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Multiple positive solutions for a class of quasi-linear elliptic equations involving concave-convex nonlinearities and Hardy terms

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Abstract

In this paper, we are concerned with the following quasilinear elliptic equation

$$-\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p*-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth domain with smooth boundary $\partial\Omega$ such that $0 \in \Omega$, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $\mu < \bar{\mu} = (\frac{N-p}{p})^p$, $\lambda > 0$, $1 < q < p$, sign-changing weight functions f and g are continuous functions on $\bar{\Omega}$, $\bar{\mu} = (\frac{N-p}{p})^p$ is the best Hardy constant and $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. By extracting the Palais-Smale sequence in the Nehari manifold, the multiplicity of positive solutions to this equation is verified.

Keywords: Multiple positive solutions, critical Sobolev exponent, concave-convex, Hardy terms, sign-changing weights

1 Introduction and main results

Let Ω be a smooth domain (not necessarily bounded) in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$ such that $0 \in \Omega$. We will study the multiplicity of positive solutions for the following quasilinear elliptic equation

$$\begin{cases} -\Delta_p u - \mu \dfrac{|u|^{p-2}u}{|x|^p} = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p*-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $\mu < \bar{\mu} = (\frac{N-p}{p})^p$, $\bar{\mu}$ is the best Hardy constant, $\lambda > 0$, $1 < q < p$, $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent and the weight functions $f, g : \bar{\Omega} \to \mathbb{R}$ are continuous, which change sign on Ω.

Let $\mathcal{D}_0^{1,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm $(\int_\Omega |\nabla u|^p dx)^{1/p}$. The energy functional of (1.1) is defined on $\mathcal{D}_0^{1,p}(\Omega)$ by

$$J_\lambda(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right) dx - \frac{\lambda}{q}\int_\Omega f|u|^q dx - \frac{1}{p^*}\int_\Omega g|u|^{p^*} dx.$$

Then $J_\lambda \in C^1(\mathcal{D}_0^{1,p}(\Omega), \mathbb{R})$. $u \in \mathcal{D}_0^{1,p}(\Omega)\backslash\{0\}$ is said to be a solution of (1.1) if $\langle J'_\lambda(u), v\rangle = 0$ for all $v \in \mathcal{D}_0^{1,p}(\Omega)$ and a solution of (1.1) is a critical point of J_λ.

Problem (1.1) is related to the well-known Hardy inequality [1,2]:

$$\int_\Omega \frac{|u|^p}{|x|^p}\mathrm{d}x \le \frac{1}{\bar{\mu}}\int_\Omega |\nabla u|^p \mathrm{d}x, \quad \forall u \in C_0^\infty(\Omega).$$

By the Hardy inequality, $\mathcal{D}_0^{1,p}(\Omega)$ has the equivalent norm $||u||_\mu$, where

$$||u||_\mu^p = \int_\Omega \left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)\mathrm{d}x, \quad \mu \in (-\infty, \bar{\mu}).$$

Therefore, for $1 < p < N$, and $\mu < \bar{\mu}$, we can define the best Sobolev constant:

$$S_\mu(\Omega) = \inf_{u\in\mathcal{D}_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx}{\left(\int_\Omega |u|^{p*}dx\right)^{\frac{p}{p*}}}. \tag{1.2}$$

It is well known that $S_\mu(\Omega) = S_\mu(\mathbb{R}^N) = S_\mu$. Note that $S_\mu = S_0$ when $\mu \le 0$ [3].

Such kind of problem with critical exponents and nonnegative weight functions has been extensively studied by many authors. We refer, e.g., in bounded domains and for $p = 2$ to [4-6] and for $p > 1$ to [7-11], while in $\mathbb{R}^N$ and for $p = 2$ to [12,13], and for $p > 1$ to [3,14-17], and the references therein.

In the present paper, our research is mainly related to (1.1) with $1 < q < p < N$, the critical exponent and weight functions f, g that change sign on Ω. When $p = 2$, $1 < q < 2$, $\mu \in [0, \bar{\mu})$, f, g are sign changing and Ω is bounded, [18] studied (1.1) and obtained that there exists $\Lambda > 0$ such that (1.1) has at least two positive solutions for all $\lambda \in (0, \Lambda)$. For the case $p \neq 2$, [19] studied (1.1) and obtained the multiplicity of positive solutions when $1 < q < p < N$, $\mu = 0$, f, g are sign changing and Ω is bounded. However, little has been done for this type of problem (1.1). Recently, Wang et al. [11] have studied (1.1) in a bounded domain Ω under the assumptions $1 < q < p < N$, $N > p^2$, $-\infty < \mu < \bar{\mu}$ and f, g are nonnegative. They also proved that there existence of $\Lambda_0 > 0$ such that for $\lambda \in (0, \Lambda_0)$, (1.1) possesses at least two positive solutions. In this paper, we study (1.1) and extend the results of [11,18,19] to the more general case $1 < q < p < N$, $-\infty < \mu < \bar{\mu}$, f, g are sign changing and Ω is a smooth domain (not necessarily bounded) in $\mathbb{R}^N$ ($N \ge 3$). By extracting the Palais-Smale sequence in the Nehari manifold, the existence of at least two positive solutions of (1.1) is verified.

The following assumptions are used in this paper:

$(\mathcal{H})$ $\mu < \bar{\mu}$, $\lambda > 0$, $1 < q < p < N$, $N \ge 3$.

(f_1) $f \in C(\bar{\Omega}) \cap L^{q*}(\Omega)$ $\left(q^* = \frac{p^*}{p^*-q}\right)$ $f^+ = \max\{f, 0\} \not\equiv 0$ in Ω.

(f_2) There exist β_0 and $\rho_0 > 0$ such that $B(x_0; 2\rho_0) \subset \Omega$ and $f(x) \ge \beta_0$ for all $x \in B(x_0; 2\rho_0)$

(g_1) $g \in C(\bar{\Omega}) \cap L^\infty(\Omega)$ and $g^+ = \max\{g, 0\} \not\equiv 0$ in Ω.

(g_2) There exist $x_0 \in \Omega$ and $\beta > 0$ such that

$$|g|_\infty = g(x_0) = \max_{x\in\bar{\Omega}} g(x), \quad g(x) > 0, \forall x \in \Omega,$$
$$g(x) = g(x_0) + o(|x - x_0|^\beta) \quad \text{as } x \to 0$$

where $|\cdot|_\infty$ denotes the $L^\infty(\Omega)$ norm.

Set

$$\Lambda_1 = \Lambda_1(\mu) = \left(\frac{p-q}{(p*-q)|g^+|_\infty}\right)^{\frac{p-q}{p*-p}} \left(\frac{p*-p}{(p*-q)|f^+|_{q*}}\right) S_\mu^{\frac{N}{p^2}(p-q)+\frac{q}{p}}. \tag{1.3}$$

The main results of this paper are concluded in the following theorems. When Ω is an unbounded domain, the conclusions are new to the best of our knowledge.

Theorem 1.1 *Suppose* $(\mathcal{H})$, (f_1) *and* (g_1) *hold. Then,* (1.1) *has at least one positive solution for all* $\lambda \in (0, \Lambda_1)$.

Theorem 1.2 *Suppose* $(\mathcal{H})$, (f_1) - (g_2) *hold, and* γ *is the constant defined as in Lemma 2.2. If* $0 \le \mu < \bar{\mu}$, $x_0 = 0$ *and* $\beta \ge p\gamma$, *then* (1.1) *has at least two positive solutions for all* $\lambda \in (0, \frac{q}{p}\Lambda_1)$.

Theorem 1.3 *Suppose* $(\mathcal{H})$, (f_1) - (g_2) *hold. If* $\mu < 0$, $x_0 \neq 0$, $\beta \ge \frac{N-p}{p-1}$ *and* $N \le p^2$, *then* (1.1) *has at least two positive solutions for all* $\lambda \in (0, \frac{q}{p}\Lambda_1(0))$.

Remark 1.4 *As* Ω *is a bounded smooth domain and* $p = 2$, *the results of Theorems 1.1, 1.2 are improvements of the main results of* [18].

Remark 1.5 *As* Ω *is a bounded smooth domain and* $p \neq 2$, $\mu = 0$, *then the results of Theorems 1.1, 1.2 in this case are the same as the known results in* [19].

Remark 1.6 *In this remark, we consider that* Ω *is a bounded domain. In* [11], *Wang et al. considered* (1.1) *with* $\mu < \bar{\mu}$, $\lambda > 0$ *and* $1 < q < p < p^2 < N$. *As* $0 \le \mu < \bar{\mu}$ *and* 1 w$<q<p<N$, *the results of Theorems 1.1, 1.2 are improvements of the main results of* [11]. *As* $\mu < 0$ *and* $1 < q < p < N \le p^2$, *Theorem 1.3 is the complement to the results in* [[11], Theorem 1.3].

This paper is organized as follows. Some preliminaries and properties of the Nehari manifold are established in Sections 2 and 3, and Theorems 1.1-1.3 are proved in Sections 4-6, respectively. Before ending this section, we explain some notations employed in this paper. In the following argument, we always employ C and C_i to denote various positive constants and omit dx in integral for convenience. $B(x_0; R)$ is the ball centered at $x_0 \in \mathbb{R}^N$ with the radius $R > 0$, $(\mathcal{D}_0^{1,p}(\Omega))^{-1}$ denotes the dual space of $\mathcal{D}_0^{1,p}(\Omega)$, the norm in $L^p(\Omega)$ is denoted by $|\cdot|_p$, the quantity $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)/\varepsilon^t| \le C$, $o(\varepsilon^t)$ means $|o(\varepsilon^t)/\varepsilon^t| \to 0$ as $\varepsilon \to 0$ and $o(1)$ is a generic infinitesimal value. In particular, the quantity $O_1(\varepsilon^t)$ means that there exist $C_1, C_2 > 0$ such that $C_1\varepsilon^t \le O_1(\varepsilon^t) \le C_2\varepsilon^t$ as ε is small enough.

2 Preliminaries

Throughout this paper, (f_1) and (g_1) will be assumed. In this section, we will establish several preliminary lemmas. To this end, we first recall a result on the extremal functions of $S_{\mu,s}$.

Lemma 2.1 [16]*Assume that* $1 < p < N$ *and* $0 \le \mu < \bar{\mu}$. *Then, the limiting problem*

$$\begin{cases} -\Delta_p u - \mu \dfrac{u^{p-1}}{|x|^p} = u^{p*-1}, \ in\ \mathbb{R}^N \backslash \{0\}, \\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \quad u > 0, \ \ in\ \mathbb{R}^N \backslash \{0\}, \end{cases} \tag{2.1}$$

has positive radial ground states

$$V_{p,\mu,\varepsilon}(x) = \varepsilon^{-\frac{N-p}{p}} U_{p,\mu}\left(\frac{x}{\varepsilon}\right) = \varepsilon^{-\frac{N-p}{p}} U_{p,\mu}\left(\frac{|x|}{\varepsilon}\right), \quad \text{for all } \varepsilon > 0,$$

that satisfy

$$\int_{\mathbb{R}^N} \left(|\nabla V_{p,\mu,\varepsilon}(x)|^p - \mu \frac{|V_{p,\mu,\varepsilon}(x)|^p}{|x|^p} \right) = \int_{\mathbb{R}^N} |V_{p,\mu,\varepsilon}(x)|^{p*} = S_\mu^{\frac{N}{p}}.$$

Furthermore, $U_{p,\mu}(|x|) = U_{p,\mu}(r)$ *is decreasing and has the following properties*:

$$U_{p,\mu}(1) = \left(\frac{N(\bar{\mu}-\mu)}{N-p}\right)^{\frac{1}{p*-p}},$$

$$\lim_{r\to 0^+} r^{a(\mu)} U_{p,\mu}(r) = c_1 > 0, \quad \lim_{r\to 0^+} r^{a(\mu)+1} |U'_{p,\mu}(r)| = c_1 a(\mu) \geq 0,$$

$$\lim_{r\to +\infty} r^{b(\mu)} U_{p,\mu}(r) = c_2 > 0, \quad \lim_{r\to +\infty} r^{b(\mu)+1} |U'_{p,\mu}(r)| = c_2 b(\mu) > 0,$$

$$c_3 \leq U_{p,\mu}(r)(r^{\frac{a(\mu)}{\delta}} + r^{\frac{b(\mu)}{\delta}})^{\delta} \leq c_4, \quad \delta := \frac{N-p}{p},$$

where c_i $(i = 1, 2, 3, 4)$ *are positive constants depending on N,* μ *and p, and* $a(\mu)$ *and* $b(\mu)$ *are the zeros of the function* $h(t) = (p-1)t^p - (N-p)t^{p-1} + \mu$, $t \geq 0$, *satisfying* $0 \leq a(\mu) < \frac{N-p}{p} < b(\mu) \leq \frac{N-p}{p-1}$.

Take $\rho > 0$ small enough such that $B(0; \rho) \subset \Omega$, and define the function

$$u_\varepsilon(x) = \eta(x) V_{p,\mu,\varepsilon}(x) = \varepsilon^{-\frac{N-p}{p}} \eta(x) U_{p,\mu}\left(\frac{|x|}{\varepsilon}\right), \tag{2.2}$$

where $\eta \in C_0^\infty(B(0;\rho)$ is a cutoff function such that $\eta(x) \equiv 1$ in $B(0, \frac{\rho}{2})$.

Lemma 2.2 [9,20]*Suppose* $1 < p < N$ *and* $0 \leq \mu < \bar{\mu}$. *Then, the following estimates hold when* $\varepsilon \to 0$.

$$||u_\varepsilon||_\mu^p = S_\mu^{\frac{N}{p}} + O(\varepsilon^{p\gamma}),$$

$$\int_\Omega |u_\varepsilon|^{p*} = S_\mu^{\frac{N}{p}} + O(\varepsilon^{p*\gamma}),$$

$$\int_\Omega |u_\varepsilon|^q = \begin{cases} O_1(\varepsilon^\theta), & \frac{N}{b(\mu)} < q < p*, \\ O_1(\varepsilon^\theta|)ln\varepsilon|, & q = \frac{N}{b(\mu)}, \\ O_1(\varepsilon^{q\gamma}), & 1 \leq q < \frac{N}{b(\mu)}, \end{cases}$$

where $\delta = \frac{N-p}{p}$, $\theta = N - \frac{N-p}{p} q$*and* $\gamma = b(\mu) - \delta$.

We also recall the following known result by Ben-Naoum, Troestler and Willem, which will be employed for the energy functional.

Lemma 2.3 [21]*Let* Ω *be an domain, not necessarily bounded, in* $\mathbb{R}^N$, $1 \leq p < N$, $k(x) \in L^{\frac{p*}{p*-q}}(\Omega)$*and* $k(x) \in L^{\frac{p*}{p*-q}}(\Omega)$*Then, the functional*

$$\mathcal{D}_0^{1,p}(\Omega) \to \mathbb{R} : u \mapsto \int_{\mathbb{R}^N} k(x)|u|^q \mathrm{d}x$$

is well-defined and weakly continuous.

3 Nehari manifold

As J_λ is not bounded below on $\mathcal{D}_0^{1,p}(\Omega)$, we need to study J_λ on the Nehari manifold

$$\mathcal{N}_\lambda = \{u \in \mathcal{D}_0^{1,p}(\Omega)\backslash\{0\} : \langle J'_\lambda(u), u\rangle = 0\}.$$

Note that $\mathcal{N}_\lambda$ contains all solutions of (1.1) and $u \in \mathcal{N}_\lambda$ if and only if

$$||u||_\mu^p - \lambda \int_\Omega f|u|^q - \int_\Omega g|u|^{p*} = 0. \tag{3.1}$$

Lemma 3.1 *J_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof Suppose $u \in \mathcal{N}_\lambda$. From ($f_1$), (3.1), the Hölder inequality and Sobolev embedding theorem, we can deduce that

$$\begin{aligned} J_\lambda(u) &= \frac{p*-p}{pp*}||u||_\mu^p - \lambda\frac{p*-q}{p*q}\int_\Omega f|u|^q \\ &\geq \frac{1}{N}||u||_\mu^p - \lambda\frac{p*-q}{p*q}|f^+|_{q*}|u|_{p*}^q \\ &\geq \frac{1}{N}||u||_\mu^p - \lambda\frac{p*-q}{p*q}|f^+|_{q*}S_\mu^{-\frac{q}{p}}||u||_\mu^q. \end{aligned} \tag{3.2}$$

Thus, J_λ is coercive and bounded below on $\mathcal{N}_\lambda$. □

Define $\psi_\lambda(u) = \langle J'_\lambda(u), u\rangle$. Then, for $u \in \mathcal{N}_\lambda$,

$$\begin{aligned} \langle \psi'_\lambda(u), u\rangle &= p||u||_\mu^p - q\lambda\int_\Omega f|u|^q - p*\int_\Omega g|u|^{p*} \\ &= (p-q)||u||_\mu^p - (p*-q)\int_\Omega g|u|^{p*} \\ &= \lambda(p^*-q)\int_\Omega f|u|^q - (p*-p)||u||_\mu^p. \end{aligned} \tag{3.3}$$

Arguing as in [22], we split $\mathcal{N}_\lambda$ into three parts:

$$\begin{aligned} \mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u\rangle > 0\}, \\ \mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u\rangle = 0\}, \\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u\rangle < 0\}. \end{aligned}$$

Lemma 3.2 *Suppose u_λ is a local minimizer of J_λ on $\mathcal{N}_\lambda$and $u_\lambda \notin \mathcal{N}_\lambda^0$.*

Then, $J'_\lambda(u_\lambda) = 0$in $(\mathcal{D}_0^{1,p}(\Omega))^{-1}$.

Proof The proof is similar to [[23], Theorem 2.3] and is omitted. □

Lemma 3.3 *$\mathcal{N}_\lambda^0 \neq \emptyset$for all $\lambda \in (0, \Lambda_1)$.*

Proof We argue by contradiction. Suppose that there exists $\lambda \in (0, \Lambda_1)$ such that $\mathcal{N}_\lambda^0 \neq \emptyset$. Then, the fact $u \in \mathcal{N}_\lambda^0$ and (3.3) imply that

$$||u||_\mu^p = \frac{p*-q}{p-q}\int_\Omega g|u|^{p*},$$

and

$$||u||_\mu^p = \lambda\frac{p^*-q}{p^*-p}\int_\Omega f|u|^q.$$

By (f_1), (g_1), the Hölder inequality and Sobolev embedding theorem, we have that

$$||u||_\mu \geq \left[\frac{p-q}{(p^*-q)|g^+|_\infty}\right]^{\frac{1}{p^*-p}} S_\mu^{\frac{N}{p^2}},$$

and

$$||u||_\mu \leq \left[\lambda \frac{p^*-q}{p^*-p}|f^+|_{q^*} S_\mu{}^{-\frac{q}{p}}\right]^{\frac{1}{p-q}}.$$

Consequently,

$$\lambda \geq \left(\frac{p-q}{(p^*-q)|g^+|_\infty}\right)^{\frac{p-q}{p^*-p}} \left(\frac{p^*-p}{(p^*-q)|f^+|_{q^*}}\right) S_\mu^{\frac{N}{p^2}(p-q)+\frac{q}{p}} = \Lambda_1,$$

which is a contradiction. □

For each $u \in \mathcal{D}_0^{1,p}(\Omega)$ with $\int_\Omega g|u|^{p^*} > 0$, we set

$$t_{\max} = \left(\frac{(p-q)||u||_\mu^p}{(p^*-q)\int_\Omega g|u|^{p^*}}\right)^{\frac{1}{p^*-p}} > 0.$$

Lemma 3.4 *Suppose that* $\lambda \in (0, \Lambda_1)$ *and* $u \in \mathcal{D}_0^{1,p}(\Omega)$*is a function satisfying with* $\int_\Omega g|u|^{p^*} > 0$.

(i) *If* $\int_\Omega f|u|^q \leq 0$, *then there exists a unique* $t^- > t_{\max}$ *such that* $t^- u \in \mathcal{N}_\lambda^-$*and*

$$J_\lambda(t^- u) = \sup_{t \geq 0} J_\lambda(tu).$$

(ii) *If* $\int_\Omega f|u|^q \leq 0$, *then there exists a unique* $t^\pm$ *such that* $0 < t^+ < t_{\max} < t^-$, $t^- u \in \mathcal{N}_\lambda^-$*and* $t^- u \in \mathcal{N}_\lambda^-$. *Moreover,*

$$J_\lambda(t^+ u) = \inf_{0 \leq t \leq t_{\max}} J_\lambda(tu), \quad J_\lambda(t^- u) = \sup_{t \geq t^+} J_\lambda(tu).$$

Proof See Brown-Wu [[24], Lemma 2.6]. □

We remark that it follows Lemma 3.3, $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ for all $\lambda \in (0, \Lambda_1)$. Furthermore, by Lemma 3.4, it follows that $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are nonempty, and by Lemma 3.1, we may define

$$\alpha_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u), \quad \alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u), \quad \alpha_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u).$$

Lemma 3.5 (i) *If* $\lambda \in (0, \Lambda_1)$, *then we have* $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.

(ii) *If* $\lambda \in (0, \frac{q}{p}\Lambda_1)$, *then* $\alpha_\lambda^- > d_0$*for some positive constant* d_0.

In particular, for each $\lambda \in (0, \frac{q}{p}\Lambda_1)$, *we have* $\alpha_\lambda = \alpha_\lambda^+ < 0 < \alpha_\lambda^-$.

Proof (i) Suppose that $u \in \mathcal{N}_\lambda^+$. From (3.3), it follows that

$$\frac{p-q}{p^*-q}||u||_\mu^p > \int_\Omega g|u|^{p^*}. \tag{3.4}$$

According to (3.1) and (3.4), we have

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{p} - \frac{1}{q}\right) ||u||_\mu^p + \left(\frac{1}{q} - \frac{1}{p^*}\right) \int_\Omega g|u|^{p^*} \\ &< \left[\left(\frac{1}{p} - \frac{1}{q}\right) + \left(\frac{1}{q} - \frac{1}{p^*}\right)\left(\frac{p-q}{p^*-q}\right)\right] ||u||_\mu^p \\ &= -\frac{p-q}{qN} ||u||_\mu^p < 0. \end{aligned}$$

By the definitions of α_λ and α_λ^+, we get that $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.

(ii) Suppose $\lambda \in (0, \frac{q}{p}\Lambda_1)$ and $u \in \mathcal{N}_\lambda^-$. Then, (3.3) implies that

$$\frac{p-q}{p^*-q} ||u||_\mu^p < \int_\Omega |u|^{p^*}. \tag{3.5}$$

Moreover, by (g_1) and the Sobolev embedding theorem, we have

$$\int_\Omega g|u|^{p^*} \leq |g^+|_\infty S_\mu^{-\frac{p^*}{p}} ||u||_\mu^{p^*}. \tag{3.6}$$

From (3.5) and (3.6), it follows that

$$||u||_\mu > \left(\frac{p-q}{(p^*-q)|g^+|_\infty}\right)^{\frac{1}{p^*-p}} S_\mu^{\frac{N}{p^2}} \text{ for all } u \in \mathcal{N}_\lambda^-. \tag{3.7}$$

By (3.2) and (3.7), we get

$$\begin{aligned} J_\lambda(u) &\geq ||u||_\mu^q \left[\frac{1}{N} ||u||_\mu^{p-q} - \lambda \frac{p^*-q}{p^*q} |f^+|_{q^*} S_\mu^{-\frac{q}{p}}\right] \\ &> \left(\frac{p-q}{(p^*-q)|g^+|_\infty}\right)^{\frac{q}{p^*-p}} S_\mu^{\frac{qN}{p^2}} \left[\frac{1}{N}\left(\frac{p-q}{(p^*-q)|g^+|_\infty}\right)^{\frac{p-q}{p^*-p}} S_\mu^{\frac{N(p-q)}{p^2}} \right. \\ &\quad \left. -\lambda \frac{p^*-q}{p^*q} |f^+|_{q^*} S_\mu^{-\frac{q}{p}}\right] \end{aligned}$$

which implies that

$$J_\lambda(u) > d_0 \text{ for all } u \in \mathcal{N}_\lambda^-,$$

for some positive constant d_0. □

Remark 3.6 *If* $\lambda \in (0, \frac{q}{p}\Lambda_0)$, *then by Lemmas 3.4 and 3.5, for each* $u \in \mathcal{D}_0^{1,p}(\Omega)$ *with* $\int_\Omega g|u|^{p^*} > 0$, *we can easily deduce that*

$$t^- u \in \mathcal{N}_\lambda^- \textit{ and } J_\lambda(t^- u) = \sup_{t \geq 0} J_\lambda(tu) \geq \alpha_\lambda^- > 0.$$

4 Proof of Theorem 1.1

First, we define the Palais-Smale (simply by (PS)) sequences, (PS)-values and (PS)-conditions in $\mathcal{D}_0^{1,p}(\Omega)$ for J_λ as follows:

Definition 4.1 (i) *For* $c \in \mathbb{R}$, *a sequence* $\{u_n\}$ *is a* $(PS)_c$*-sequence in* $\mathcal{D}_0^{1,p}(\Omega)$ *for* J_λ *if* $J_\lambda(u_n) = c + o(1)$ *and* $(J_\lambda)'(u_n) = o(1)$ *strongly in* $(\mathcal{D}_0^{1,p}(\Omega))^{-1}$ *as* $n \to \infty$.

(ii) $c \in \mathbb{R}$ *is a* (PS)*-value in* $\mathcal{D}_0^{1,p}(\Omega)$*for* J_λ *if there exists a* $(PS)_c$*-sequence in* $\mathcal{D}_0^{1,p}(\Omega)$*for* J_λ.

(iii) J_λ *satisfies the* $(PS)_c$*-condition in* $\mathcal{D}_0^{1,p}(\Omega)$*if any* $(PS)_c$*-sequence* $\{u_n\}$ *in* $\mathcal{D}_0^{1,p}(\Omega)$*for* J_λ *contains a convergent subsequence.*

Lemma 4.2 (i) *If* $\lambda \in (0, \Lambda_1)$, *then* J_λ *has a* $(PS)_{\alpha_\lambda}$*-sequence* $\{u_n\} \subset \mathcal{N}_\lambda$.

(ii) *If* $\lambda \in (0, \frac{q}{p}\Lambda_1)$, *then* J_λ *has a* $(PS)_{\alpha_\lambda^-}$*-sequence* $\{u_n\} \subset \mathcal{N}_\lambda^-$.

Proof The proof is similar to [19,25] and the details are omitted. □

Now, we establish the existence of a local minimum for J_λ on $\mathcal{N}_\lambda$.

Theorem 4.3 *Suppose that* $N \geq 3$, $\mu < \bar{\mu}$, $1 < q < p < N$ *and the conditions* (f_1), (g_1) *hold. If* $\lambda \in (0, \Lambda_1)$, *then there exists* $u_\lambda \in \mathcal{N}_\lambda^+$ *such that*

(i) $J_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+$,

(ii) u_λ *is a positive solution of* (1.1),

(iii) $||u_\lambda||_\mu \to 0$ *as* $\lambda \to 0^+$.

Proof By Lemma 4.2 (*i*), there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ such that

$$J_\lambda(u_n) = \alpha_\lambda + o(1) \quad \text{and} \quad J'_\lambda(u_n) = o(1) \quad \text{in } (\mathcal{D}_0^{1,p}(\Omega))^{-1}. \tag{4.1}$$

Since J_λ is coercive on $\mathcal{N}_\lambda$ (see Lemma 2.1), we get that (u_n) is bounded in $\mathcal{D}_0^{1,p}(\Omega)$. Passing to a subsequence, there exists $u_\lambda \in \mathcal{D}_0^{1,p}(\Omega)$ such that as $n \to \infty$

$$\begin{cases} u_n \rightharpoonup u_\lambda \text{ weakly in } \mathcal{D}_0^{1,p}(\Omega), \\ u_n \rightharpoonup u_\lambda \text{ weakly in } L^{p^*}(\Omega), \\ u_n \to u_\lambda \text{ strongly in } L^r_{loc}(\Omega) \text{ for all } 1 \leq r < p^*, \\ u_n \to u_\lambda \text{ a.e. in } \Omega. \end{cases} \tag{4.2}$$

By (f_1) and Lemma 2.3, we obtain

$$\lambda \textstyle\int_\Omega f|u_n|^q = \lambda \int_\Omega f|u_\lambda|^q + o(1) \text{ as } n \to \infty. \tag{3}$$

From (4.1)-(4.3), a standard argument shows that u_λ is a critical point of J_λ. Furthermore, the fact $\{u_n\} \subset \mathcal{N}_\lambda$ implies that

$$\lambda \int_\Omega f|u_n|^q = \frac{q(p^* - p)}{p(p^* - q)}||u_n||_\mu^p - \frac{p^* q}{p^* - q}J_\lambda(u_n). \tag{4.4}$$

Taking $n \to \infty$ in (4.4), by (4.1), (4.3) and the fact $\alpha_\lambda < 0$, we get

$$\lambda \int_\Omega f|u_\lambda|^q \geq -\frac{p^* q}{p^* - q}\alpha_\lambda > 0. \tag{4.5}$$

Thus, $u_\lambda \in \mathcal{N}_\lambda$ is a nontrivial solution of (1.1).

Next, we prove that $u_n \to u_\lambda$ strongly in $\mathcal{D}_0^{1,p}(\Omega)$ and $J_\lambda(u_\lambda) = \alpha_\lambda$. From (4.3), the fact $u_n, u_\lambda \in \mathcal{N}_\lambda$ and the Fatou's lemma it follows that

$$\begin{aligned} \alpha_\lambda \leq J_\lambda(u_\lambda) &= \frac{1}{N}||u_\lambda||_\mu^p - \lambda\frac{p^* - q}{p^* q}\int_\Omega f|u_\lambda|^q \\ &\leq \liminf_{n\to\infty}\left(\frac{1}{N}||u_n||_\mu^p - \lambda\frac{p^* - q}{p^* q}\int_\Omega f|u_n|^q\right) \\ &= \liminf_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda, \end{aligned}$$

which implies that $J_\lambda(u_\lambda) = \alpha_\lambda$ and $\lim_{n\to\infty}||u_n||_\mu^p = ||u_\lambda||_\mu^p$. Standard argument shows that $u_n \to u_\lambda$ strongly in $\mathcal{D}_0^{1,p}(\Omega)$. Moreover, $u_\lambda \in \mathcal{N}_\lambda^+$. Otherwise, if $u_\lambda \in \mathcal{N}_\lambda^-$, by Lemma 3.4, there exist unique t_λ^+ and t_λ^- such that $t_\lambda^+ u_\lambda \in \mathcal{N}_\lambda^+$, $t_\lambda^- u_\lambda \in \mathcal{N}_\lambda^-$ and $t_\lambda^+ < t_\lambda^- = 1$. Since

$$\frac{d}{dt}J_\lambda(t_\lambda^+ u_\lambda) = 0 \quad \text{and} \quad \frac{d^2}{dt^2}J_\lambda(t_\lambda^+ u_\lambda) > 0,$$

there exists $\bar{t} \in (t_\lambda^+, t_\lambda^-)$ such that $J_\lambda(t_\lambda^+ u_\lambda) < J_\lambda(\bar{t}u_\lambda)$. By Lemma 3.4, we get that

$$J_\lambda(t_\lambda^+ u_\lambda) < J_\lambda(\bar{t}u_\lambda) \le J_\lambda(t_\lambda^- u_\lambda) = J_\lambda(u_\lambda),$$

which is a contradiction. If $u \in \mathcal{N}_\lambda^+$, then $|u| \in \mathcal{N}_\lambda^+$, and by $J_\lambda(u_\lambda) = J_\lambda(|u_\lambda|) = \alpha_\lambda$, we get $|u_\lambda| \in \mathcal{N}_\lambda^+$ is a local minimum of J_λ on $\mathcal{N}_\lambda$. Then, by Lemma 3.2, we may assume that u_λ is a nontrivial nonnegative solution of (1.1). By Harnack inequality due to Trudinger [26], we obtain that $u_\lambda > 0$ in Ω. Finally, by (3.3), the Hölder inequality and Sobolev embedding theorem, we obtain

$$||u_\lambda||_\mu^{p-q} < \lambda \frac{p^* - q}{p^* - p}|f^+|_{q^*} S_\mu^{-\frac{q}{p}}.$$

which implies that $||u_\lambda||_\mu \to 0$ as $\lambda \to 0^+$. □

Proof of Theorem 1.1 From Theorem 4.3, it follows that the problem (1.1) has a positive solution $u_\lambda \in \mathcal{N}_\lambda^+$ for all $\lambda \in (0, \Lambda_0)$. □

5 Proof of Theorem 1.2

For $1 < p < N$ and $\mu < \bar{\mu}$, let

$$c^* = \frac{1}{N}|g^+|_\infty^{-\frac{N-p}{p}} S_\mu^{\frac{N}{p}}.$$

Lemma 5.1 *Suppose* $\{u_n\}$ *is a bounded sequence in* $\mathcal{D}_0^{1,p}(\Omega)$. *If* $\{u_n\}$ *is a* $(PS)_c$*-sequence for* J_λ *with* $c \in (0, c^*)$, *then there exists a subsequence of* $\{u_n\}$ *converging weakly to a nonzero solution of* (1.1).

Proof Let $\{u_n\} \subset \mathcal{D}_0^{1,p}(\Omega)$ be a $(PS)_c$-sequence for J_λ with $c \in (0, c^*)$. Since $\{u_n\}$ is bounded in $\mathcal{D}_0^{1,p}(\Omega)$, passing to a subsequence if necessary, we may assume that as $n \to \infty$

$$\begin{cases} u_n \rightharpoonup u_0 \text{ weakly in } \mathcal{D}_0^{1,p}(\Omega), \\ u_n \rightharpoonup u_0 \text{ weakly in } L^{p^*}(\Omega), \\ u_n \to u_0 \text{ strongly in } L_{loc}^r(\Omega) \text{ for } 1 \le r < p^*, \\ u_n \to u_0 \text{ a.e. in } \Omega. \end{cases} \tag{5.1}$$

By (f_1), (g_1), (5.1) and Lemma 2.3, we have that $J'_\lambda(u_0) = 0$ and

$$\lambda \int_\Omega f|u_n|^q = \lambda \int_\Omega f|u_0|^q + o(1) \text{ as } n \to \infty. \tag{5.2}$$

Next, we verify that $u_0 \not\equiv 0$. Arguing by contradiction, we assume $u_0 \equiv 0$. Since $J'_\lambda(u_n) = o(1)$ as $n \to \infty$ and $\{u_n\}$ is bounded in $\mathcal{D}_0^{1,p}(\Omega)$, then by (5.2), we can deduce that

$$0 = \langle \lim_{n\to\infty} J'_\lambda(u_n), u_n\rangle = \lim_{n\to\infty}\left(||u_n||_\mu^p - \int_\Omega g|u_n|^{p^*}\right).$$

Then, we can set

$$\lim_{n\to\infty} ||u_n||_\mu^p = \lim_{n\to\infty}\int_\Omega g|u_n|^{p^*} = l. \tag{5.3}$$

If $l = 0$, then we get $c = \lim_{n\to\infty} J_\lambda(u_n) = 0$, which is a contradiction. Thus, we conclude that $l > 0$. Furthermore, the Sobolev embedding theorem implies that

$$\begin{aligned}||u_n||_\mu^p &\geq S_\mu\left(\int_\Omega g|u_n|^{p^*}\right)^{\frac{p}{p^*}}\\ &\geq S_\mu\left(\int_\Omega \frac{g}{|g^+|_\infty}|u_n|^{p^*}\right)^{\frac{p}{p^*}}\\ &= S_\mu|g^+|_\infty^{-\frac{N-p}{N}}\left(\int_\Omega g|u_n|^{p^*}\right)^{\frac{p}{p^*}}.\end{aligned}$$

Then, as $n \to \infty$ we have $l = \lim_{n\to\infty}||u_n||_\mu^p \geq S_\mu|g^+|_\infty^{-\frac{N-p}{N}} l^{\frac{p}{p^*}}$, which implies that

$$l \geq |g^+|_\infty^{-\frac{N-p}{p}} S_\mu^{\frac{N}{p}}. \tag{5.4}$$

Hence, from (5.2)-(5.4), we get

$$\begin{aligned}c &= \lim_{n\to\infty} J_\lambda(u_n)\\ &= \frac{1}{p}\lim_{n\to\infty}||u_n||_\mu^p - \frac{\lambda}{q}\lim_{n\to\infty}\int_\Omega f|u_n|^q - \frac{1}{p^*}\lim_{n\to\infty}\int_\Omega g|u_n|^{p^*}\\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right) l\\ &\geq \frac{1}{N}|g^+|_\infty^{-\frac{N-p}{p}} S_\mu^{\frac{N}{p}}.\end{aligned}$$

This is contrary to $c < c^*$. Therefore, u_0 is a nontrivial solution of (1.1). □

Lemma 5.2 *Suppose* $(\mathcal{H})$*and* (f_1) - (g_2) *hold. If* $0 < \mu < \bar{\mu}$, $x_0 = 0$ *and* $\beta \geq p\gamma$, *then for any* $\lambda > 0$, *there exists* $v_\lambda \in \mathcal{D}_0^{1,p}(\Omega)$*such that*

$$\sup_{t\geq 0} J_\lambda(tv_\lambda) < c^*. \tag{5.5}$$

In particular, $\alpha_\lambda^- < c^*$*for all* $\lambda \in (0, \Lambda_1)$.

Proof From [[11], Lemma 5.3], we get that if ε is small enough, there exist $t_\varepsilon > 0$ and the positive constants C_i $(i = 1, 2)$ independent of ε, such that

$$\sup_{t\geq 0} J_\lambda(tu_\varepsilon) = J_\lambda(t_\varepsilon u_\varepsilon) \text{ and } 0 < C_1 \leq t_\varepsilon \leq C_2 < \infty. \tag{5.6}$$

By (g_2), we conclude that

$$\begin{aligned}\left|\int_\Omega g(x)|u_\varepsilon|^{p^*} - \int_\Omega g(0)|u_\varepsilon|^{p^*}\right| &\le \int_\Omega |g(x)-g(0)|\ |u_\varepsilon|^{p^*} \\ &= O\left(\int_{B(0;\rho)} |x|^\beta |u_\varepsilon|^{p^*}\right) \\ &= O(\varepsilon^\beta),\end{aligned}$$

which together with Lemma 2.2 implies that

$$\int_\Omega g(x)|u_\varepsilon|^{p^*} = g(0)S_\mu^{\frac{N}{p}} + O(\varepsilon^{p^*\gamma}) + O(\varepsilon^\beta). \tag{5.7}$$

From the fact $\lambda > 0$, $1 < q < p$, $\beta \ge p\gamma$ and

$$\max_{t\ge 0}\left(\frac{t^p}{p}B_1 - \frac{t^{p^*}}{p^*}B_2\right) = \frac{1}{N}B_1^{\frac{N}{p}}B_2^{-\frac{N-p}{p}}, \quad B_1 > 0, B_2 > 0,$$

and by Lemma 2.2, (5.7) and (f_2), we get

$$\begin{aligned} J_\lambda(t_\varepsilon u_\varepsilon) &= \frac{t_\varepsilon^p}{p}||u_\varepsilon||_\mu^p - \frac{t_\varepsilon^{p^*}}{p^*}\int_\Omega g|u_\varepsilon|^{p^*} - \lambda\frac{t_\varepsilon^q}{q}\int_\Omega f|u_\varepsilon|^q \\ &\le \frac{1}{N}||u_\varepsilon||_\mu^{\frac{N}{p}}\left(\int_\Omega g|u_\varepsilon|^{p^*}\right)^{-\frac{N-p}{p}} - \lambda\frac{C_1^q}{q}\beta_0\int_\Omega |u_\varepsilon|^q \\ &= \frac{1}{N}\left(S_\mu^{\frac{N}{p}} + O(\varepsilon^{p\gamma})\right)^{\frac{N}{p}}\left(g(0)S_\mu^{\frac{N}{p}} + O(\varepsilon^{p^*\gamma}) + O(\varepsilon^\beta)\right)^{-\frac{N-p}{p}} \\ &\quad - \lambda\frac{C_1^q}{q}\beta_0\int_\Omega |u_\varepsilon|^q \\ &= \frac{1}{N}g(0)^{-\frac{N-p}{p}}S_\mu^{\frac{N}{p}} + O(\varepsilon^{p\gamma}) + O(\varepsilon^\beta) - \lambda\frac{C_1^q}{q}\beta_0\int_\Omega |u_\varepsilon|^q. \end{aligned} \tag{5.8}$$

By (5.6) and (5.8), we have that

$$\sup_{t\ge 0} J_\lambda(tu_\varepsilon) \le c^* + O(\varepsilon^{p\gamma}) + O(\varepsilon^\beta) - \lambda\frac{C_1^q}{q}\beta_0\int_\Omega |u_\varepsilon|^q. \tag{5.9}$$

(i) If $1 < q < \frac{N}{b(\mu)}$, then by Lemma 2.2 and $\gamma = b(\mu) - \delta = b(\mu) - \frac{N-p}{p} > 0$ we have that

$$\int_\Omega |u_\varepsilon|^q = O_1(\varepsilon^{q\gamma}).$$

Combining this with (5.9), for any $\lambda > 0$, we can choose ε_λ small enough such that

$$\sup_{t\ge 0} J_\lambda(tu_{\varepsilon_\lambda}) < c^*.$$

(ii) If $\frac{N}{b(\mu)} \le q < p$, then by Lemma 2.2 and $\gamma > 0$ we have that

$$\int_\Omega |u_\varepsilon|^q = \begin{cases} O_1(\varepsilon^\theta), & q > \frac{N}{b(\mu)}, \\ O_1(\varepsilon^\theta|ln\varepsilon|), & q = \frac{N}{b(\mu)}, \end{cases}$$

and

$$p\gamma = b(\mu)p + p - N > N + (1 - \frac{N}{p})q = \theta.$$

Combining this with (5.9), for any $\lambda > 0$, we can choose ε_λ small enough such that

$$\sup_{t\geq 0} J_\lambda(tu_{\varepsilon_\lambda}) < c^*.$$

From (i) and (ii), (5.5) holds by taking $v_\lambda = u_{\varepsilon_\lambda}$.

In fact, by (f_2), (g_2) and the definition of u_{ε_λ}, we have that

$$\int_\Omega f|u_{\varepsilon_\lambda}|^q > 0 \quad \text{and} \quad \int_\Omega g|u_{\varepsilon_\lambda}|^{p^*} > 0.$$

From Lemma 3.4, the definition of α_λ^- and (5.5), for any $\lambda \in (0, \Lambda_0)$, there exists $t_{\varepsilon_\lambda} > 0$ such that $t_{\varepsilon_\lambda} u_{\varepsilon_\lambda} \in \mathcal{N}_\lambda^-$ and

$$\alpha_\lambda^- \leq J_\lambda(t_{\varepsilon_\lambda} u_{\varepsilon_\lambda}) \leq \sup_{t\geq 0} J_\lambda(tt_{\varepsilon_\lambda} u_{\varepsilon_\lambda}) < c^*.$$

The proof is thus complete. □

Now, we establish the existence of a local minimum of J_λ on $\mathcal{N}_\lambda^-$.

Theorem 5.3 *Suppose* $(\mathcal{H})$*and* (f_1) - (g_2) *hold. If* $0 < \mu < \bar{\mu}$, $x_0 = 0$, $\beta \geq p\gamma$ *and* $\lambda \in (0, \frac{q}{p}\Lambda_1)$, *then there exists* $U_\lambda \in \mathcal{N}_\lambda^-$*such that*

(i) $J_\lambda(U_\lambda) = \alpha_\lambda^-$,
(ii) U_λ *is a positive solution of* (1.1).

Proof If $\lambda \in (0, \frac{q}{p}\Lambda_1)$, then by Lemmas 3.5 *(ii)*, 4.2 *(ii)* and 5.2, there exists a $\{u_n\} \subset \mathcal{N}_\lambda^-$-sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ in $\mathcal{D}_0^{1,p}(\Omega)$ for J_λ with $\alpha_\lambda^- \in (0, c^*)$. Since J_λ is coercive on $\mathcal{N}_\lambda^-$ (see Lemma 3.1), we get that $\{u_n\}$ is bounded in $\mathcal{D}_0^{1,p}(\Omega)$. From Lemma 5.1, there exists a subsequence still denoted by $\{u_n\}$ and a nontrivial solution $U_\lambda \in \mathcal{D}_0^{1,p}(\Omega)$ of (1.1) such that $u_n \rightharpoonup U_\lambda$ weakly in $\mathcal{D}_0^{1,p}(\Omega)$.

First, we prove that $U_\lambda \in \mathcal{N}_\lambda^-$. On the contrary, if $U_\lambda \in \mathcal{N}_\lambda^+$, then by $\mathcal{N}_\lambda^- \cup \{0\}$ is closed in $\mathcal{D}_0^{1,p}(\Omega)$, we have $||U_\lambda||_\mu < \liminf_{n\to\infty} ||u_n||_\mu$. From (g_2) and $U_\lambda \not\equiv 0$ in Ω, we have $\int_\Omega g|U_\lambda|^{p^*} > 0$. Thus, by Lemma 3.4, there exists a unique t_λ such that $t_\lambda U_\lambda \in \mathcal{N}_\lambda^-$. If $u \in \mathcal{N}_\lambda$, then it is easy to see that

$$J_\lambda(u) = \frac{1}{N}||u||_\mu^p - \lambda\left(\frac{p^* - q}{p^* q}\right)\int_\Omega f|u|^q. \tag{5.10}$$

From Remark 3.6, $u_n \in \mathcal{N}_\lambda^-$ and (5.10), we can deduce that

$$\alpha_\lambda^- \leq J_\lambda(t_\lambda U_\lambda) < \lim_{n\to\infty} J_\lambda(t_\lambda u_n) \leq \lim_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda^-.$$

This is a contradiction. Thus, $U_\lambda \in \mathcal{N}_\lambda^-$.

Next, by the same argument as that in Theorem 4.3, we get that $u_n \to U_\lambda$ strongly in $\mathcal{D}_0^{1,p}(\Omega)$ and $J_\lambda(U_\lambda) = \alpha_\lambda^- > 0$ for all $\lambda \in (0, \frac{q}{p}\Lambda_1)$. Since $J_\lambda(U_\lambda) = J_\lambda(|U_\lambda|)$ and $|U_\lambda| \in \mathcal{N}_\lambda^-$, by Lemma 3.2, we may assume that U_λ is a nontrivial nonnegative solution of (1.1). Finally, by Harnack inequality due to Trudinger [26], we obtain that U_λ is a positive solution of (1.1). □

Proof of Theorem 1.2 From Theorem 4.3, we get the first positive solution $u_\lambda \in \mathcal{N}_\lambda^+$ for all $\lambda \in (0, \Lambda_0)$. From Theorem 5.3, we get the second positive solution $U_\lambda \in \mathcal{N}_\lambda^+$ for all $\lambda \in (0, \frac{q}{p}\Lambda_0)$. Since $\mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^- = \emptyset$, this implies that u_λ and U_λ are distinct. □

6 Proof of Theorem 1.3

In this section, we consider the case $\mu \le 0$. In this case, it is well-known $S_\mu = S_0$ where S_μ is defined as in (1.2). Thus, we have $c^* = \frac{1}{N}|g^+|_\infty^{-\frac{N-p}{p}} S_0^{\frac{N}{p}}$ when $\mu \le 0$.

Lemma 6.1 *Suppose* $(\mathcal{H})$*and* (f_1) - (g_2) *hold. If* $N \le p^2$, $\mu < 0$, $x_0 \ne 0$ *and* $\frac{\beta}{p} \ge \tilde{\gamma} := \frac{N-p}{p(p-1)}$, *then for any* $\lambda > 0$ *and* $\mu < 0$, *there exists* $v_{\lambda,\mu} \in \mathcal{D}_0^{1,p}(\Omega)$*such that*

$$\sup_{t\ge 0} J_\lambda(tv_{\lambda,\mu}) < c^*. \tag{6.1}$$

In particular, $\alpha_\lambda^- < c^*$*for all* $\lambda \in (0, \Lambda_1)$.

Proof Note that S_0 has the following explicit extremals [27]:

$$V_\varepsilon(x) = \bar{C}\varepsilon^{-\frac{N-p}{p}}\left(1 + \left(\frac{|x-x_0|}{\varepsilon}\right)^{\frac{p}{p-1}}\right)^{-\frac{N-p}{p}}, \quad \forall \varepsilon > 0, x_0 \in \mathbb{R}^N,$$

where $\bar{C} > 0$ is a particular constant. Take $\rho > 0$ small enough such that $B(x_0; \rho) \subset \Omega\backslash\{0\}$ and set $\tilde{u}_\varepsilon(x) = \varphi(x)V_\varepsilon(x)$, where $\varphi(x) \in C_0^\infty(B(x_0;\rho)$ is a cutoff function such that $\phi(x) \equiv 1$ in $B(x_0; \rho/2)$. Arguing as in Lemma 2.2, we have

$$\int_\Omega |\nabla \tilde{u}_\varepsilon|^p = S_0^{\frac{N}{p}} + O(\varepsilon^{p\tilde{\gamma}}), \tag{6.2}$$

$$\int_\Omega |\tilde{u}_\varepsilon|^{p^*} = S_0^{\frac{N}{P}} + +O(\varepsilon^{p^*\tilde{\gamma}}), \tag{6.3}$$

$$\int_\Omega |\tilde{u}_\varepsilon|^q = \begin{cases} O_1(\varepsilon^\theta), & \frac{N(p-1)}{N-p} < q < p^*, \\ O_1(\varepsilon^\theta |ln\varepsilon|), & q = \frac{N(p-1)}{N-p}, \\ O_1(\varepsilon^{q\tilde{\gamma}}), & 1 \le q < \frac{N(p-1)}{N-p}, \end{cases} \tag{6.4}$$

where $\theta = N - \frac{N-p}{p}q$. Note that $\beta \ge p\tilde{\gamma}$, $p^*\tilde{\gamma} > p\tilde{\gamma}$. Arguing as in Lemma 5.2, we deduce that there exists $\tilde{t}_\varepsilon$ satisfying $0 < C_1 \le \tilde{t}_\varepsilon \le C_2$, such that

$$\begin{aligned} J_\lambda(t\tilde{u}_\varepsilon) &\le \sup_{t\ge 0} J_\lambda(t\tilde{u}_\varepsilon) = J_\lambda(\tilde{t}_\varepsilon \tilde{u}_\varepsilon) \\ &= \frac{\tilde{t}_\varepsilon^p}{p}\int_\Omega |\nabla \tilde{u}_\varepsilon|^p - \frac{\tilde{t}_\varepsilon^{p^*}}{p^*}\int_\Omega g|\tilde{u}_\varepsilon|^{p^*} - \lambda\frac{\tilde{t}_\varepsilon^q}{q}\int_\Omega f|\tilde{u}_\varepsilon|^q - \mu\frac{\tilde{t}_\varepsilon^p}{p}\int_\Omega \frac{|\tilde{u}_\varepsilon|^p}{|x|^p} \\ &\le \frac{1}{N}g(x_0)^{-\frac{N-p}{p}} S_\mu^{\frac{N}{p}} + O(\varepsilon^{p\tilde{\gamma}}) - \lambda\frac{C_1^q}{q}\beta_0\int_\Omega |\tilde{u}_\varepsilon|^q \\ &\quad - \mu||x_0| - \rho|^{-p}\frac{C_2^p}{p}\int_\Omega |\tilde{u}_\varepsilon|^p. \end{aligned} \tag{6.5}$$

From $(\mathcal{H})$, $N \le p^2$ and (6.4), we can deduce that

$$1 < q\tilde{\gamma} < p\tilde{\gamma} = \frac{N-p}{p-1} \le p \le \frac{N(p-1)}{N-p}$$

and

$$\int_\Omega |\tilde{u}_\varepsilon|^q = O_1(\varepsilon^{q\tilde{\gamma}}) \text{ and } \int_\Omega |\tilde{u}_\varepsilon|^p = \begin{cases} O_1(\varepsilon^p|ln\varepsilon|), & p = \frac{N(p-1)}{N-p}, \\ O_1(\varepsilon^{p\tilde{\gamma}}), & 1 < p < \frac{N(p-1)}{N-p}. \end{cases}$$

Combining this with (6.5), for any $\lambda > 0$ and $\mu < 0$, we can choose $\varepsilon_{\lambda,\mu}$ small enough such that

$$\sup_{t\ge 0} J_\lambda(t\tilde{u}_{\varepsilon_{\lambda,\mu}}) < \frac{1}{N} g(x_0)^{-\frac{N-p}{p}} S_0^{\frac{N}{p}} = c^*.$$

Therefore, (6.1) holds by taking $v_{\lambda,\mu} = \tilde{u}_{\varepsilon_{\lambda,\mu}}$.

In fact, by (f_2), (g_2) and the definition of $\tilde{u}_{\varepsilon_{\lambda,\mu}}$, we have that

$$\int_\Omega f|\tilde{u}_{\varepsilon_{\lambda,\mu}}|^q > 0 \quad \text{and} \quad \int_\Omega g|\tilde{u}_{\varepsilon_{\lambda,\mu}}|^{p^*} > 0.$$

From Lemma 3.4, the definition of α_λ^- and (6.1), for any $\lambda \in (0, \Lambda_0)$ and $\mu < 0$, there exists $t_{\varepsilon_{\lambda,\mu}} > 0$ such that $t_{\varepsilon_{\lambda,\mu}}\tilde{u}_{\varepsilon_{\lambda,\mu}} \in \mathcal{N}_\lambda^-$ and

$$\alpha_\lambda^- \le J_\lambda(t_{\varepsilon_{\lambda,\mu}}\tilde{u}_{\varepsilon_{\lambda,\mu}}) \le \sup_{t\ge 0} J_\lambda(tt_{\varepsilon_{\lambda,\mu}}\tilde{u}_{\varepsilon_{\lambda,\mu}}) < c^*.$$

The proof is thus complete. □

Proof of Theorem 1.3 Let $\Lambda_1(0)$ be defined as in (1.3). Arguing as in Theorems 4.3 and 5.3, we can get the first positive solution $\tilde{u}_\lambda \in \mathcal{N}_\lambda^+$ for all $\lambda \in (0, \Lambda_1(0))$ and the second positive solution $\tilde{U}_\lambda \in \mathcal{N}_\lambda^-$ for all $\lambda \in (0, \frac{q}{p}\Lambda_1(0))$. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, this implies that $\tilde{u}_\lambda$ and $\tilde{U}_\lambda$ are distinct. □

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