}_y+(\frac{v\boldsymbol{b}_y}{v})_{tt}\boldsymbol{b}_y\right]\theta_{tt}(y,\ t)dy\\
&=B_1+B_2+B_3+B_4+B_5+B_6+B_7.
\end{aligned}$$

By virtue of Theorem 1.1 and Lemmas 3.1-3.2, using the embedding theorem, we deduce for any $\varepsilon \in (0, 1)$,

$$\begin{aligned}
B_1 &\le -C^{-1}\|\theta_{tty}(t)\|^2+C_2\|\theta_{ty}(t)\|_{L^\infty}(\|u_y(t)\|^2+\|\theta_t(t)\|^2)\|\theta_{tty}(t)\|\\
&\quad +C_2\|(\frac{\kappa}{v})_{tt}(t)\|^2\|\theta_y(t)\|_{L^\infty}\|\theta_{tty}(t)\|\\
&\le -2C^{-1}\|\theta_{tty}(t)\|^2+C_2(\|u_y(t)\|^2+\|\theta_t(t)\|^2+\|u_{ty}(t)\|^2\\
&\quad +\|\theta_{ty}(t)\|^2+\|\theta_{tt}(t)\|^2+\|\theta_{tyy}(t)\|^2),\\
B_2 &\le \varepsilon\|u_{tty}(t)\|^2+C_2\varepsilon^{-1}\|\theta_{tt}(t)\|^2,\\
B_3 &\le \varepsilon(\|\boldsymbol{w}_{tty}(t)\|^2+\|\boldsymbol{b}_{tty}(t)\|^2)+C_2\varepsilon^{-1}\|\theta_{tt}(t)\|^2
\end{aligned}$$

and

$$\begin{aligned}
B_4 &\le C_1\int_0^1(|\theta_t|+|u_y|+|u_{ty}|+|u_y|^2)|\theta_{tt}\|u_{ty}|(y,t)dy\\
&\le C_2\|u_{ty}(t)\|^{\frac{1}{2}}\|u_{tyy}(t)\|^{\frac{1}{2}}(\|\theta_t(t)\|+\|u_y(t)\|+\|u_{ty}(t)\|)\|\theta_{tt}(t)\|
\end{aligned}$$

which implies

$$\begin{aligned}
\int_0^t B_4ds &\le C_2\sup_{0\le s\le t}\|\theta_{tt}(s)\|\left(\int_0^t\|u_{ty}(s)\|^2ds\right)^{\frac{1}{4}}\left(\int_0^t\|u_{tyy}(s)\|^2ds\right)^{\frac{1}{4}}\\
&\qquad\times\left(\int_0^t(\|u_{ty}\|^2+\|\theta_t\|^2+\|u_y\|^2)(s)ds\right)^{\frac{1}{2}}\\
&\le \varepsilon\left(\sup_{0\le s\le t}\|\theta_{tt}(s)\|^2+\int_0^t\|u_{tyy}(s)\|^2ds\right)+C_3\varepsilon^{-3}.
\end{aligned}$$

$$\begin{aligned}
B_5 &\le C_1\int_0^1\left[(|\boldsymbol{w}_y|^2+|\boldsymbol{w}_{ty}|)|\boldsymbol{w}_{ty}|+(|\boldsymbol{w}_y|^2+|\boldsymbol{w}_{ty}|)|\boldsymbol{w}_{ty}|\right]|\theta_{tt}|(y,t)dy\\
&\le C_2\|\boldsymbol{w}_{ty}(t)\|_{L^\infty}(\|\boldsymbol{w}_y(t)\|^2+\|\boldsymbol{w}_{ty}(t)\|)\|\theta_{tt}(t)\|+C_2\|\boldsymbol{w}_{ty}(t)\|_{L^\infty}(\|\boldsymbol{b}_y(t)\|^2+\|\boldsymbol{b}_{ty}(t)\|)\|\theta_{tt}(t)\|
\end{aligned}$$

which implies

$$\int_0^t B_5 \mathrm{d}s \leq \varepsilon \left(\sup_{0\leq s\leq t} \| \theta_{tt}(s)\|^2 + \int_0^t (\| \boldsymbol{w}_{ty}\|^2 + \| \boldsymbol{w}_{tyy}\|^2(s)\mathrm{d}s \right) + C_3\varepsilon^{-3},$$

$$\begin{aligned}
B_6 &\leq C_2 \| u_y(t)\|_{L^\infty} \| \theta_{tt}(t) \| \big[(\| \theta_t(t)\|_{L^\infty} + \| u_y(t)\|_{L^\infty})(\| \theta_t(t) \| + \| u_y(t) \|) + \| \theta_{tt}(t) \| \\
&\quad + \| u_{ty}(t) \| + \| u_y(t) \| + \| u_{tty}(t) \| \big] \\
&\leq C_2 \| \theta_{tt}(t) \| (\| \theta_t(t) \| + \| u_y(t)\|_{H^1} + \| \theta_{ty}(t) \| + \| \theta_{tt}(t) \| + \| u_{ty}(t) \| + \| u_{tty}(t) \|) \\
&\leq \varepsilon \| u_{tty}(t)\|^2 + C_2\varepsilon^{-1}(\| \theta_t(t)\|^2 + \| u_y(t) \|_{H^1}^2 + \| \theta_{ty}(t)\|^2 + \| \theta_{tt}(t)\|^2 + \| u_{ty}(t)\|^2), \\
B_7 &\leq C_2 \| \boldsymbol{w}_y(t)\|_{L^\infty} \| \theta_{tt}(t) \| (\| \boldsymbol{w}_{tty}(t) \| + \| \boldsymbol{w}_{ty}(t) \| + \| \boldsymbol{w}_y(t) \|) + C_2 \| \boldsymbol{b}_y(t)\|_{L^\infty} \| \theta_{tt}(t) \| \\
&\quad \times (\| \boldsymbol{b}_{tty}(t) \| + \| \boldsymbol{b}_{ty}(t) \| + \| \boldsymbol{b}_y(t) \|) \\
&\leq C_2 \| \theta_{tt}(t) \| (\| \boldsymbol{w}_{ty}(t) \| + \| \boldsymbol{w}_y(t)\|_{H^1} + \| \boldsymbol{w}_{tty}(t) \| + \| \boldsymbol{b}_{ty}(t) \| + \| \boldsymbol{b}_y(t)\|_{H^1} + \| \boldsymbol{b}_{tty}(t) \|) \\
&\leq \varepsilon(\| \boldsymbol{w}_{tty}(t)\|^2 + \| \boldsymbol{b}_{tty}(t)\|^2) + C_2\varepsilon^{-1}(\| \theta_{tt}(t)\|^2 + \| \boldsymbol{w}_y(t) \|_{H^1}^2 \\
&\quad + \| \boldsymbol{w}_{ty}(t)\|^2 + \| \boldsymbol{b}_y(t) \|_{H^1}^2 + \| \boldsymbol{b}_{ty}(t)\|^2).
\end{aligned}$$

Thus, for $\varepsilon \in (0, 1)$ small enough, we derive from above estimates

$$\begin{aligned}
&\| \theta_{tt}(t) \|^2 + \int_0^t \| \theta_{tty}(s) \|^2 \mathrm{d}s \leq C_2\varepsilon^{-1} \int_0^t (\| \theta_{tyy}(s) \|^2 + \| \theta_{tt}(s) \|^2) \,\mathrm{d}s + C_3\varepsilon^{-3} \\
&\quad + C_1\varepsilon \left[\sup_{0\leq s\leq t} \| \theta_{tt}(s)\|^2 + \int_0^t (\| u_{tyy}\|^2 + \| u_{tty}\|^2 + \| \boldsymbol{w}_{tyy}\|^2 \right. \\
&\quad \left. + \| \boldsymbol{w}_{tty}\|^2 + \| \boldsymbol{b}_{tyy}\|^2 + \| \boldsymbol{b}_{tty}\|^2) \,(s)\mathrm{d}s \right].
\end{aligned} \tag{3.29}$$

Thus, taking supremum in t on the left-hand side of (3.29), picking $\varepsilon \in (0, 1)$ small enough, and using (3.23), we can derive estimate (3.26).

Lemma 3.3 *Under the assumptions in Theorem 1.2, the following estimates hold, for any $t \in [0,T]$,*

$$\begin{aligned}
&\| u_{ty}(t) \|^2 + \int_0^t \| u_{tyy}(s) \|^2 ds \\
&\quad \leq C_3\varepsilon^{-6} + C_2\varepsilon^2 \int_0^t (\| \boldsymbol{b}_{tyy}\|^2 + \| \theta_{tyy}\|^2 + \| u_{tty}\|^2) \,(s)ds,
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
&\| \boldsymbol{w}_{ty}(t) \|^2 + \| \boldsymbol{b}_{ty}(t) \|^2 + \int_0^t (\| \boldsymbol{w}_{tyy}\|^2 + \| \boldsymbol{b}_{tyy}\|^2)(s) \,ds \\
&\quad \leq C_3\varepsilon^{-6} + C_2\varepsilon^2 \int_0^t (\| \boldsymbol{w}_{tty}\|^2 + \| \boldsymbol{b}_{tty}\|^2) \,(s) \,ds,
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
&\| \theta_{ty}(t) \|^2 + \int_0^t \| \theta_{tyy}(s) \|^2 ds \leq C_3\varepsilon^{-6} \\
&\quad + C_2\varepsilon^2 \int_0^t (\| \boldsymbol{b}_{tyy}\|^2 + \| u_{tyy}\|^2 + \| \boldsymbol{w}_{tyy}\|^2 + \| \theta_{tty}\|^2 + \| \theta_{yyy}\|^2 \, \| \theta_{ty}\|^2) \,(s)ds.
\end{aligned} \tag{3.32}$$

Proof. Differentiating (1.2) with respect to y and t, multiplying the resulting equation by u_{ty}, and integrating by parts, we arrive at

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\parallel u_{ty}(t)\parallel^2 = D_0(y,\ t) + D_1(t), \tag{3.33}$$

where

$$D_0(y,\ t) = \sigma_{ty}u_{ty}|_{y=0}^{y=1},\ D_1(t) = -\int_0^1 \sigma_{ty}u_{tyy}\mathrm{d}y,\ \sigma = -\left(p + \frac{1}{2}|\boldsymbol{b}|^2 - \frac{\lambda u_y}{v}\right).$$

We use Theorem 1.1, Lemma 2.1, the interpolation inequality and Poincaré's inequality to obtain

$$\begin{aligned} D_0 \le\ & C_1\big[(\parallel u_y(t)\parallel_{L^\infty} + \parallel \theta_t(t)\parallel_{L^\infty})(\parallel v_y(t)\parallel_{L^\infty} + \parallel \theta_y(t)\parallel_{L^\infty}) \\ & + \parallel \boldsymbol{b}_t(t)\parallel_{L^\infty}\parallel \boldsymbol{b}_y(t)\parallel_{L^\infty} + \parallel \boldsymbol{b}_{ty}(t)\parallel_{L^\infty}\parallel \boldsymbol{b}(t)\parallel_{L^\infty} + \parallel \theta_{ty}(t)\parallel_{L^\infty} + \parallel u_y(t)\parallel_{L^\infty}^2 \\ & + \parallel u_y(t)\parallel_{L^\infty}\parallel u_{yy}(t)\parallel_{L^\infty} + \parallel u_{ty}(t)\parallel_{L^\infty}\parallel v_y(t)\parallel_{L^\infty} + \parallel u_{tyy}(t)\parallel_{L^\infty}\big]\parallel u_{ty}(t)\parallel_{L^\infty} \\ \le\ & C_3(D_{01} + D_{01})\parallel u_{ty}(t)\parallel^{\frac{1}{2}}\parallel u_{tyy}(t)\parallel^{\frac{1}{2}}, \end{aligned} \tag{3.34}$$

where

$$\begin{aligned} D_{01} =& \parallel u_y(t)\parallel_{H^2} + \parallel \theta_t(t)\parallel + \parallel \theta_{ty}(t)\parallel + \parallel \boldsymbol{b}_t(t)\parallel + \parallel \boldsymbol{b}_{ty}(t)\parallel, \\ D_{02} =& \parallel \theta_{ty}(t)\parallel^{\frac{1}{2}}\parallel \theta_{tyy}(t)\parallel^{\frac{1}{2}} + \parallel u_{tyy}(t)\parallel^{\frac{1}{2}}\parallel u_{tyyy}(t)\parallel^{\frac{1}{2}} + \parallel u_{tyy}(t)\parallel \\ & + \parallel u_{ty}(t)\parallel^{\frac{1}{2}}\parallel u_{tyy}(t)\parallel^{\frac{1}{2}} + \parallel \boldsymbol{b}_{ty}(t)\parallel^{\frac{1}{2}}\parallel \boldsymbol{b}_{tyy}(t)\parallel^{\frac{1}{2}}. \end{aligned}$$

Using the Young inequality several times, we derive

$$\begin{aligned} & C_3D_{01}\parallel u_{ty}(t)\parallel^{\frac{1}{2}}\parallel u_{tyy}(t)\parallel^{\frac{1}{2}} \le \frac{\varepsilon^2}{2}\parallel u_{tyy}(t)\parallel^2 \\ & + C_3\varepsilon^{-\frac{2}{3}}(\parallel u_{ty}(t)\parallel^2 + \parallel u_y(t)\parallel_{H^2}^2 + \parallel \theta_t(t)\parallel^2 + \parallel \theta_{ty}(t)\parallel^2 + \parallel \boldsymbol{b}_t(t)\parallel^2 + \parallel \boldsymbol{b}_{ty}(t)\parallel^2) \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} C_3D_{02}\parallel u_{ty}(t)\parallel^{\frac{1}{2}}\parallel u_{tyy}(t)\parallel^{\frac{1}{2}} \le\ & \frac{\varepsilon^2}{2}\parallel u_{tyy}(t)\parallel^2 + \varepsilon^2(\parallel u_{tyyy}(t)\parallel^2 + \parallel \boldsymbol{b}_{tyy}(t)\parallel^2 + \parallel \theta_{tyy}(t)\parallel^2) \\ & + C_3\varepsilon^{-6}(\parallel u_{ty}(t)\parallel^2 + \parallel \theta_{ty}(t)\parallel^2 + \parallel \boldsymbol{b}_{ty}(t)\parallel^2). \end{aligned} \tag{3.36}$$

Thus, we infer from (3.34)-(3.36) that

$$\begin{aligned} D_0 \le\ & \varepsilon^2(\parallel u_{tyyy}(t)\parallel^2 + \parallel u_{tyy}(t)\parallel^2 + \parallel \boldsymbol{b}_{tyy}(t)\parallel^2 + \parallel \theta_{tyy}(t)\parallel^2) \\ & + C_3\varepsilon^{-6}(\parallel u_{ty}(t)\parallel^2 + \parallel \theta_{ty}(t)\parallel^2 + \parallel \boldsymbol{b}_{ty}(t)\parallel^2 + \parallel \theta_t(t)\parallel^2 + \parallel u_y(t)\parallel_{H^2}^2 + \parallel \boldsymbol{b}_t(t)\parallel^2), \end{aligned}$$

which, together with Theorem 1.1, Lemma 2.1, and Lemmas 3.1-3.2, yields

$$\int_0^t D_0\mathrm{d}s \le \varepsilon^2\int_0^t (\parallel u_{tyyy}\parallel^2 + \parallel u_{tyy}\parallel^2 + \parallel \boldsymbol{b}_{tyy}\parallel^2 + \parallel \theta_{tyy}\parallel^2)(s)\mathrm{d}s + C_3\varepsilon^{-6}. \tag{3.37}$$

Similarly, by Theorem 1.1, Lemma 2.1, and Lemmas 3.1-3.2 and the embedding theorem, we have

$$D_1 \le (2C_3)^{-1}\parallel u_{tyy}(t)\parallel^2 + C_3(\parallel u_{ty}(t)\parallel^2 + \parallel \boldsymbol{b}_{ty}(t)\parallel^2 + \parallel \theta_t(t)\parallel_{H^1}^2 + \parallel u_y(t)\parallel_{H^1}^2), \tag{3.38}$$

which, combined with (3.33), (3.37)-(3.38), Theorem 1.1, Lemma 2.1, and Lemmas 3.1-3.2, gives that for $\varepsilon \in (0, 1)$ small enough,

$$\| u_{ty}(t) \|^2 + \int_0^t \| u_{tyy}(s) \|^2 ds \le C_3\varepsilon^{-6} + C_2\varepsilon^2 \int_0^t (\| \boldsymbol{b}_{tyy}\|^2 + \| \theta_{tyy}\|^2 + \| u_{tyyy}\|^2)(s) ds. \quad (3.39)$$

On the other hand, differentiating (1.2) with respect to x and t, using Theorem 1.1 and Lemmas 3.1-3.2, we have

$$\begin{aligned} \| u_{tyyy}(t) \| \le C_1 \| u_{tty}(t) \| + C_2(\| u_{yy}(t) \|^2_{H^2} + \| \theta_y(t) \|^2_{H^1} + \| v_y(t) \|^2_{H^1} \\ + \| \boldsymbol{b}_y(t) \|^2_{H^1} + \| \theta_t(t) \|^2_{H^2} + \| \boldsymbol{b}_t(t) \|^2_{H^2}). \end{aligned} \quad (3.40)$$

Thus, inserting (3.40) into (3.39) implies estimate (3.30).

Analogously, we can obtain estimates (3.31)-(3.32). □

Lemma 3.4 *Under the assumptions in Theorem 1.2, the following estimates hold for any $t \in [0,T]$,*

$$\begin{aligned} &\| u_{tt}(t) \|^2 + \| u_{ty}(t) \|^2 + \| \boldsymbol{w}_{tt}(t) \|^2 + \| \boldsymbol{w}_{ty}(t) \|^2 + \| \boldsymbol{b}_{tt}(t) \|^2 + \| \boldsymbol{b}_{ty}(t) \|^2 + \| \theta_{tt}(t) \|^2 \\ &+ \| \theta_{ty}(t) \|^2 + \int_0^t (\| u_{tty}\|^2 + \| u_{tyy}\|^2 + \| \boldsymbol{w}_{tty}\|^2 + \| \boldsymbol{w}_{tyy}\|^2 + \| \boldsymbol{b}_{tty}\|^2 \\ &+ \| \boldsymbol{b}_{tyy}\|^2 + \| \theta_{tty}\|^2 + \| \theta_{tyy}\|^2)(s)\, ds \le C_4, \end{aligned} \quad (3.41)$$

$$\| v_{yyy}(t) \|^2_{H^1} + \| v_{yy}(t) \|^2_{W^{1,\infty}} + \int_0^t (\| v_{yyy} \|^2_{H^1} + \| v_{yy} \|^2_{W^{1,\infty}})(s)\, ds \le C_4, \quad (3.42)$$

$$\begin{aligned} &\| u_{yyy}(t) \|^2_{H^1} + \| u_{yy}(t) \|^2_{W^{1,\infty}} + \| \boldsymbol{w}_{yyy}(t) \|^2_{H^1} + \| \boldsymbol{w}_{yy}(t) \|^2_{W^{1,\infty}} + \| \boldsymbol{b}_{yyy}(t) \|^2_{H^1} + \| \boldsymbol{b}_{yy}(t) \|^2_{W^{1,\infty}} \\ &+ \| \theta_{yyy}(t) \|^2_{H^1} + \| \theta_{yy}(t) \|^2_{W^{1,\infty}} + \| v_{tyyy}(t) \|^2 + \| u_{tyy}(t) \|^2 + \| \boldsymbol{w}_{tyy}(t) \|^2 + \| \boldsymbol{b}_{tyy}(t) \|^2 \\ &+ \| \theta_{tyy}(t) \|^2 + \int_0^t (\| u_{tt}\|^2 + \| \boldsymbol{w}_{tt}\|^2 + \| \boldsymbol{b}_{tt}\|^2 + \| \theta_{tt}\|^2 + \| u_{yy} \|^2_{W^{2,\infty}} + \| \boldsymbol{w}_{yy} \|^2_{W^{2,\infty}} \\ &+ \| \boldsymbol{b}_{yy} \|^2_{W^{2,\infty}} + \| \theta_{yy} \|^2_{W^{2,\infty}} + \| \theta_{tyy} \|^2_{H^1} + \| u_{tyy} \|^2_{H^1} + \| \boldsymbol{w}_{tyy} \|^2_{H^1} + \| \boldsymbol{b}_{tyy} \|^2_{H^1} \\ &+ \| \theta_{ty} \|^2_{W^{1,\infty}} + \| u_{ty} \|^2_{W^{1,\infty}} + \| \boldsymbol{w}_{ty} \|^2_{W^{1,\infty}} + \| \boldsymbol{b}_{ty} \|^2_{W^{1,\infty}} + \| v_{tyyy} \|^2_{H^1})(s)\, ds \le C_4, \end{aligned} \quad (3.43)$$

$$\int_0^t (\| u_{yyyy} \|^2_{H^1} + \| \boldsymbol{w}_{yyyy} \|^2_{H^1} + \| \boldsymbol{b}_{yyyy} \|^2_{H^1} + \| \theta_{yyyy} \|^2_{H^1})(s)\, ds \le C_3. \quad (3.44)$$

Proof. Adding up (3.30)-(3.32), picking $\varepsilon \in (0, 1)$ enough small, by Lemmas 3.1-3.3, and Gronwall's inequality, we get

$$\begin{aligned} &\| u_{ty}(t) \|^2 + \| \boldsymbol{w}_{ty}(t) \|^2 + \| \boldsymbol{b}_{ty}(t) \|^2 + \| \theta_{ty}(t) \|^2 + \int_0^t (\| u_{tyy}\|^2 + \| \boldsymbol{w}_{tyy}\|^2 \\ &+ \| \boldsymbol{b}_{tyy}\|^2 + \| \theta_{tyy}\|^2)(s) ds \le C_3\varepsilon^{-6} + C_2\varepsilon^2 \int_0^t (\| u_{tty}\|^2 + \| \boldsymbol{w}_{tty}\|^2 \\ &+ \| \boldsymbol{b}_{tty}\|^2 + \| \theta_{tty}\|^2 + \| \theta_{ty}\|^2 \; \| \theta_{yyy}\|^2)(s) ds. \end{aligned} \quad (3.45)$$

Now multiplying (3.24)-(3.26) by ε, ε, and $\varepsilon^{\frac{3}{2}}$, adding the resultant to (3.45), and choosing $\varepsilon \in (0, 1)$ small enough, we obtain

$$\begin{aligned}&\| u_{tt}(t) \|^2 + \| u_{ty}(t) \|^2 + \| \boldsymbol{w}_{tt}(t) \|^2 + \| \boldsymbol{w}_{ty}(t) \|^2 + \| \boldsymbol{b}_{tt}(t) \|^2 + \| \boldsymbol{b}_{ty}(t) \|^2 + \| \theta_{tt}(t) \|^2 \\ &\quad + \| \theta_{ty}(t) \|^2 + \int_0^t (\| u_{tty}\|^2 + \| u_{tyy}\|^2 + \| \boldsymbol{w}_{tty}\|^2 + \| \boldsymbol{w}_{tyy}\|^2 + \| \boldsymbol{b}_{tty}\|^2 \\ &\quad + \| \boldsymbol{b}_{tyy}\|^2 + \| \theta_{tty}\|^2 + \| \theta_{tyy}\|^2)(s)\mathrm{d}s \le C_4\varepsilon^{-6} + C_2\varepsilon^2 \int_0^t \| \theta_{ty}(s) \|^2 \| \theta_{yyy}(s) \|^2 \mathrm{d}s,\end{aligned}$$

which, by Gronwall's inequality, gives the estimate (3.41).

Differentiating (2.12) with respect to y, and using $v_{tyy} = u_{yyy}$, we obtain

$$\lambda \frac{\partial}{\partial t}\left(\frac{v_{yyy}}{v}\right) + \frac{\theta}{v^2} v_{yyy} = E_1(y, t), \tag{3.46}$$

with

$$E_1(y, t) = E_y(y, t) + u_{tyy} + \left(\frac{\theta}{v^2}\right)_y v_{yy} + \lambda\left(\frac{v_y v_{yy}}{v}\right)_t.$$

Obviously, we can infer from Lemmas 3.1-3.3 that

$$\|E_1(t)\| \le C_2(\| u_{tyy}(t) \| + \| \theta_y(t) \|_{H^2} + \| u_y(t) \|_{H^2} + \| \boldsymbol{b}_y(t) \|_{H^2} + \| v_y(t) \|_{H^1}) \tag{3.47}$$

leading to

$$\int_0^t \| E_1(s) \|^2 \mathrm{d}s \le C_3. \tag{3.48}$$

Multiplying (3.46) by $\frac{v_{yyy}}{v}$, we get

$$\frac{d}{\mathrm{d}t}\left\|\frac{v_{yyy}}{v}(t)\right\|^2 + C_1^{-1}\left\|\frac{v_{yyy}}{v}(t)\right\|^2 \le C_1 \| E_1(t) \|^2, \tag{3.49}$$

which, combined with (3.48), gives

$$\| v_{yyy}(t) \|^2 + \int_0^t \| v_{yyy}(s) \|^2 \mathrm{d}s \le C_3. \tag{3.50}$$

By (3.4), (3.6), (3.8), (3.10), (3.12), (3.14), (3.17), (3.19), (3.41), (3.50), and Lemmas 3.1-3.3, and using the embedding theorem, we have

$$\begin{aligned}&\| u_{yyy}(t) \|^2 + \| u_{yy}(t) \|^2_{L^\infty} + \| \boldsymbol{w}_{yyy}(t) \|^2 + \| \boldsymbol{w}_{yy}(t) \|^2_{L^\infty} + \| \boldsymbol{b}_{yyy}(t) \|^2 + \| \boldsymbol{b}_{yy}(t) \|^2_{L^\infty} \\ &+ \| \theta_{yyy}(t) \|^2 + \| \theta_{yy}(t) \|^2_{L^\infty} + \int_0^t (\| u_{yy} \|^2_{W^{1,\infty}} + \| \boldsymbol{w}_{yy} \|^2_{W^{1,\infty}} + \| \boldsymbol{b}_{yy} \|^2_{W^{1,\infty}} + \| \theta_{yy} \|^2_{W^{1,\infty}} \\ &+ \| \theta_{yyy} \|^2_{H^1} + \| u_{yyy} \|^2_{H^1} + \| \boldsymbol{w}_{yyy} \|^2_{H^1} + \| \boldsymbol{b}_{yyy} \|^2_{H^1})(s)\mathrm{d}s \le C_3.\end{aligned} \tag{3.51}$$

Differentiating (1.2)-(1.5) with respect to t, using (3.41) and Lemmas 3.1-3.3, we get

$$\|u_{tyy}(t)\| \le C_1 \| u_{tt}(t) \| + C_1(\| u_{ty}(t) \| + \| \boldsymbol{b}_{ty}(t) \| + \| \theta_{ty}(t) \|) \le C_4, \tag{3.52}$$

$$\|\boldsymbol{w}_{tyy}(t)\| \le C_1 \| \boldsymbol{w}_{tt}(t) \| + C_1(\| \boldsymbol{w}_{ty}(t) \| + \| \boldsymbol{b}_{ty}(t) \|) \le C_4, \tag{3.53}$$

$$\|\boldsymbol{b}_{tyy}(t)\| \le C_1 \| \boldsymbol{b}_{tt}(t) \| + C_1(\| \boldsymbol{w}_{ty}(t) \| + \| \boldsymbol{b}_{ty}(t) \|) \le C_4, \tag{3.54}$$

$$\|\theta_{tyy}(t)\| \le C_1 \| \theta_{tt}(t) \| + C_1(\| u_{ty}(t) \| + \| \boldsymbol{w}_{ty}(t) \| + \| \boldsymbol{b}_{ty}(t) \| + \| \theta_{ty}(t) \|) \le C_4, \tag{3.55}$$

which, combined with (3.6), (3.10), (3.14) and (3.19), yields

$$\begin{aligned} &\| u_{yyyy}(t) \| + \| \boldsymbol{w}_{yyyy}(t) \| + \| \boldsymbol{b}_{yyyy}(t) \| + \| \theta_{yyyy}(t) \| + \int_0^t (\| u_{tyy}\|^2 + \| \boldsymbol{w}_{tyy}\|^2 + \| \boldsymbol{b}_{tyy}\|^2 \\ &\quad + \| \theta_{tyy}\|^2 + \| u_{yyyy}\|^2 + \| \boldsymbol{w}_{yyyy}\|^2 + \| \boldsymbol{b}_{yyyy}\|^2 + \| \theta_{yyyy}\|^2)(s)\mathrm{d}s \le C_3. \end{aligned} \tag{3.56}$$

Therefore, it follows from (3.51), (3.56), and the embedding theorem, we obtain

$$\begin{aligned} &\| u_{yyy}(t) \|_{L^\infty} + \| \boldsymbol{w}_{yyy}(t) \|_{L^\infty} + \| \boldsymbol{b}_{yyy}(t) \|_{L^\infty} + \| \theta_{yyy}(t) \|_{L^\infty} \\ &\quad + \int_0^t (\| u_{yyy}\|_{L^\infty} + \| \boldsymbol{w}_{yyy}\|_{L^\infty} + \| \boldsymbol{b}_{yyy}\|_{L^\infty} + \| \theta_{yyy}\|_{L^\infty})(s)\mathrm{d}s \le C_3. \end{aligned} \tag{3.57}$$

Differentiating (3.46) with respect to y, we obtain

$$\lambda \frac{\partial}{\partial t}\left(\frac{v_{yyyy}}{v}\right) + \frac{\theta}{v^2} v_{yyyy} = E_2(y,\ t) \tag{3.58}$$

where $E_2(y,\ t) = E_{1y}(y,\ t) + (\frac{\theta}{v^2})_y v_{yyy} + \lambda \frac{\partial}{\partial t}(\frac{v_y v_{yyy}}{v^2})$.
Using the embedding theorem and Lemmas 3.1-3.3, we can conclude

$$\|E_2(t)\| \le C_1 \| u_{tyyy}(t) \| + C_4(\| \theta_y(t) \|_{H^3} + \| u_y(t) \|_{H^3} + \| \boldsymbol{b}_y(t) \|_{H^3} + \| v_y(t) \|_{H^2}). \tag{3.59}$$

We infer from (3.20)-(3.23) that

$$\int_0^t (\| u_{tt}\|^2 + \| \boldsymbol{w}_{tt}\|^2 + \| \boldsymbol{b}_{tt}\|^2 + \| \theta_{tt}\|^2)(s)\mathrm{d}s \le C_4, \tag{3.60}$$

which, together with Lemma 3.3, gives

$$\int_0^t (\| u_{tyyy}\|^2 + \| \boldsymbol{w}_{tyyy}\|^2 + \| \boldsymbol{b}_{tyyy}\|^2 + \| \theta_{tyyy}\|^2)(s)\mathrm{d}s \le C_3. \tag{3.61}$$

Thus, it follows from (3.40), (3.59), (3.61), and Lemmas 3.1-3.3 that

$$\int_0^t \| E_2(s) \|^2 \mathrm{d}s \le C_3. \tag{3.62}$$

Multiplying (3.58) by $\frac{v_{yyy}}{v}$, we get

$$\frac{d}{\mathrm{d}t}\left\| \frac{v_{yyyy}}{v}(t)\right\|^2 + C_1 \left\| \frac{v_{yyyy}}{v}(t)\right\|^2 \le C_1 \| E_2(t) \|^2, \tag{3.63}$$

whence by (3.62),

$$\| v_{yyyy}(t) \|^2 + \int_0^t \| v_{yyyy}(s) \|^2 \mathrm{d}s \le C_3. \tag{3.64}$$

Differentiating (1.2) with respect to y there times, using Lemmas 3.1-3.3 and Poincaré's inequality, we have

$$\|u_{yyyyy}(t)\| \leq C_3 \| u_{tyyy}(t) \| + C_3(\| v_y(t) \|_{H^3} + \| u_y(t) \|_{H^3} + \| \theta_y(t) \|_{H^3} + \| \boldsymbol{b}_y(t) \|_{H^3}). \quad (3.65)$$

Thus, we conclude from (1.2), (3.56), (3.61), (3.64), and (3.65) that

$$\int_0^t (\| u_{yyyyy}\|^2 + \| v_{tyyy} \|_{H^1}^2)(s)\mathrm{d}s \leq C_3. \quad (3.66)$$

Similarly, we can deduce from (1.3)-(1.5) that

$$\int_0^t (\| \boldsymbol{b}_{yyyyy}\|^2 + \| \boldsymbol{w}_{yyyyy}\|^2 + \| \theta_{yyyyy}\|^2)(s)\mathrm{d}s \leq C_4. \quad (3.67)$$

which, along with (3.51) and (3.66), gives

$$\int_0^t (\| u_{yy} \|_{W^{2,\infty}}^2 + \| \boldsymbol{w}_{yy} \|_{W^{2,\infty}}^2 + \| \boldsymbol{b}_{yy} \|_{W^{2,\infty}}^2 + \| \theta_{yy} \|_{W^{2,\infty}}^2)(s)\mathrm{d}s \leq C_3. \quad (3.68)$$

Finally, using (1.1), (3.50)-(3.56), (3.64), (3.66)-(3.68), and Sobolev's interpolation inequality, we can get the desired estimates (3.42)-(3.44).

Proof of Theorem 1.2. By Lemma 2.1, Lemmas 3.1-3.4, and Theorem 1.1, we complete the proof of Theorem 1.2.

Acknowledgements
The work in part was supported by the NNSF of China (No. 11031003) and the Doctoral Innovational Fund of Donghua University (No. BC20101220).

Author details
[1]College of Information Sciences and Technology, Donghua University, Songjiang Shanghai, 201620, People's Republic of China [2]Department of Applied Mathematics, College of Sciences, Donghua University, Songjiang Shanghai 201620, People's Republic of China [3]Songyin School, Jinshan Shanghai, 201504, People's Republic of China

Authors' contributions
All authors contributed to each part of this work equally.

Competing interests
The authors declare that they have no competing interests.

Received: 10 June 2011 Accepted: 10 October 2011 Published: 10 October 2011

References

1. Clermmow, PC, Dougherty, JP: Electrodynamics of Particles and Plasmas. Addison-Wesley, New York (1990)
2. Woods, LC: Principle of Magnetoplasma Dynamics. Oxford University Press, New York (1987)
3. Kazhikhov, AV: Sur la solubilité globale des problémes monodimensionnels aux valeurs initiales-limités pour les équations d'un gaz visqueux et calorifére. pp. 317–320. C R Acad Sci Paris**284 A**, (1977)
4. Kazhikhov, AV: To a theory of boundary value problems for equations of one-dimensional nonstationary motion of viscous heat-conduction gases. pp. 37–62. Boundary Value Problems for Hydrodynamical Equations (in Russian) No. 50, Inst. Hydrodynamics, Siberian Branch Akad. USSR (1981)
5. Kazhikhov, AV, Sheluhin, VV: Unique global solution with respect to time of the initial-boundary value problems for one-dimensional equations of a viscous gas. J Appl Math Mech. **41**, 273–282 (1977). doi:10.1016/0021-8928(77)90011-9
6. Kawashima, S, Nishida, T: Global solutions to the initial value problem for the equations of one-dimensional motion of viscous polytropic gases. J Math Kyoto Univ. **21**, 825–837 (1981)
7. Zheng, S, Qin, Y: Universal attractor for the Navier-Stocks equations of compressible and heat-conductive fluid in bounded annular domains in $\mathbb{R}^n$. Arch Rat Mech Anal. **160**, 153–179 (2001). doi:10.1007/s002050100163
8. Kawohl, B: Global existence of large solutions to initial boundary value problems for the equations of one-dimensional motion of viscous polytropic gases. J Differ Equ. **58**, 76–103 (1985). doi:10.1016/0022-0396(85)90023-3
9. Jiang, S: On initial boundary value problems for a viscous heat-conducting one-dimensional real gas. J Differ Equ. **110**, 157–181 (1994). doi:10.1006/jdeq.1994.1064

10. Qin, Y: Nonlinear Parabolic-Hyperbolic Coupled Systems and Their Attractors. Operator Theory, Advances in PDEs, Basel, Boston-Berlin, Birkhäuser. **184** (2008)
11. Ducomet, B: A model of thermal dissipation for a one-dimensional viscous reactive and radiative. Math Meth Appl Sci. **22**, 1323–1349 (1999). doi:10.1002/(SICI)1099-1476(199910)22:153.0.CO;2-8
12. Umehara, M, Tani, A: Global solution to the one-dimensional equations for a self-gravitating viscous radiative and reactive gas. J Differ Equ. **234**, 439–463 (2007). doi:10.1016/j.jde.2006.09.023
13. Ducomet, B, Feireisl, E: The equations of magnetohydrodynamics: On the interaction between matter and radiation in the evolution of gaseous stars. Comm Math Phys. **266**, 595–629 (2006). doi:10.1007/s00220-006-0052-y
14. Zhang, J, Xie, F: Global solution for a one-dimensional model problem in thermally radiative magnetohydrodynamics. J Differ Equ. **245**, 1853–1882 (2008). doi:10.1016/j.jde.2008.07.010
15. Chen, G-Q, Wang, D: Global solutions of nonlinear magnetohydrodynamics with large initial data. J Differ Equ. **182**, 344–376 (2002). doi:10.1006/jdeq.2001.4111
16. Chen, G-Q, Wang, D: Existence and continuous dependence of large solutions for the magnetohydrodynamics equations. Z Angew Math Phys. **54**, 608–632 (2003). doi:10.1007/s00033-003-1017-z
17. Hoff, D, Tsyganov, E: Uniqueness and continuous dependence of weak solutions in compressible magnetohydrodynamics. Z Angew Math Phys. **56**, 791–840 (2005). doi:10.1007/s00033-005-4057-8
18. Fan, J, Jiang, S, Nakamura, G: Stability of weak solutions to equations of magnetohydro-dynamics with Lebesgue initial data. (2005) Preprint
19. Fan, J, Jiang, S, Nakamura, G: Vanishing sheer viscosity limit in the magnetohydrodynamic equations. Comm Math Phys. **270**, 691–708 (2007). doi:10.1007/s00220-006-0167-1
20. Ströhmer, G: About compressible viscous fluid flow in a bounded region. Pacific J Math. **143**, 359–375 (1990)
21. Vol'pert, AI, Hudjaev, SI: On the Cauchy problem for composite systems of nonlinear differential equations. Math USSR-Sb. **16**, 517–544 (1972). doi:10.1070/SM1972v016n04ABEH001438
22. Wang, D: Large solutions to the initial-boundary value problem for planar magnetohydrodynamics. SIAM J Appl Math. **63**, 1424–1441 (2003). doi:10.1137/S0036139902409284

doi:10.1186/1687-2770-2011-30
Cite this article as: Liu *et al.*: Regularity of large solutions for the compressible magnetohydrodynamic equations. *Boundary Value Problems* 2011 **2011**:30.