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Infinitely many periodic solutions for some second-order differential systems with p(t)-Laplacian

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Abstract

In this article, we investigate the existence of infinitely many periodic solutions for some nonautonomous second-order differential systems with *p*(*t*)-Laplacian. Some multiplicity results are obtained using critical point theory.

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1. Introduction

Consider the second-order differential system with $p(t)$-Laplacian

$$\begin{cases} -\frac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)) + |u(t)|^{p(t)-2}u(t) = \nabla F(t, u(t)) \quad \text{a. e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, F: $[0, T] \times \mathbb{R}^N \to \mathbb{R}$, and $p(t) \in C([0, T], \mathbb{R}^+)$ satisfies the following assumptions:

(A) $p(0) = p(T)$ and $p^- := \min_{0\le t\le T} p(t) > 1$, where $q^+ > 1$ which satisfies $1/p^- + 1/q^+ = 1$.

Moreover, we suppose that F: $[0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumptions:

(A') $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$, such that

$$|F(t, x)| \le a(|x|)b(t), \quad |\nabla F(t, x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

The operator $\frac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t))$ is said to be $p(t)$-Laplacian, and becomes p-Laplacian when $p(t) \equiv p$ (a constant). The $p(t)$-Laplacian possesses more complicated nonlinearity than p-Laplacian; for example, it is inhomogeneous. The study of various mathematical problems with variable exponent growth conditions has received considerable attention in recent years. These problems are interesting in applications and raise many mathematical problems. One of the most studied models leading to problem of this type is the model of motion of electro-rheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electromagnetic field. Another field of application of equations with variable exponent growth

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conditions is image processing (see [1,2]). The variable nonlinearity is used to outline the borders of the true image and to eliminate possible noise. We refer the reader to [3-12] for an overview on this subject.

In 2003, Fan and Fan [13] studied the ordinary $p(t)$-Laplacian system and introduced a generalized Orlicz-Sobolev space $W_T^{1,p(t)}$, which is different from the usual space $W_T^{1,p}$, then Wang and Yuan [14] obtained the existence and multiplicity of periodic solutions for ordinary $p(t)$-Laplacian system under the generalized Ambrosetti-Rabinowitz conditions. Fountain and Dual Fountain theorems were established by Bartsch and Willem [15,16], and both theorems are effective tools for studying the existence of infinitely many large energy solutions and small energy solutions. When we impose some suitable conditions on the growth of the potential function at origin or at infinity, we get three multiplicity results of infinitely many periodic solutions for system (1.1) using the Fountain theorem, the Dual Fountain theorem, and the Symmetric Mountain Pass theorem.

The rest of the article is divided as follows: Basic definitions and preliminary results are collected in Second 2. The main results and proofs are given in Section 3. The three examples are presented in Section 4 for illustrating our results.

In this article, we denote by $p^+ := \max_{0\le t\le T} p(t) > 1$ throughout this article, and we use $\langle \cdot, \cdot \rangle$ and $|\cdot|$ to denote the usual inner product and norm in $\mathbb{R}^N$, respectively.

2. Preliminaries

In this section, we recall some known results in nonsmooth critical point theory, and the properties of space $W_T^{1,p(t)}$ are listed for the convenience of readers.

Definition 2.1 [14]. Let $p(t)$ satisfies the condition (A), define

$$L^{p(t)}([0,T],\mathbb{R}^N) = \left\{ u \in L^1([0,T],\mathbb{R}^N) : \int_0^T |u|^{p(t)} dt < \infty \right\}$$

with the norm

$$|u|_{p(t)} := \inf\left\{ \lambda > 0 : \int_0^T \left|\frac{u}{\lambda}\right|^{p(t)} dt \le 1 \right\}.$$

For $u \in L^1_{loc}([0,T],\mathbb{R}^N)$, let u' denote the weak derivative of u, if $u' \in L^1_{\text{loc}}([0,T],\mathbb{R}^N)$ and satisfies

$$\int_0^T u'\phi dt = -\int_0^T u\phi' dt, \quad \forall \phi \in C_0^\infty([0,T],\mathbb{R}^N).$$

Define

$$W^{1,p(t)}([0,T],\mathbb{R}^N) = \{u \in L^{p(t)}([0,T],\mathbb{R}^N) : \quad u' \in L^{p(t)}([0,T],\mathbb{R}^N)\} \triangleright$$

with the norm $\|u\|_{W^{1,p(t)}} := |u|_{p(t)} + |u'|_{p(t)}$.

In this article, we will use the following equivalent norm on $W^{1,\,p(t)}$ ([0, T], $\mathbb{R}^N$), i.e.,

$$\|u\| := \inf\left\{ \lambda > 0 : \int_0^T \left(\left|\frac{u}{\lambda}\right|^{p(t)} + \left|\frac{\dot{u}}{\lambda}\right|^{p(t)} \right) dt \le 1 \right\},$$

and some lemmas given in the following section have been proven under the norm of $\|u\|_{W^{1,p(t)}}$, and it is obvious that they also hold under the norm $||u||$.

Remark 2.1. If $p(t) = p$, where $p \in (1, \infty)$ is a constant, by the definition of $|u|_{p(t)}$, it is easy to get $|u|_p = (\int_0^T |u(t)|^p dt)^{1/p}$, which is the same with the usual norm in space L^p.

The space $L^{p(t)}$ is a generalized Lebesgue space, and the space $W^{1,\,p(t)}$ is a generalized Sobolev space. Because most of the following lemmas have appeared in [13,14,17,18], we omit their proofs.

Lemma 2.1 [13]. $L^{p(t)}$ and $W^{1,\,p(t)}$ are both Banach spaces with the norms defined above, when $p^- > 1$, they are reflexive.

Lemma 2.2 [14]. (i) The space $L^{p(t)}$ is a separable, uniform convex Banach space, its conjugate space is $L^{q(t)}$, for any $u \in L^{p(t)}$ and $v \in L^{q(t)}$, we have

$$\left|\int_0^T uv dt\right| \le 2|u|_{p(t)}|v|_{q(t)},$$

where $\dfrac{1}{p(t)} + \dfrac{1}{q(t)} = 1$.

(ii) If $p_1(t)$ and $p_2(t) \in C([0, T], \mathbb{R}^+)$ and $p_1(t) \le p_2(t)$ for any $t \in [0, T]$, then $L^{p_2(t)} \to L^{p_1(t)}$, and the embedding is continuous.

Lemma 2.3 [14]. If we denote $\rho(u) = \displaystyle\int_0^T |u(t)|^{p(t)} dt$, $\forall\ u \in L^{p(t)}$, then

(i) $|u|_{p(t)} < 1$ $(= 1; > 1) \Leftrightarrow \rho(u) < 1$ $(= 1; > 1)$;

(ii) $|u|_{p(t)} > 1 \Rightarrow |u|_{p(t)}^{p^-} \le \rho(u) \le |u|_{p(t)}^{p^+}$, $|u|_{p(t)} < 1 \Rightarrow |u|_{p(t)}^{p^+} \le \rho(u) \le |u|_{p(t)}^{p^-}$;

(iii) $|u|_{p(t)} \to 0 \Leftrightarrow \rho(u) \to 0$; $|u|_{p(t)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

(iv) For $u \ne 0$, $|u|_{p(t)} = \lambda \Leftrightarrow \rho(\dfrac{u}{\lambda}) = 1$.

Similar to Lemma 2.3, we have

Lemma 2.4. If we denote $I(u) = \int_0^T (|u(t)|^{p(t)} + |\dot{u}(t)|^{p(t)})dt$, $\forall\ u \in W^{1,p(t)}$, then

(i) $||u|| < 1$ $(= 1; > 1) \Leftrightarrow I(u) < 1$ $(= 1; > 1)$;

(ii) $\|u\| > 1 \Rightarrow \|u\|^{p^-} \le I(u) \le \|u\|^{p^+}$, $\|u\| < 1 \Rightarrow \|u\|^{p^+} \le I(u) \le \|u\|^{p^-}$;

(iii) $||u|| \to 0 \Leftrightarrow I(u) \to 0$; $||u|| \to \infty \Leftrightarrow I(u) \to \infty$.

(iv) For $u \ne 0$, $\|u\| = \lambda \Leftrightarrow I(\dfrac{u}{\lambda}) = 1$.

Defnition 2.2 [17].

$$C_T^\infty = C_T^\infty(\mathbb{R}, \mathbb{R}^N) := \{u \in C^\infty(\mathbb{R}, \mathbb{R}^N) : \quad u \text{ is } T\text{ - periodic}\}$$

with the norm $\|u\|_\infty := \max\limits_{t\in[0,T]} |u(t)|$.

For a constant $p \in (1, \infty)$, using another conception of weak derivative which is called T-weak derivative, Mawhin and Willem gave the definition of the space $W_T^{1,p}$ by the following way.

Definition 2.3 [17]. Let $u \in L^1([0, T], \mathbb{R}^N)$ and $v \in L^1([0, T], \mathbb{R}^N)$, if

$$\int_0^T v\phi dt = -\int_0^T u\phi' dt \quad \forall \phi \in C_T^\infty,$$

then v is called a T-weak derivative of u and is denoted by $\dot{u}$.

Definition 2.4 [17]. Define

$$W_T^{1,p}([0,T],\mathbb{R}^N) = \{u \in L^p([0,T],\mathbb{R}^N) : \dot{u} \in L^p([0,T],\mathbb{R}^N)\}$$

with the norm $\|u\|_{W_T^{1,p}} = (|u|_p^p + |\dot{u}|_p^p)^{1/p}$.

Definition 2.5 [13]. Define

$$W_T^{1,p(t)}([0,T],\mathbb{R}^N) = \{u \in L^{p(t)}([0,T],\mathbb{R}^N) : \dot{u} \in L^{p(t)}([0,T],\mathbb{R}^N)\}$$

and $H_T^{1,p(t)}([0,T],\mathbb{R}^N)$ to be the closure of C_T^∞ in $W^{1,p(t)}$ ([0, T], $\mathbb{R}^N$).

Remark 2.2. From Definition 2.4, if $u \in W_T^{1,p(t)}([0,T],\mathbb{R}^N)$, it is easy to conclude that $u \in W_T^{1,p^-}([0,T],\mathbb{R}^N)$.

Lemma 2.5 [13].

(i) $C_T^\infty([0,T],\mathbb{R}^N)$ is dense in $W_T^{1,p(t)}([0,T],\mathbb{R}^N)$;

(ii) $W_T^{1,p(t)}([0,T],\mathbb{R}^N) = H_T^{1,p(t)}([0,T],\mathbb{R}^N) := \{u \in W^{1,p(t)}([0,T],\mathbb{R}^N) : u(0) = u(T)\}$;

(iii) If $u \in H_T^{1,1}$, then the derivative u' is also the T-weak derivative $\dot{u}$, i.e., $u' = \dot{u}$.

Lemma 2.6 [17]. Assume that $u \in W_T^{1,1}$, then

(i) $\int_0^T \dot{u}dt = 0$,

(ii) u has its continuous representation, which is still denoted by $u(t) = \int_0^t \dot{u}(s)ds + u(0)$, $u(0) = u(T)$,

(iii) $\dot{u}$ is the classical derivative of u, if $\dot{u} \in C([0,T],\mathbb{R}^N)$.

Since every closed linear subspace of a reflexive Banach space is also reflexive, we have

Lemma 2.7 [13]. $H_T^{1,p(t)}([0,T],\mathbb{R}^N)$ is a reflexive Banach space if $p^- > 1$.

Obviously, there are continuous embeddings $L^{p(t)} \to L^{p^-}, W^{1,p(t)} \to W^{1,p^-}$ and $H_T^{1,p(t)} \to H_T^{1,p^-}$. By the classical Sobolev embedding theorem, we obtain

Lemma 2.8 [13]. There is a continuous embedding

$$W^{1,p(t)}(\text{or } H_T^{1,p(t)}) \to C([0,T],\mathbb{R}^N),$$

when $p^- > 1$, the embedding is compact.

Lemma 2.9 [13]. Each of the following two norms is equivalent to the norm in $W_T^{1,p(t)}$:

(i) $|\dot{u}|_{p(t)} + |u|_q$, $1 \le q \le \infty$;

(ii) $|\dot{u}|_{p(t)} + |\bar{u}|$, where $\bar{u} = (1/T)\int_0^T u(t)dt$.

Lemma 2.10 [13]. If u, $u_n \in L^{p(t)}$ (n = 1,2,...), then the following statements are equivalent to each other

(i) $\lim\limits_{n\to\infty} |u_n - u|_{p(t)} = 0$;

(ii) $\lim\limits_{n\to\infty} \rho(u_n - u) = 0$;

(iii) $u_n \to u$ in measure in [0, T] and $\lim\limits_{n\to\infty} \rho(u_n) = \rho(u)$.

Lemma 2.11 [14]. The functional J defined by

$$J(u) = \int_0^T \frac{1}{p(t)} |\dot{u}(t)|^{p(t)} dt$$

is continuously differentiable on $W_T^{1,p(t)}$ and J' is given by

$$\langle J'(u), v\rangle = \int_0^T (|\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{v}(t))dt, \tag{2.1}$$

and J' is a mapping of (S_+), i.e., if $u_n \rightharpoonup u$ weakly in $W_T^{1,p(t)}$ and

$$\limsup_{n\to\infty} (J'(u_n) - J'(u), u_n - u) \le 0,$$

then u_n has a convergent subsequence on $W_T^{1,p(t)}$.

Lemma 2.12 [18]. Since $W_T^{1,p(t)}$ is a separable and reflexive Banach space, there exist $\{e_n\}_{n=1}^{\infty} \subset W_T^{1,p(t)}$ and $\{f_n\}_{n=1}^{\infty} \subset (W_T^{1,p(t)})^*$ such that

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1, & n = m, \\ 0, & n \ne m, \end{cases}$$

$W_T^{1,p(t)} = \overline{\mathrm{span}\{e_n : n = 1, 2, \ldots\}}$ and $(W_T^{1,p(t)})^* = \overline{\mathrm{span}\{f_n : n = 1, 2, \ldots\}}^{W^*}$.

For $k = 1, 2, \ldots$, denote

$$X_k = \mathrm{span}\{e_k\}, \quad Y_k = \oplus_{j=1}^k X_j, \quad Z_k = \overline{\oplus_{j=k}^{\infty} X_j}. \tag{2.2}$$

Lemma 2.13 [19]. Let X be a reflexive infinite Banach space, $\varphi \in C^1(X, \mathbb{R})$ is an even functional with the (C) condition and $\varphi(0) = 0$. If $X = Y \oplus V$ with $\dim Y < \infty$, and φ satisfies

(i) there are constants $\sigma, \alpha > 0$ such that $\varphi|_{\partial B_\sigma \cap V} \ge \alpha$,

(ii) for any finite-dimensional subspace W of X, there exists positive constants $R_2(W)$ such that $\varphi(u) \le 0$ for $u \in W \backslash B_r(0)$, where $B_r(0)$ is an open ball in W of radius r centered at 0. Then φ possesses an unbounded sequence of critical values.

Lemma 2.14 [15]. Suppose

(A1) $\varphi \in C^1(X, \mathbb{R})$ is an even functional, then the subspace X_k, Y_k, and Z_k are defined by (2.2);

If for every $k \in \mathbb{N}$, there exists $\rho_k > r_k > 0$ such that

(A2) $a_k := \max_{u\in Y_k,\ \|u\|=\rho_k} \varphi(u) \le 0$, where $Y_k := \oplus_{j=0}^k X_j$;

(A3) $b_k := \inf_{u\in Z_k,\ \|u\|=r_k} \varphi(u) \to \infty$, as $k \to \infty$, where $Z_k := \overline{\oplus_{j=k}^{\infty} X_j}$;

(A4) φ satisfies the $(\mathrm{PS})_c$ condition for every $c > 0$.

Then φ has an unbounded sequence of critical values.

Lemma 2.15 [16]. Assume (A1) is satisfied, and there is a $k_0 > 0$ so as to for each $k \ge k_0$, there exist $\rho_k > r_k > 0$ such that

(A5) $d_k := \inf_{u\in Z_k,\ \|u\|\le\rho_k} \varphi(u) \to 0$, as $k \to \infty$;

(A6) $i_k := \max_{u\in Y_k,\ \|u\|=r_k} \varphi(u) < 0$;

(A7) $\inf_{u\in Z_k,\ \|u\|=\rho_k} \varphi(u) \ge 0$;

(A8) φ satisfies the $(\mathrm{PS})_c^*$ condition for every $c \in [d_{k0}, 0)$.

Then φ has a sequence of negative critical values converging to 0.

Remark 2.3. φ satisfies the $(\mathrm{PS})_c^*$ condition means that if any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to \infty$, $u_{n_j} \in Y_{n_j}, \varphi(u_{n_j}) \to c$ and $(\varphi|_{Y_{n_j}})'(u_{n_j}) \to 0$, then $\{u_{n_j}\}$ contains a subsequence converging to critical point of φ. It is obvious that if φ satisfies the $(\mathrm{PS})_c^*$ condition, then φ satisfies the $(\mathrm{PS})_c$ condition.

3. Main results and proofs of the theorems

Theorem 3.1. Let $F(t, x)$ satisfies the condition (A'), and suppose the following conditions hold:

(B1) there exist $\beta > p^+$ and $r > 0$ such that

$$\beta F(t,x) \leq (\nabla F(t,x), x)$$

for a.e. $t \in [0, T]$ and all $|x| \geq r$ in $\mathbb{R}^N$;

(B2) there exist positive constants $\mu > p^+$ and $Q > 0$ such that

$$\limsup_{|x|\to+\infty} \frac{F(t,x)}{|x|^{\mu}} \leq Q$$

uniformly for a.e. $t \in [0, T]$;

(B3) there exists $\mu' > p^+$ and $Q' > 0$ such that

$$\liminf_{|x|\to+\infty} \frac{F(t,x)}{|x|^{\mu'}} \geq Q'$$

uniformly for a.e. $t \in [0, T]$;

(B4) $F(t, x) = F(t, -x)$ for $t \in [0, T]$ and all x in $\mathbb{R}^N$.

Then system (1.1) has infinite solutions u_k in $W_T^{1,p(t)}$ for every positive integer k such that $||u_k||_\infty \to +\infty$, as $k \to \infty$.

Remark 3.1. Suppose that $F(t, \cdot)$ is continuously differentiable in x and $p(t) \equiv p$, then condition (B1) reduces to the well-known Ambrosetti-Rabinowitz condition (see [19]), which was introduced in the context of semi-linear elliptic problems. This condition implies that $F(t, x)$ grows at a superquadratic rate as $|x| \to \infty$. This kind of technical condition often appears as necessary to use variational methods when we solve superlinear differential equations such as elliptic problems, Dirac equations, Hamiltonian systems, wave equations, and Schrödinger equations.

Theorem 3.2. Assume that $F(t, x)$ satisfies (A'), (B1), (B3), and (B4) and the following assumption:

(B5) $\int_0^T F(t,0)dt = 0$, and there exists $r_1 > p^+$ and $M > 0$ such that

$$\limsup_{|x|\to 0} \frac{|F(t,x)|}{|x|^{r_1}} \leq M.$$

Then system (1.1) has infinite solutions u_k in $W_T^{1,p(t)}$ for every positive integer k such that $||u_k||_\infty \to +\infty$, as $k \to \infty$.

Theorem 3.3. Assume that $F(t, x)$ satisfies the following assumption:

(B6) $F(t, x) := a(t)|x|^\gamma$, where $a(t) \in L^\infty(0, T; \mathbb{R}^+)$ and $1 < \gamma < p^-$ is a constant. Then system (1.1) has infinite solutions u_k in $W_T^{1,p(t)}$ for every positive integer k.

The proof of Theorem 3.1 is organized as follows: first, we show the functional φ defined by

$$\varphi(u) = \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt + \int_0^T \frac{1}{p(t)}|u(t)|^{p(t)}dt - \int_0^T F(t,u(t))dt$$

satisfies the (PS) condition, then we verify for φ the conditions in Lemma 2.14 item-by-item, then φ has an unbounded sequence of critical values.

Proof of Theorem 3.1. Let $\{u_n\} \subset W_T^{1,p(t)}$ such that $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$. First, we prove $\{u_n\}$ is a bounded sequence, otherwise, $\{u_n\}$ would be unbounded sequence, passing to a subsequence, still denoted by $\{u_n\}$, such that $||u_n|| \geq 1$ and $||u_n|| \to \infty$. Note that

$$\langle\varphi'(u), v\rangle = \int_0^T (|\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{v}(t))dt + \int_0^T (|u(t)|^{p(t)-2}u(t) - \nabla F(t, u(t)), v(t))dt \tag{3.1}$$

for all $v \in W_T^{1,p(t)}$.

It follows from (3.1) that

$$\begin{aligned}
&\int_0^T \left(\frac{\beta}{p(t)} - 1\right)(|\dot{u}_n(t)|^{p(t)} + |u_n(t)|^{p(t)})dt = \beta\varphi(u_n) - \langle\varphi'(u_n), u_n\rangle + \int_0^T [\beta F(t, u_n(t)) \\
&-(\nabla F(t, u_n(t)), u_n(t))]dt \\
&= \beta\varphi(u_n) - \langle\varphi'(u_n), u_n\rangle + \int_{\Omega_1} [\beta F(t, u_n(t)) - (\nabla F(t, u_n(t)), u_n(t))]dt + \int_{\Omega_2} [\beta F(t, u_n(t)) \\
&-(\nabla F(t, u_n(t)), u_n(t))]dt \\
&\leq \beta\varphi(u_n) - \langle\varphi'(u_n), u_n\rangle + \int_{\Omega_1} [\beta F(t, u_n(t)) - (\nabla F(t, u_n(t)), u_n(t))]dt \\
&\leq \beta\varphi(u_n) - \langle\varphi'(u_n), u_n\rangle + C_0,
\end{aligned} \tag{3.2}$$

where $\Omega_1 := \{t \in [0, T]; |u_n(t)| \leq r\}$, $\Omega_2 := [0, T] \setminus \Omega_1$ and C_0 is a positive constant.

However, from (3.2), we have

$$\beta\varphi(u_n) + C_0 \geq \left(\frac{\beta}{p^+} - 1\right)\|u_n\|^{p^-} - \left\|\varphi'(u_n)\right\| \|u_n\|,$$

Thus $||u_n||$ is a bounded sequence in $W_T^{1,p(t)}$.

By Lemma 2.8, the sequence $\{u_n\}$ has a subsequence, also denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \text{ weakly in } W_T^{1,p(t)} \quad \text{and} \quad u_n \to u \text{ strongly in } C([0, T]; \mathbb{R}^N) \tag{3.3}$$

and $||u||_\infty \leq C_1||u||$ by Lemma 2.8, where C_1 is a positive constant.

Therefore, we have

$$\langle\varphi'(u_n) - \varphi'(u), u_n - u\rangle \to 0 \text{ as } n \to \infty, \tag{3.4}$$

i.e.,

$$\begin{aligned}
&\langle\varphi'(u_n) - \varphi'(u), u_n - u\rangle = \int_0^T (\nabla F(t, u_n(t)) - \nabla F(t, u(t)), u_n(t) - u(t))dt + \int_0^T (|u_n(t)|^{p(t)-2} \\
&u_n(t) - |u(t)|^{p(t)-2}u(t), u_n(t) - u(t))dt + \int_0^T (|\dot{u}_n(t)|^{p(t)-2}\dot{u}_n(t) - |\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{u}_n(t) - \dot{u}(t))dt.
\end{aligned} \tag{3.5}$$

By (3.4) and (3.5), we get $\langle J'(u) - J'(u_n), u - u_n\rangle \to 0$, i.e.,

$$\int_0^T (|\dot{u}_n(t)|^{p(t)-2}\dot{u}_n(t) - |\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{u}_n(t) - \dot{u}(t))dt \to 0,$$

so it follows Lemma 2.11 that $\{u_n\}$ admits a convergent subsequence.

For any $u \in Y_k$, let

$$\|u\|_* := \left(\int_0^T |u(t)|^{\mu'}dt\right)^{1/\mu'}, \tag{3.6}$$

and it is easy to verify that $||\cdot||_*$ defined by (3.6) is a norm of Y_k. Since all the norms of a finite dimensional normed space are equivalent, so there exists positive constant C_2 such that

$$C_2\,\|u\| \leq \|u\|_* \quad \text{for} \quad u \in Y_k. \tag{3.7}$$

In view of (B3), there exist two positive constants M_1 and C_3 such that

$$F(t,x) \geq M_1|x|^{\mu'} \tag{3.8}$$

for a.e. $t \in [0, T]$ and $|x| \geq C_3$.

It follows (3.7) and (3.8) that

$$\begin{aligned}
\varphi(u) &= \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt + \int_0^T \frac{1}{p(t)}|u(t)|^{p(t)}dt - \int_0^T F(t,u(t))dt \\
&\leq \frac{1}{p^-}(\|u\|^{p^+}+1) - \int_{\Omega_3} F(t,u(t))dt - \int_{\Omega_4} F(t,u(t))dt \\
&\leq \frac{1}{p^-}(\|u\|^{p^+}+1) - M_1\int_{\Omega_3} |u(t)|^{\mu'}dt - \int_{\Omega_4} F(t,u(t))dt \\
&= \frac{1}{p^-}(\|u\|^{p^+}+1) - M_1\int_0^T |u(t)|^{\mu'}dt + M_1\int_{\Omega_4} |u(t)|^{\mu'}dt - \int_{\Omega_4} F(t,u(t))dt \\
&\leq \frac{1}{p^-}(\|u\|^{p^+}+1) - C_2^{\mu'}M_1\|u\|^{\mu'} + C_4,
\end{aligned}$$

where $\Omega_3 := \{t \in [0, T]; |u(t)| \geq C_3\}$, $\Omega_4 := [0, T] \setminus \Omega_3$ and C_4 is a positive constant.

Since $\mu' > p^+$, there exist positive constants d_k such that

$$\varphi(u) \leq 0 \quad \text{for all} \quad u \in Y_k \quad \text{and} \quad \|u\| \geq d_k. \tag{3.9}$$

For any $u \in Z_k$, let

$$\|u\|_\mu := \left(\int_0^T |u(t)|^\mu dt\right)^{1/\mu} \quad \text{and} \quad \beta_k := \sup_{u\in Z_k,\ \|u\|=1} \|u\|_\mu, \tag{3.10}$$

then we conclude $\beta_k \to 0$ as $k \to \infty$.

In fact, it is obvious that $\beta_k \geq \beta_{k+1} > 0$, so $\beta_k \to \beta \geq 0$ as $k \to \infty$. For every $k \in \mathbb{N}$, there exists $u_k \in Z_k$ such that

$$\|u_k\| = 1 \quad \text{and} \quad \|u_k\|_\mu > \beta_k/2. \tag{3.11}$$

As $W_T^{1,p(t)}$ is reflexive, $\{u_k\}$ has a weakly convergent subsequence, still denoted by $\{u_k\}$, such that $u_k \rightharpoonup u$. We claim $u = 0$.

In fact, for any $f_m \in \{f_n: n = 1, 2...,\}$, we have $f_m(u_k) = 0$, when $k > m$, so

$$f_m(u_k) \to 0, \quad \text{as} \quad k \to \infty$$

for any $f_m \in \{f_n: n = 1, 2 ...,\}$, therefore $u = 0$.

By Lemma 2.8, when $u_k \rightharpoonup 0$ in $W_T^{1,p(t)}$, then $u_k \to 0$ strongly in $C([0, T]; \mathbb{R}^N)$. So, we conclude $\beta = 0$ by (3.11).

In view of (B2), there exist two positive constants M_2 and C_{10} such that

$$F(t,x) \leq M_2|x|^\mu \tag{3.12}$$

uniformly for a.e. $t \in [0, T]$ and $|x| \geq C_5$.

When $||u|| \geq 1$, we conclude

$$\begin{aligned}\varphi(u) &= \int_0^T \frac{1}{p(t)}|u(t)|^{p(t)}dt + \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt - \int_0^T F(t,u(t))dt \\ &\geq \frac{1}{p^+}\int_0^T (|u(t)|^{p(t)} + |\dot{u}(t)|^{p(t)})dt - \int_{\Omega_5} F(t,u(t))dt - \int_{\Omega_6} F(t,u(t))dt \\ &\geq \frac{1}{p^+}\|u\|^{p^-} - M_2\int_0^T |u(t)|^{\mu}dt + M_2\int_{\Omega_6} |u(t)|^{\mu}dt - \int_{\Omega_6} F(t,u(t))dt \\ &\geq \frac{1}{p^+}\|u\|^{p^-} - M_2\beta_k^{\mu}\|u\|^{\mu} - C_6,\end{aligned}$$

where $\Omega_5 := \{t \in [0, T]; |u(t)| \geq C_5\}$, $\Omega_6 := [0, T] \setminus \Omega_5$ and C_6 is a positive constant.

Choosing $r_k = 1/\beta_k$, it is obvious that

$$r_k \to \infty \quad \text{as} \quad k \to \infty,$$

then

$$b_k := \inf_{u\in Z_k,\ \|u\|=r_k} \varphi(u) \to \infty \quad \text{as} \quad k \to \infty, \tag{3.13}$$

i.e., the condition (A3) in Lemma 2.14 is satisfied.

In view of (3.9), let $\rho_k := \max\{d_k, r_k + 1\}$, then

$$a_k := \max_{u\in Y_k,\ \|u\|=\rho_k} \varphi(u) \leq 0,$$

and this shows the condition of (A2) in Lemma 2.14 is satisfied.

We have proved the functional φ satisfies all the conditions of Lemma 2.14, then φ has an unbounded sequence of critical values $c_k = \varphi(u_k)$ by Lemma 2.14, we only need to show $||u_k||_\infty \to \infty$ as $k \to \infty$.

In fact, since u_k is a critical point of the functional φ, we have

$$\int_0^T |\dot{u}_k(t)|^{p(t)}dt + \int_0^T |u_k(t)|^{p(t)}dt - \int_0^T (\nabla F(t,u_k(t)), u_k(t))dt = 0.$$

Hence, we have

$$\begin{aligned}c_k = \varphi(u_k) &= \int_0^T \frac{1}{p(t)}|\dot{u}_k(t)|^{p(t)}dt + \int_0^T \frac{1}{p(t)}|u_k(t)|^{p(t)}dt - \int_0^T F(t,u_k(t))dt, \\ &\leq \frac{1}{p^-}\int_0^T |\dot{u}_k(t)|^{p(t)}dt + \frac{1}{p^-}\int_0^T |u_k(t)|^{p(t)}dt - \int_0^T F(t,u_k(t))dt, \\ &= \int_0^T (\nabla F(t,u_k(t)), u_k(t))dt - \int_0^T F(t,u_k(t))dt,\end{aligned} \tag{3.14}$$

since $c_k \to \infty$, we conclude

$$\|u_k\|_\infty \to \infty \quad \text{as} \quad k \to \infty$$

by (3.14). In fact, if not, going to a subsequence if necessary, we may assume that

$$\|u_k\|_\infty \leq C_7$$

for all $k \in \mathbb{N}$ and some positive constant C_7.

Combining (A') and (3.14), we have

$$\begin{aligned} c_k &\leq \int_0^T (\nabla F(t, u_k(t)), u_k(t))dt - \int_0^T F(t, u_k(t))dt, \\ &\leq (C_7 + 1) \max_{0 \leq s \leq C_7} a(s) \int_0^T b(t)dt, \end{aligned}$$

which contradicts $c_k \to \infty$. This completes the proof of Theorem 3.1.

Proof of Theorem 3.2. To prove $\{u_n\}$ has a convergent subsequence in space $W_T^{1,p(t)}$ is the same as that in the proof of Theorem 3.1, thus we omit it. It is obvious that φ is even and $\varphi(0) = 0$ under condition (B5), and so we only need to verify other conditions in Lemma 2.13.

Proposition 3.1. Under the condition (B5), there exist two positive constants σ and α such that $\varphi(u) \geq \alpha$ for all $u \in \tilde{W}_T^{1,p(t)}$ and $||u|| = \sigma$.

Proof. In view of condition (B5), there exist two positive constants ε and δ such that

$$0 < \varepsilon < C_1 \quad \text{and} \quad 0 < \delta < \varepsilon,$$

where C_1 is the same as in (3.3), and

$$|F(t, x)| \leq (M + \varepsilon)|x|^{r_1} \tag{3.15}$$

for a.e. $t \in [0, T]$ and $|x| \leq \delta$.

Let $\sigma := \delta/C_1$ and $||u|| = \sigma$, since $\sigma < 1$, we have

$$\|u\|^{p^+} \leq I(u) \quad \text{and} \quad \|u\|_\infty \leq C_1 \|u\|. \tag{3.16}$$

by Lemmas 2.4 and 2.8.

Combining (3.15) and (3.16), we have

$$\begin{aligned} \varphi(u) &= \int_0^T \frac{1}{p(t)} |u(t)|^{p(t)} dt + \int_0^T \frac{1}{p(t)} |\dot{u}(t)|^{p(t)} dt - \int_0^T F(t, u(t))dt \\ &\geq \frac{1}{p^+} \int_0^T (|u(t)|^{p(t)} + |\dot{u}(t)|^{p(t)})dt - (M + \varepsilon) \int_0^T |u(t)|^{r_1} dt \\ &\geq \frac{1}{p^+} \|u\|^{p^+} - (M + \varepsilon) T C_1^{r_1} \|u\|^{r_1} \\ &= \left[\frac{1}{p^+} - (M + \varepsilon) T C_1^{r_1} \sigma^{r_1 - p^+} \right] \sigma^{p^+}, \end{aligned}$$

so we can choose σ small enough, such that

$$\frac{1}{p^+} - (M + \varepsilon) T C_1^{r_1} \sigma^{r_1 - p^+} \geq \frac{1}{2p^+} \quad \text{and} \quad \alpha := \frac{1}{2p^+} \sigma^{p^+},$$

and this completes the proof of Proposition 3.1.

Proposition 3.2. For any finite dimensional subspace W of $W_T^{1,p(t)}$, there is $r_2 = r_2(W) > 0$ such that $\varphi(u) \leq 0$ for $u \in W \backslash B_{r_2}(0)$, where $B_{r_2}(0)$ is an open ball in W of radius r_2 centered at 0.

Proof. The proof of Proposition 3.2 is the same as the proof of the condition (A2) in the proof of Theorem 3.1.

We have proved the functional φ satisfies all the conditions of Lemma 2.13, φ has an unbounded sequence of critical values $c_k = \varphi(u_k)$ by Lemma 2.13. Arguing as in the proof of Theorem 3.1, system (1.1) has infinite solutions $\{u_k\}$ in $W_T^{1,p(t)}$ for every positive integer k such that $||u_k||_\infty \to +\infty$, as $k \to \infty$. The proof of Theorem 3.2 is complete.

Proof of Theorem 3.3. First, we show that φ satisfies the $(\mathrm{PS})_c^*$ for every $c \in \mathbb{R}$. Suppose $n_j \to \infty$, $u_{n_j} \in Y_{n_j}, \varphi(u_{n_j}) \to c$ and $(\varphi|_{Y_{n_j}})'(u_{n_j}) \to 0$, then $\{u_{n_j}\}$ is a bounded sequence, otherwise, $\{u_{n_j}\}$ would be unbounded sequence, passing to a subsequence, still denoted by $\{u_{n_j}\}$ such that $\|u_{n_j}\| \geq 1$ and $\|u_{n_j}\| \to \infty$. Note that

$$\int_0^T (1 - \frac{\gamma}{p(t)})(|\dot{u}_{n_j}(t)|^{p(t)} + |u_{n_j}(t)|^{p(t)})dt = \langle \varphi'(u_{n_j}), u_{n_j} \rangle - \gamma \varphi(u_{n_j}). \tag{3.17}$$

However, from (3.17), we have

$$-\gamma\varphi(u_{n_j}) \geq (1 - \frac{\gamma}{p^-})\|u_{n_j}\|^{p^-} - \left\|(\varphi|_{Y_{n_j}})'(u_{n_j})\right\| \|u_{n_j}\|,$$

thus $||u_n||$ is a bounded sequence in $W_T^{1,p(t)}$. Going, if necessary, to a subsequence, we can assume that $u_{n_j} \rightharpoonup u$ in $W_T^{1,p(t)}$. As $X = \overline{\bigcup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence

$$\begin{aligned} &\lim_{n_j \to \infty} \langle \varphi'(u_{n_j}), u_{n_j} - u \rangle \\ &= \lim_{n_j \to \infty} \langle \varphi'(u_{n_j}), u_{n_j} - v_{n_j} \rangle + \lim_{n_j \to \infty} \langle \varphi'(u_{n_j}), v_{n_j} - u \rangle \\ &= \lim_{n_j \to \infty} \langle (\varphi|_{Y_{n_j}})'(u_{n_j}), u_{n_j} - v_{n_j} \rangle = 0. \end{aligned}$$

In view of (3.4) and (3.5), we can also conclude $u_{n_j} \to u$, furthermore, we have $\varphi'(u_{n_j}) \to \varphi'(u)$.

Let us prove $\varphi'(u) = 0$ below. Taking arbitrarily $\omega_k \in Y_k$, notice when $n_j \leq k$ we have

$$\begin{aligned} \langle \varphi'(u), \omega_k \rangle &= \langle \varphi'(u) - \varphi'(u_{n_j}), \omega_k \rangle + \langle \varphi'(u_{n_j}), \omega_k \rangle \\ &= \langle \varphi'(u) - \varphi'(u_{n_j}), \omega_k \rangle + \langle (\varphi|_{Y_{n_j}})'(u_{n_j}), \omega_k \rangle. \end{aligned}$$

Going to limit in the right side of above equation reaches

$$\langle \varphi'(u), \omega_k \rangle = 0, \ \forall \omega_k \in Y_k,$$

so $\varphi'(u) = 0$, this shows that φ satisfies the $(\mathrm{PS})_c^*$ for every $c \in \mathbb{R}$.

For any finite dimensional subspace $W \subset W_T^{1,p(t)}$, there exists $\varepsilon_1 > 0$ such that

$$\operatorname{meas}\{t \in [0, T] : a(t)|u(t)|^\gamma \geq \varepsilon_1 \|u\|^\gamma\} \geq \varepsilon_1, \quad \forall u \in W \backslash \{0\}. \tag{3.18}$$

Otherwise, for any positive integer n, there exists $u_n \in W \setminus \{0\}$ such that

$$\operatorname{meas}\{t \in [0, T] : a(t)|u_n(t)|^\gamma \geq \frac{1}{n}\|u_n\|^\gamma\} < \frac{1}{n}.$$

Set $v_n(t) := \dfrac{u_n(t)}{\|u_n\|} \in W\backslash\{0\}$, then $||v_n|| = 1$ for all $n \in \mathbb{N}$ and

$$\operatorname{meas}\{t \in [0, T] : a(t)|v_n(t)|^\gamma \geq \frac{1}{n}\} < \frac{1}{n}. \tag{3.19}$$

Since $\dim W < \infty$, it follows from the compactness of the unit sphere of W that there exists a subsequence, denoted also by $\{v_n\}$, such that $\{v_n\}$ converges to some v_0 in W. It is obvious that $||v_0|| = 1$.

By the equivalence of the norms on the finite dimensional space W, we have $v_n \to v_0$ in $L^{p^-}(0, T; \mathbb{R}^N)$, i.e.,

$$\int_0^T |v_n - v_0|^{p^-} dt \to 0 \quad \text{as} \quad n \to \infty. \tag{3.20}$$

By (3.20) and Hölder inequality, we have

$$\begin{aligned}\int_0^T a(t)|v_n - v_0|^{\gamma} dt &\le (\int_0^T a(t)^{\frac{p^-}{p^- - \gamma}} dt)^{\frac{p^- - \gamma}{p^-}} (\int_0^T |v_n - v_0|^{p^-} dt)^{\frac{\gamma}{p^-}} \\ &= \|a\|_{\frac{p^- - \gamma}{p^-}} (\int_0^T |v_n - v_0|^{p^-} dt)^{\frac{\gamma}{p^-}} \to 0, \text{ as } n \to \infty.\end{aligned} \tag{3.21}$$

Thus, there exist $\xi_1, \xi_2 > 0$ such that

$$\text{meas}\{t \in [0, T] : a(t)|v_0(t)|^{\gamma} \ge \xi_1\} \ge \xi_2. \tag{3.22}$$

In fact, if not, we have

$$\text{meas}\{t \in [0, T] : a(t)|v_0(t)|^{\gamma} \ge \frac{1}{n}\} = 0$$

for all positive integer n.

It implies that

$$0 \le \int_0^T a(t)|v_0|^{\gamma + p^-} dt < \frac{T}{n} \|v_0\|_{\infty}^{p^-} \le \frac{C_6^{p^-} T}{n} \|v_0\|^{p^-} \to 0$$

as $n \to \infty$, where C_6 is the same in (3.3). Hence $v_0 = 0$ which contradicts that $||v_0|| = 1$. Therefore, (3.22) holds. Now let

$$\Omega_0 = \{t \in [0, T] : a(t)|v_0(t)|^{\gamma} \ge \xi_1\}, \quad \Omega_n = \{t \in [0, T] : a(t)|v_n(t)|^{\gamma} < \frac{1}{n}\},$$

and $\Omega_n^c = [0, T] \backslash \Omega_n = \{t \in [0, T] : a(t)|v_n(t)|^{\gamma} \ge \frac{1}{n}\}$.

By (3.19) and (3.22), we have

$$\begin{aligned}\text{meas}(\Omega_n \cap \Omega_0) &= \text{meas}(\Omega_0 \backslash (\Omega_n^c \cap \Omega_0) \\ &\ge \text{meas}(\Omega_0) - \text{meas}(\Omega_n^c \cap \Omega_0) \\ &\ge \xi_2 - \frac{1}{n}\end{aligned}$$

for all positive integer n. Let n be large enough such that

$$\xi_2 - \frac{1}{n} \ge \frac{1}{2}\xi_2 \quad \text{and} \quad \frac{1}{2^{\gamma-1}}\xi_1 - \frac{1}{n} \ge \frac{1}{2^{\gamma}}\xi_1,$$

then we have

$$\begin{aligned}\int_0^T a(t)|v_n - v_0|^{\gamma} dt &\ge \int_{\Omega_n \cap \Omega_0} a(t)|v_n - v_0|^{\gamma} dt \\ &\ge \frac{1}{2^{\gamma-1}} \int_{\Omega_n \cap \Omega_0} a(t)|v_0|^{\gamma} dt - \int_{\Omega_n \cap \Omega_0} a(t)|v_n|^{\gamma} dt \\ &\ge (\frac{1}{2^{\gamma-1}}\xi_1 - \frac{1}{n}) \text{meas}(\Omega_n \cap \Omega_0) \\ &\ge \frac{\xi_1}{2^{\gamma}} . \frac{\xi_2}{2} = \frac{\xi_1 \xi_2}{2^{\gamma+1}} > 0\end{aligned}$$

for all large n, which is a contradiction to (3.21). Therefore, (3.18) holds.

For any $u \in Z_k$, let

$$\|u\|_{p^-} := \left(\int_0^T |u(t)|^{p^-} dt\right)^{1/p^-} \quad \text{and} \quad \gamma_k := \sup_{u \in Z_k,\ \|u\|=1} \|u\|_{p^-},$$

then we conclude $\gamma_k \to 0$ as $k \to \infty$ as in the proof of Theorem 3.1.

$$\begin{aligned}
\varphi(u) &= \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt + \int_0^T \frac{1}{p(t)}|u(t)|^{p(t)}dt - \int_0^T F(t,u(t))dt \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - \int_0^T a(t)|u(t)|^{\gamma} dt \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - \left(\int_0^T a(t)^{\frac{p^-}{p^- - \gamma}} dt\right)^{\frac{p^- - \gamma}{p^-}} \|u\|_{p^-}^{\gamma} \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - \left(\int_0^T a(t)^{\frac{p^-}{p^- - \gamma}} dt\right)^{\frac{p^- - \gamma}{p^-}} \gamma_k^{\gamma} \|u\|^{\gamma}.
\end{aligned} \tag{3.23}$$

Let $\rho_k := (2cp^+\gamma_k^{\gamma})^{\frac{1}{p^+ - \gamma}}$, where $c := \left(\int_0^T a(t)^{\frac{p^-}{p^- - \gamma}} dt\right)^{\frac{p^- - \gamma}{p^-}}$, it is obvious that $\rho_k \to 0$, as $k \to \infty$. In view of (3.23), We conclude

$$\inf_{u \in Z_k,\ \|u\|=\rho_k} \varphi(u) \geq \frac{1}{2p^+}\rho_k^{p^+} > 0,$$

so the condition (A7) in Lemma 2.15 is satisfied.

Furthermore, by (3.23), for any $u \in Z_k$ with $||u|| \leq \rho_k$, we have

$$\varphi(u) \geq -c\gamma_k^{\gamma}\|u\|^{\gamma}.$$

Therefore,

$$-c\gamma_k^{\gamma}\rho_k^{\gamma} \leq \inf_{u \in Z_k,\ \|u\| \leq \rho_k} \varphi(u) \leq 0,$$

so we have

$$\inf_{u \in Z_k,\ \|u\| \leq \rho_k} \varphi(u) \to 0$$

for $\rho_k, \gamma_k \to 0$, as $k \to \infty$.

For any $u \in Y_k \setminus \{0\}$ with $||u|| \leq 1$,

$$\begin{aligned}
\varphi(u) &= \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt + \int_0^T \frac{1}{p(t)}|u(t)|^{p(t)}dt - \int_0^T a(t)|u(t)|^{\gamma} dt \\
&\leq \frac{1}{p^-}\|u\|^{p^-} - \int_0^T a(t)|u(t)|^{\gamma} dt \\
&\leq \frac{1}{p^-}\|u\|^{p^-} - \varepsilon_1\|u\|^{\gamma}\mathrm{meas}(\Omega_u) \\
&\leq \frac{1}{p^-}\|u\|^{p^-} - \varepsilon_1^2\|u\|^{\gamma},
\end{aligned}$$

where ε_1 is given in (3.18), and

$$\Omega_u := \mathrm{meas}\{t \in [0,T] : a(t)|u(t)|^{\gamma} \geq \varepsilon_1\|u\|^{\gamma}\} \geq \varepsilon_1, \quad \forall u \in Y_k \setminus \{0\}.$$

Choosing $0 < r_k < \min\{\rho_k, (\frac{p^- \varepsilon^2}{2})^{\frac{1}{p^- - \gamma}}\}$, we conclude

$$i_k := \max_{u \in Y_k,\ \|u\| = r_k} \varphi(u) < -\frac{1}{p^-} r_k^{p^-} < 0 \quad \forall\, k \in \mathbb{N},$$

i.e., the condition (A6) in Lemma 2.15 is satisfied. The proof of Theorem 3.3 is complete.

4. Example

In this section, we give three examples to illustrate our results.

Example 4.1. In system (1.1), let $F(t, x) = |x|^{8+\frac{T}{2}}$ and

$$p(t) = \begin{cases} 7 + t, & 0 \le t \le T/2, \\ -t + T + 7, & T/2 < t \le T. \end{cases}$$

Choose

$$\beta = 8 + \frac{T}{2}, \quad r = 2, \quad \mu = \mu' = 8 + \frac{T}{2} \quad \text{and} \quad Q = Q' = 1,$$

so it is easy to verify that all the conditions (B1)-(B4) are satisfied. Then by Theorem 3.1, system (1.1) has infinite solutions $\{u_k\}$ in $W_T^{1,p(t)}$ for every positive integer k such that $||u_k||_\infty \to +\infty$, as $k \to \infty$.

Example 4.2. In system (1.1), let $F(t, x) = |x|^8$ and

$$p(t) = \begin{cases} 5, & 0 \le t \le T/2 \\ 5 + \sin\frac{2\pi t}{T}, & T/2 < t \le T. \end{cases}$$

We choose $\beta = \frac{13}{2}$, $r = 2$, $\mu' = 8$, $r_1 = 7$, $Q' = 1$ and $M = 1$, so it is easy to verify that all the conditions of Theorem 3.2 are satisfied. Then by Theorem 3.2, so system (1.1) has infinite solutions $\{u_k\}$ in $W_T^{1,p(t)}$ for every positive integer k such that $||u_k||_\infty \to +\infty$, as $k \to \infty$.

Example 4.3. In system (1.1), let $F(t, x) = a(t)|x|^3$ where

$$a(t) = \begin{cases} T, \ t = 0 \\ t, \ 0 < t \le T, \end{cases}$$

and

$$p(t) = \begin{cases} 5, & 0 \le t \le T/2 \\ 5 + \sin\frac{2\pi t}{T}, & T/2 < t \le T. \end{cases}$$

It is easy to verify that all the conditions of Theorem 3.3 are satisfied. Then by Theorem 3.3, so system (1.1) has infinite solutions $\{u_k\}$ in $W_T^{1,p(t)}$ for every positive integer k.

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Authors' contributions

All the authors typed, read, and approved the final manuscript.

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The authors declare that they have no competing interests.

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