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A stabilized mixed discontinuous Galerkin method for the incompressible miscible displacement problem

Yan Luo[1], Minfu Feng[2] and Youcai Xu[2*]

* Correspondence: xyc@scu.edu.cn
[2]School of Mathematics, Sichuan University, Chengdu, Sichuan 610064, PR China
Full list of author information is available at the end of the article

Abstract

A new fully discrete stabilized discontinuous Galerkin method is proposed to solve the incompressible miscible displacement problem. For the pressure equation, we develop a mixed, stabilized, discontinuous Galerkin formulation. We can obtain the optimal priori estimates for both concentration and pressure.

Keywords: Discontinuous Galerkin methods, a priori error estimates, incompressible miscible displacement

1 Introduction

We consider the problem of miscible displacement which has considerable and practical importance in petroleum engineering. This problem can be considered as the result of advective-diffusive equation for concentrations and the Darcy flow equation. The more popular approach in application so far has been based on the mixed formulation. In a previous work, Douglas and Roberts [1] presented a mixed finite element (MFE) method for the compressible miscible displacement problem. For the Darcy flow, Masud and Hughes [2] introduced a stabilized finite element formulation in which an appropriately weighted residual of the Darcy law is added to the standard mixed formulation. Recently, discontinuous Galerkin for miscible displacement has been investigated by numerical experiments and was reported to exhibit good numerical performance [3,4]. In Hughes-Masud-Wan [5], the method of [2] was extended to the discontinuous Galerkin framework for the Darcy flow. A family of mixed finite element discretizations of the Darcy flow equations using totally discontinuous elements was introduced in [6]. In [7] primal semi-discrete discontinuous Galerkin methods with interior penalty are proposed to solve the coupled system of flow and reactive transport in porous media, which arises from many applications including miscible displacement and acid-stimulated flow. In [8], stable Crank-Nicolson discretization was given for incompressible miscible displacement problem.

The discontinuous Galerkin (DG) method was introduced by Reed and Hill [9], and extended by Cockburn and Shu [10-12] to conservation law and system of conservation laws,respectively. Due to localizability of the discontinuous Galerkin method, it is easy to construct higher order element to obtain higher order accuracy and to derive highly parallel algorithms. Because of these advantages, the discontinuous Galerkin method has become a very active area of research [4-7,13-18]. Most of the literature concerning discontinuous Galerkin methods can be found in [13].

In this paper, we analyze a fully discrete finite element method with the stabilized mixed discontinuous Galerkin methods for the incompressible miscible displacement problem in porous media. For the pressure equation, we develop a mixed, stabilized, discontinuous Galerkin formulation. To some extent, we develop a more general stabilized formulation and because of the proper choose of the parameters γ and β, this paper includes the methods of [2,6] and [5]. All the schemes are stable for any combination of discontinuous discrete concentration, velocity and pressure spaces. Based on our results, we can assert that the mixed stabilized discontinuous Galerkin formulation of the incompressible miscible displacement problem is mathematically viable, and we also believe it may be practically useful. It generalizes and encompasses all the successful elements described in [2,6] and [5]. Optimal error estimate are obtained for the concentration, velocity and pressure.

An outline of the remainder of the paper follows: In Section 2, we describe the modeling equations. The DG schemes for the concentration and some of their properties are introduced in Section 3. Stabilized mixed DG methods are introduced for the velocity and pressure in Section 4. In Section 5, we propose the numerical approximation scheme of incompressible miscible displacement problems with a fully discrete in time, combined with a mixed, stabilized and discontinuous Galerkin method. The boundedness and stability of the finite element formulation are studied in Section 6. Error estimates for the incompressible miscible displacement problem are obtained in Section 7.

Throughout the paper, we denote by C a generic positive constant that is independent of h and Δt, but might depend on the partial differential equation solution; we denote by ε a fixed positive constant that can be chosen arbitrarily small.

2 Governing equations

Miscible displacement of one incompressible fluid by another in a porous medium $\Omega \in R^d(d = 2, 3)$ over time interval $J = (0, T]$ is modeled by the system concentration equation:

$$\phi\frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c - \nabla\cdot(\mathbf{D}(\mathbf{u})\nabla c) = qc^*, \quad (x,t)\in\Omega\times J. \tag{2.1}$$

Pressure equation:

$$\mathbf{u} = -a(c)\nabla p, \quad (x,t)\in\Omega\times J, \tag{2.2}$$

$$\nabla\cdot\mathbf{u} = q, \quad (x,t)\in\Omega\times J. \tag{2.3}$$

The initial conditions

$$c(x,0) = c_0(x), \quad x\in\Omega. \tag{2.4}$$

The no-flow boundary conditions

$$\begin{cases} \mathbf{u}\cdot\mathbf{n} = 0, & x\in\partial\Omega, \\ (\mathbf{D}(\mathbf{u})\nabla c - c\mathbf{u})\cdot\mathbf{n} = 0, & x\in\partial\Omega. \end{cases} \tag{2.5}$$

Dispersion/diffusion tensor

$$\mathbf{D}(\mathbf{u}) = \phi d_m \mathbf{I} + |\mathbf{u}|(d_l \mathbf{E}(\mathbf{u}) + d_t(\mathbf{I} - \mathbf{E}(\mathbf{u}))), \tag{2.6}$$

where the unknowns are p (the pressure in the fluid mixture), $\mathbf{u}$ (the Darcy velocity of the mixture, i.e., the volume of fluid flowing cross a unit across-section per unit time) and c (the concentration of the interested species, i.e., the amount of the species per unit volume of the fluid mixture). $\varphi = \varphi(x)$ is the porosity of the medium, uniformly bounded above and below by positive numbers. The $\mathbf{E}(\mathbf{u})$ is the tensor that projects onto the $\mathbf{u}$ direction, whose (i,j) component is $(\mathbf{E}(\mathbf{u}))_{ij} = \frac{u_i u_j}{|\mathbf{u}|^2}$; d_m is the molecular diffusivity and assumed to be strictly positive; d_l and d_t are the longitudinal and the transverse dispersivities, respectively, and are assumed to be nonnegative. The imposed external total flow rate q is sum of sources (injection) and sinks (extraction) and is assumed to be bounded. Concentration c^* in the source term is the injected concentration c_w if $q \geq 0$ and is the resident concentration c if $q < 0$. Here, we assume that the $a(c)$ is a globally Lipschitz continuous function of c, and is uniformly symmetric positive definite and bounded.

3 Discontinuous Galerkin method for the concentration

3.1 Notation

Let $T_h = (K)$ be a sequence of finite element partitions of Ω. Let Γ_I denote the set of all interior edges, Γ_B the set of the edges e on $\partial\Omega$, and $\Gamma_h = \Gamma_B + \Gamma_I$. K^+, K^- be two adjacent elements of T_h; let x be an arbitrary point of the set $e = \partial K^+ \cap \partial K^-$, which is assumed to have a nonzero $(d - 1)$ dimensional measure; and let n^+, n^- be the corresponding outward unit normals at that point. Let $(\mathbf{u}, p)$ be a function smooth inside each element $K^\pm$ and let us denote by $(\mathbf{u}^\pm, p^\pm)$ the traces of $(\mathbf{u}, p)$ on e from the interior of $K^\pm$. Then we define the mean values $\{\{\cdot\}\}$ and jumps $[[\cdot]]$ at $x \in \{e\}$ as

$$[[\mathbf{u}]] = \mathbf{u}^+ \cdot \mathbf{n}^+ + \mathbf{u}^- \cdot \mathbf{n}^-, \ \{\{\mathbf{u}\}\} = \frac{1}{2}(\mathbf{u}^+ + \mathbf{u}^-), \ \{\{p\}\} = \frac{1}{2}(p^+ + p^-), \ [[p]] = p^+ n^+ + p^- n^-.$$

For $e \in \Gamma_B$, the obvious definitions is $\{\{p\}\} = p$, $[[\mathbf{u}]] = \mathbf{u}\cdot\mathbf{n}$, with $\mathbf{n}$ denoting the outward unit normal vector on $\partial\Omega$. we define the set $\langle K, K' \rangle$ as

$$\langle K, K' \rangle := \begin{cases} \emptyset & \text{if } \operatorname{meas}_{d-1}(\partial K \cap \partial K') = 0, \\ \text{interior of } \partial K \cap \partial K' & \text{otherwise.} \end{cases}$$

For $s \geq 0$, we define

$$H^s(T_h) = \{v \in L^2(\Omega) : v|_K \in H^s(K).K \in T_h\}. \tag{3.1}$$

The usual Sobolev norm on Ω is denoted by $||\cdot||_{m,\Omega}$ [19]. The broken norms are defined, for a positive number m, as

$$|||v|||_m^2 = \sum_{K \in T_h} \|v\|_{m,K}^2. \tag{3.2}$$

The discontinuous finite element space is taken to be

$$D_r(T_h) = \{v \in L^2(\Omega) : v|_K \in P_r(K), K \in T_h\}, \tag{3.3}$$

where $P_r(K)$ denotes the space of polynomials of (total) degree less than or equal to r ($r \geq 0$) on K. Note that we present error estimators in this paper for the local space P_r, but the results also apply to the local space Q_r (the tensor product of the polynomial spaces of degree less than or equal to r in each spatial dimension) because $P_r(K) \subset Q_r(K)$.

The cut-off operator $\mathcal{M}$ is defined as

$$\begin{aligned} &\mathcal{M}(c)(x) = \min(c(x), M), \\ &\mathcal{M}(\mathbf{u})(x) = \begin{cases} \mathbf{u}(x) & \text{if } |\mathbf{u}(x)| \leq M, \\ M\mathbf{u}(x)/|\mathbf{u}(x)| & \text{if } |\mathbf{u}(x)| > M, \end{cases} \end{aligned} \tag{3.4}$$

where M is a large positive constant. By a straightforward argument, we can show that the cut-off operator $\mathcal{M}$ is uniformly Lipschitz continuous in the following sense.

Lemma 3.1 [7] (Property of operator $\mathcal{M}$) *The cut-off operator $\mathcal{M}$defined as in Equation 3.4 is uniformly Lipschitz continuous with a Lipschitz constant one, that is*

$$\begin{aligned} &\|\mathcal{M}(c) - \mathcal{M}(w)\|_{L^\infty(\Omega)} \leq \|c - w\|_{L^\infty(\Omega)}, \quad \forall c \in L^\infty(\Omega), w \in L^\infty(\Omega), \\ &\|\mathcal{M}(\mathbf{u}) - \mathcal{M}(\mathbf{v})\|_{(L^\infty(\Omega))^d} \leq \|\mathbf{u} - \mathbf{v}\|_{(L^\infty(\Omega))^d}, \quad \forall \mathbf{u} \in (L^\infty(\Omega))^d, \mathbf{v} \in (L^\infty(\Omega))^d. \end{aligned}$$

We shall also use the following inverse inequalities, which can be derived using the method in [20]. *Let $K \in T_h$, $v \in P_r(K)$ and h_K is the diameter of K. Then there exists a constant C independent of v and h_K, such that*

$$\begin{cases} \|D^q v\|_{0,\partial K} \leq C h_K^{-1/2} \|D^q v\|_K, & q \geq 0. \\ \|D^{q+1} v\|_{0,K} \leq C h_K^{-1} \|D^q v\|_{0,K}, & q \geq 0. \end{cases} \tag{3.5}$$

3.2 Discontinuous Galerkin schemes

Let $\nabla_h \cdot v$ and $\nabla_h v$ be the functions whose restriction to each element $K \in \Im$ are equal to $\nabla \cdot v$, ∇v, respectively. We introduce the bilinear form $B(c, w; \mathbf{u})$ and the linear functional $L(w; \mathbf{u}, c)$

$$\begin{aligned} B(c, w; \mathbf{u}) = {} & (\mathbf{D}(\mathbf{u})\nabla_h c, \nabla_h w) + \int_{\Gamma_h} \{\{\mathbf{D}(\mathbf{u})\nabla_h w\}\}[[c]]\mathrm{d}s - \int_{\Gamma_h} \{\{\mathbf{D}(\mathbf{u})\nabla_h c\}\}[[w]]\mathrm{d}s \\ & + \int_{\Gamma_h} C_{11}[[c]][[w]]\mathrm{d}s + (\mathbf{u} \cdot \nabla_h c, w) - \int_\Omega c q^- w \mathrm{d}x, \\ L(w; \mathbf{u}, c) = {} & \int_\Omega c_w q^+ w \mathrm{d}x, \end{aligned}$$

with

$$C_{11} = \begin{cases} c_{11} \max\{h_{K^+}^{-1}, h_{K^-}^{-1}\} & x \in \langle K^+, K^- \rangle, \\ c_{11} h_{K^+}^{-1} & x \in \partial K^+ \cap \partial\Omega, \end{cases} \tag{3.6}$$

here $c_{11} > 0$ is a constant independent of the meshsize.

We now define the weak formulation on which our mixed discontinuous method is based

$$(\phi c_t, w) + B(c, w; \mathbf{u}) = L(w; \mathbf{u}, c), \quad \forall w \in H^k(T_h). \tag{3.7}$$

Let N be a positive integer, $\triangle t = \dfrac{T}{N}$ and $t_m = m\Delta t$ for m = 0, 1, ..., N. The approximation of c_t at $t = t^{n+1}$ can be discreted by the forward difference. The DG schemes for approximating concentration are as follows. We seek $c_h \in W^{1,\infty}(0, T; D_{k-1}(T_h))$

satisfying

$$\left(\phi\frac{c_h^{n+1}-c_h^n}{\triangle t}, w_h\right) + B(c_h^{n+1}, w_h; \mathbf{u}_M^n) = L(w_h; \mathbf{u}_M^n, c_h^{n+1}),$$
$$\forall w_h \in W^{1,\infty}(0, T; D_{k-1}(T_h)), \tag{3.8}$$

where $\mathbf{u}_M^n = \mathcal{M}(\mathbf{u}_h^n)$ with the DG velocity $\mathbf{u}_h$ defined below

$$\mathbf{u}_h^n = -a(\mathcal{M}(c_h^n))\nabla p_h^n, \quad x \in K, K \in T_h.$$

4 A stabilized mixed DG method for the velocity and pressure

4.1 Elimination for the flux variable *u*

Letting $\alpha(c) = a(c)^{-1}$. For the velocity and pressure, we define the following forms

$$a(\mathbf{u}, \mathbf{v}; c) = (\alpha(c)\mathbf{u}, \mathbf{v}), \tag{4.1}$$

$$b(p, \mathbf{v}) = (p, \nabla_h \cdot \mathbf{v}) - \int_{\Gamma_I} \{\{p\}\}[[\mathbf{v}]]\mathrm{d}s - \int_{\Gamma_B} \{\{\mathbf{v}\}\}[[p]]\mathrm{d}s. \tag{4.2}$$

The discrete problem for the velocity and pressure can be written as: find $\mathbf{u}_h \in (D_{l-2}(T_h))^d$, $(l \geq 2)$, $p_h \in D_{l-1}(T_h)$ such as

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}; c) - b(p_h, \mathbf{v}) = 0, & \forall \mathbf{v} \in (D_{l-2}(T_h))^d, \\ \qquad\qquad b(\psi, \mathbf{u}_h) = (\psi, q), & \forall \psi \in D_{l-1}(T_h). \end{cases} \tag{4.3}$$

In order to eliminate the flux variable, we first recall a useful identity, that holds for vectors $\mathbf{u}$ and scalars ψ piecewise smooth on T_h:

$$\sum_{K\in T_h} \int_{\partial K} \mathbf{v} \cdot \mathbf{n}\psi \,\mathrm{d}s = \int_{\Gamma_h} \{\{\mathbf{v}\}\} \cdot [[\psi]]\mathrm{d}s + \int_{\Gamma_I} [[\mathbf{v}]]\{\{\psi\}\}\mathrm{d}s. \tag{4.4}$$

Using (4.4) we have

$$\sum_K \int_K (\nabla \cdot \mathbf{u}_h\psi + \mathbf{u}_h \cdot \nabla\psi)\mathrm{d}x = \int_{\Gamma_h} \{\{\mathbf{u}_h\}\} \cdot [[\psi]]\mathrm{d}s + \int_{\Gamma_I} [[\mathbf{u}_h]]\{\{\psi\}\}\mathrm{d}s. \tag{4.5}$$

Substituting (4.5) in the first equation of (4.3) we obtain

$$(\alpha(c)\mathbf{u}_h + \nabla_h p_h, \mathbf{v}) - \int_{\Gamma_I} [[p_h]] \cdot \{\{\mathbf{v}\}\}\mathrm{d}s = 0. \tag{4.6}$$

We introduce the lift operator $R{:}L^1(\cup\partial K) \to (D_{l-2}(T_h))^d$ defined by

$$\int_\Omega R[[\psi]] \cdot \mathbf{v}\mathrm{d}x = -\int_{\Gamma_I} [[\psi]] \cdot \{\{\mathbf{v}\}\}\mathrm{d}s, \quad \forall v \in (D_{l-2}(T_h))^d. \tag{4.7}$$

From (4.6) and (4.7) we have

$$(\alpha(c)\mathbf{u}_h + \nabla_h p_h + R[[p_h]], \mathbf{v}) = 0. \tag{4.8}$$

We also introduce the L^2-projection π onto $(D_{l-2}(T_h))^d$

$$(\pi\mathbf{w}, \mathbf{v}) = (\mathbf{w}, \mathbf{v}), \quad \forall \mathbf{v} \in (D_{l-2}(T_h))^d. \tag{4.9}$$

Equation 4.8 gives now

$$\alpha(c)\mathbf{u}_h = -(\pi\nabla_h p_h + R[[p_h]]). \tag{4.10}$$

Noting that $\nabla_h D_{l-1}(T_h) \subset (D_{l-2}(T_h))^d$, we have $\pi\nabla_h p_h \equiv \nabla_h p_h$ for all $p_h \in D_{l-1}(T_h)$. The Equation 4.10 gives

$$\alpha(c)\mathbf{u}_h = -(\nabla_h p_h + R[[p_h]]). \tag{4.11}$$

Using (4.5) and the lifting operator R defined in (4.7) we have

$$\begin{aligned} b(\psi, \mathbf{u}_h) &= -(\mathbf{u}_h, \nabla_h\psi) + \int_{\Gamma_I} [[\psi]] \cdot \{\{\mathbf{u}\}\}\mathrm{d}s, \\ &= -(\mathbf{u}_h, \nabla_h\psi + R[[\psi]]). \end{aligned} \tag{4.12}$$

Substituting (4.12) in the second equation of (4.3) and using (4.11) we have

$$(a(c)(\nabla_h p_h + R[[p_h]]), \nabla_h\psi + R[[\psi]]) = (q, \psi). \tag{4.13}$$

For future reference, it is convenient to rewrite (4.13) as follows

$$A_{\mathrm{BR}}(p_h, \psi) = (q, \psi), \quad \forall\psi \in D_{l-1}(T_h), \tag{4.14}$$

where

$$A_{\mathrm{BR}}(p_h, \psi) = (a(c)(\nabla_h p_h + R[[p_h]]), \nabla_h\psi + R[[\psi]]). \tag{4.15}$$

4.2 Stabilization of formulation (4.3)

We write first (4.3) in the equivalent form: find $(\mathbf{u}_h, p_h) \in (D_{l-2}(T_h))^d \times D_{l-1}(T_h)$ such that

$$A(\mathbf{u}_h, \mathbf{v}; p_h, \psi; c) = l(\psi), \quad \forall(\mathbf{v}, \psi) \in (D_{l-2}(T_h))^d \times D_{l-1}(T_h), \tag{4.16}$$

where

$$A(\mathbf{u}_h, \mathbf{v}; p_h, \psi; c) = a(\mathbf{u}_h, \mathbf{v}; c) - b(p_h, \mathbf{v}) + b(\psi, \mathbf{u}_h), \quad l(\psi) = (q, \psi). \tag{4.17}$$

In a sense, (4.16) can be seen as a Darcy problem. The usual way to stabilized it is to introduce penalty terms on the jumps of p and/or on the jumps of u. In [2], Masud and Hughes introduced a stabilized finite element formulation in which an appropriately weighted residual of the Darcy law is added to the standard mixed formulation. In Hughes-Masud-Wan [5], the method was extend within the discontinuous Galerkin framework. A family of mixed finite element discretizations of the Darcy flow equations using totally discontinuous elements was introduced in [6]. In this paper, we consider the following stabilized formulation which includes the methods of [2,6] and [5].

The stabilized formulation of (4.16) is

$$A_{\mathrm{stab}}(\mathbf{u}_h, \mathbf{v}; p_h, \psi; c) = l_{\mathrm{stab}}(\psi), \quad \forall(\mathbf{v}, \psi) \in (D_{l-2}(T_h))^d \times D_{l-1}(T_h), \tag{4.18}$$

where

$$\begin{aligned} A_{\mathrm{stab}}(\mathbf{u}, \mathbf{v}; p, \psi; c) &= A(\mathbf{u}, \mathbf{v}; p, \psi; c) + \gamma e(p, \psi) \\ &\quad + \beta\theta\left(\mathbf{u} + a(c)\nabla_h p, -\alpha(c)\mathbf{v} + \delta\nabla_h\psi\right), \\ l_{\mathrm{stab}}(\psi) &= l(\psi), \\ e(p, \psi) &= a(c)\int_{\Gamma_h} C_{11}[[p]][[\psi]]\mathrm{d}s, \end{aligned} \tag{4.19}$$

where γ and β are chosen as the following (i) $\gamma = 1$, $\beta = 1$. (ii) $\gamma = 0$, $\beta = 1$, δ could assume either the value +1 or the value -1. The definition of θ will be given in the following content.

5 A mixed stabilized DG method for the incompressible miscible displacement problem

By combining (3.8) with (4.18), we have the stabilized DG for the approximating (2.1)-(2.5): seek $c_h \in W^{1,\infty}(0, T; D_{k-1}(T_h)) =: W_h$, $p_h \in W^{1,\infty}(0, T; D_{l-1}(T_h)) =: Q_h$ and $\mathbf{u}_h \in (W^{1,\infty}(0, T; D_{l-2}(T_h)))^d =: V_h$ satisfying

$$\begin{cases} (\phi \dfrac{c_h^{n+1} - c_h^n}{\triangle t}, w) + B(c_h^{n+1}, w; \mathbf{u}_M^n) = L(w; \mathbf{u}_M^n, c_h^{n+1}), \quad \forall w \in W_h, \\ A_{\text{stab}}(\mathbf{u}_h^n, \mathbf{v}; p_h^n, \psi; \mathcal{M}(c_h^n)) = l_{\text{stab}}(\psi), \quad \forall (\mathbf{v} \times \psi) \in (V_h \times Q_h). \end{cases} \tag{5.1}$$

We define the "stability norm" by

$$\|(\mathbf{u}, p)\|_{\text{stab}} = \left\{ \frac{1}{2} \| |\alpha^{1/2}(c)\mathbf{u}| \|_0^2 + \|p\|_{1,h}^2 \right\}^{1/2}, \tag{5.2}$$

where

$$\begin{aligned} \|p\|_{1,h}^2 &= \frac{1}{2}\|a^{1/2}(c)\nabla_h p\|_0^2 + \|a^{1/2}(c)[[p]]\|_{0,\Gamma_h}^2, \\ \|a^{1/2}(c)[[p]]\|_{0,\Gamma_h}^2 &= \int_{\Gamma_h} a(c)C_{11}[[p]] \cdot [[p]] \mathrm{d}s, \quad \|\nabla_h p\|_0^2 = \sum_K \|\nabla p\|_{0,K}^2. \end{aligned} \tag{5.3}$$

6 Stability and consistency

From [6], we can state the following results.

Lemma 6.1 [6] *There exist two positive constants C_1 and C_2, depending only on the minimum angle of the decomposition and on the polynomial degree*

$$C_1 \|R[[\psi]]\|_{0,\Omega}^2 \le \sum_{e \in \Gamma_I} h_e^{-1} \|[[\psi]]\|_{0,e}^2 \le C_2 \|R[[\psi]]\|_{0,\Omega}. \tag{6.1}$$

Lemma 6.2 [6] *There exists two positive constants C_1 and C_2, depending only on the minimum angle of the decomposition such that*

$$C_1 \|R[[\psi]]\|_{0,\Omega}^2 \le \sum_{e \in \Gamma_I} h_e^{-1} \|[[\psi]]\|_{0,e}^2 \le C_2 (\|R[[\psi]]\|_{0,\Omega}^2 + \|\nabla_h \psi\|_0^2), \quad \psi \in H^2(T_h). \tag{6.2}$$

Lemma 6.3 [6] *Let $\mathcal{H}$ be a Hilbert spaces, and λ and μ positive constants. Then, for every ξ and η in $\mathcal{H}$ we have*

$$\lambda \|\xi + \eta\|_{\mathcal{H}}^2 + \mu \|\eta\|_{\mathcal{H}}^2 \ge \frac{\lambda\mu}{2(\lambda + \mu)} (\|\xi\|_{\mathcal{H}}^2 + \|\eta\|_{\mathcal{H}}^2). \tag{6.3}$$

Theorem 6.1 (Stability) *For $\delta = 1$, problem (4.18) is stable for all $\theta \in (0,1)$.*

Proof Consider first the case $\gamma = 1$, $\beta = 1$. From the definition of $A_{\text{stab}}(\cdot,\cdot;\cdot,\cdot;\cdot)$, we have

$$A_{\text{stab}}(\mathbf{u}_h, \mathbf{u}_h; p_h, p_h; c) = a(\mathbf{u}_h, \mathbf{u}_h; c) + e(p_h, p_h) + \theta(\mathbf{u}_h + a(c)\nabla_h p_h, -\alpha(c)\mathbf{u}_h + \nabla_h p_h). \tag{6.4}$$

We remark that (6.4) can be rewritten as

$$A_{\text{stab}}(\mathbf{u}_h, \mathbf{u}_h; p_h, p_h; c) = (1-\theta)\| |\alpha^{1/2}(c)\mathbf{u}| \|_0^2 + \theta \|a^{1/2}(c)\nabla_h p\|_0^2 + \|a^{1/2}(c)[[p]]\|_{0,\Gamma_h}^2, \tag{6.5}$$

and the stability in the norm (5.2) follows from $\theta = \frac{1}{2}$.

Consider now the case $\gamma = 0$, $\beta = 1$. Using the equivalent expressions (4.11) and (4.12) for the first and second equation of (4.3), respectively, the problem (4.18) for $\gamma = 0$ can be rewritten as: find $\mathbf{u}_h \in (D_{l-2}(T_h))^d$, $p_h \in D_{l-1}(T_h)$ such that

$$\begin{cases} (\alpha(c)\mathbf{u}_h + \nabla_h p_h + R[[p_h]], v) - \theta(\alpha(c)\mathbf{u}_h + \nabla_h p_h, v) = 0, \\ \qquad -(\mathbf{u}_h, \nabla_h \psi + R[[\psi]]) + \delta\theta(\mathbf{u}_h + a(c)\nabla_h \psi, \nabla_h \psi) = (q, \psi). \end{cases} \tag{6.6}$$

From the first equation in (6.6) and (4.9) we have

$$\alpha(c)\mathbf{u}_h = -(\nabla_h p_h + \frac{1}{1-\theta} R[[p_h]]). \tag{6.7}$$

Substituting the expression (6.7) in the second equation of (6.6) for $\delta = 1$, we have

$$A_{\mathrm{BR}}(p_h, \psi) + \frac{\theta}{1-\theta}\int_\Omega a(c)R[[p_h]] \cdot R[[\psi]]\mathrm{d}x = (q, \psi), \quad \forall \psi \in D_{l-1}(T_h). \tag{6.8}$$

Denote by $B_{1h}(\cdot,\cdot)$ the bilinear form (6.8), we have

$$B_{1h}(\psi, \psi) = \|a(c)(\nabla_h \psi + R[[\psi]])\|_{0,\Omega} + \frac{\theta}{1-\theta}\|a(c)^{1/2} R[[\psi]]\|^2_{0,\Omega}, \tag{6.9}$$

and the stability in the norm (5.3) follows from Lemma 6.1. This completes the proof. □

Theorem 6.2 For δ = -1, problem (4.18) is stable for all $\theta < 0$.

Proof Consider first the case $\gamma = 1$, $\beta = 1$. The problem (4.18) for δ = -1 reads

$$\begin{aligned} A_{\mathrm{stab}}(\mathbf{u}_h, \mathbf{u}_h; p_h, p_h; c) = a(\mathbf{u}_h, \mathbf{u}_h; c) + \theta(\mathbf{u}_h + a(c)\nabla_h p_h, -\alpha(c)\mathbf{u}_h - \nabla_h p_h) \\ + e(p_h, p_h). \end{aligned} \tag{6.10}$$

Using the arithmetic-geometric mean inequality, we have

$$\begin{aligned} A_{\mathrm{stab}}(\mathbf{u}_h, \mathbf{u}_h; p_h, p_h; c) \geq (1-2\theta)\||\alpha^{1/2}(c)\mathbf{u}\|^2_0 - 2\theta\|a^{1/2}(c)\nabla_h p\|^2_0 \\ + \|a^{1/2}(c)[[p]]\|^2_{0,\Gamma_h}, \end{aligned} \tag{6.11}$$

and since $\theta < 0$ the result follows.

Consider now the case $\gamma = 0$, $\beta = 1$. From (6.7) the second equation of (6.6) for δ = -1 can be written as

$$A_{\mathrm{BR}}(p_h, \psi) + \frac{2\theta}{1-\theta}(R[[p_h]], a(c)\nabla_h \psi) + \frac{\theta}{1-\theta}\int_\Omega a(c)R[[p_h]] \cdot R[[\psi]]\mathrm{d}x = (q, \psi). \tag{6.12}$$

We remark that formulation (6.12) can be rewritten as

$$\frac{1}{1-\theta}A_{\mathrm{BR}}(p_h, \psi) - \frac{\theta}{1-\theta}A_{\mathrm{BO}}(p_h, \psi) = (q, \psi), \tag{6.13}$$

where $A_{BO}(p_h, \psi)$ is introduced by Baumann and Oden [14], and given by

$$A_{\mathrm{BO}}(p_h, \psi) := \int_\Omega a(c)(\nabla_h p_h - R[[p_h]]) \cdot (\nabla_h \psi + R[[\psi]])\mathrm{d}x + \int_\Omega a(c)R[[p_h]] \cdot R[[\psi]]\mathrm{d}x. \tag{6.14}$$

Denote by $B_{2h}(\cdot,\cdot)$ the bilinear form (6.13), we have

$$B_{2h}(\psi, \psi) = \frac{1}{1-\theta}\|a^{1/2}(c)(\nabla_h \psi + R[[\psi]])\|_{0,\Omega} - \frac{\theta}{1-\theta}\|a^{1/2}(c)\nabla_h \psi\|^2_{0,\Omega}, \tag{6.15}$$

and since $\theta < 0$ the result follows again from Lemma 6.3 and 6.1. □

Theorem 6.3 (Consistency) *If p,c and* $\mathbf{u}$ *are the solution of (2.1)-(2.5) and are essentially bounded, then*

$$\begin{cases} (\phi c_t, w) + B(c, w; \mathbf{u}) = L(w; \mathbf{u}, c), & \forall w \in L^2(0,T; H^k(T_h)) \\ A_{\text{stab}}(\mathbf{u}, \mathbf{v}; p, \psi; c) = l_{\text{stab}}(\psi), & \forall (\mathbf{v} \times \psi) \in ((L^2(0,T; H^{l-1}(T_h)))^d \times L^2(0,T; H^l(T_h))) \end{cases} \tag{6.16}$$

provided that the constant M for the cut-off operator is sufficiently large.

To summarize, for all the bilinear forms in (6.4), (6.10), (6.8) or (6.13) we have: $\exists C > 0$ such that

$$B_{1h}(\psi, \psi) \geq C\|\psi\|_{1,h}^2, \quad B_{2h}(\psi, \psi) \geq C\|\psi\|_{1,h}^2, \quad \forall \psi \in D_{l-1}(T_h), \tag{6.17}$$

and $\exists C > 0$ such that

$$A(\mathbf{v}, \mathbf{v}; \psi, \psi; c)_{\text{stab}} \geq C\|(\mathbf{v}, \psi)\|_{\text{stab}}^2, \quad \forall (\mathbf{v}, \psi) \in (D_{l-2}(T_h))^d \times D_{l-1}(T_h), \tag{6.18}$$

where (6.17) clearly holds for every $\theta \in (0,1)$ for the case ((6.4), (6.8)), and for every $\theta < 0$ for the case ((6.10), (6.13)). On the other hand, since $\nabla_h D_{l-1}(T_h) \subset (D_{l-2}(T_h))^d$ holds, boundedness of the bilinear form in (6.8) and (6.13) follows directly from the boundedness of the bilinear forms A_{BR} and A_{BO}, as proved in [13], thanks to the equivalence of the norms (6.1) and (6.2). Thus, we have: $\exists C > 0$ such that

$$B_{1h}(p_h, \psi) \leq C\|p_h\|_{1,h}\|\psi\|_{1,h}, \; B_{2h}(p_h, \psi) \leq C\|p_h\|_{1,h}\|\psi\|_{1,h}, \quad \forall p_h, \psi \in D_{l-1}(T_h). \tag{6.19}$$

7 Error estimates

Let $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{c})$ be an interpolation of the exact solution ($\mathbf{u}$, p, c) such that

$$\begin{cases} a(\tilde{\mathbf{u}}, v; c) - b(\tilde{p}, v) = 0, & \forall v \in (D_{l-2}(T_h))^d, \\ b(\psi, \tilde{u}) + e(\tilde{p}, \psi) = (q, \psi), & \forall \psi \in D_{l-1}(T_h), \\ (\tilde{c} - c, w) = 0, & \forall w \in D_{k-1}(T_h). \end{cases} \tag{7.1}$$

Let us define interpolation errors, finite element solution errors and auxiliary errors

$$\begin{aligned} &\xi_1 = \tilde{\mathbf{u}} - \mathbf{u}_h, \; \xi_2 = \tilde{\mathbf{u}} - \mathbf{u}, \; e_{\mathbf{u}} = \mathbf{u} - \mathbf{u}_h = \xi_1 - \xi_2; \\ &\eta_1 = \tilde{p} - p_h, \; \eta_2 = \tilde{p} - p, \; e_p = p - p_h = \eta_1 - \eta_2; \\ &\tau_1 = \tilde{c} - c_h, \; \tau_2 = \tilde{c} - c, \; e_c = c - c_h = \tau_1 - \tau_2. \end{aligned}$$

It was proven in [18] that

$$\||\alpha^{1/2}(c)\xi_2|\|_0^2 + \|a^{1/2}(c)[[\eta_2]]\|_{0,\Gamma_h}^2 \leq Ch^{2l-2}(\|\mathbf{u}\|_{l-1}^2 + \|p\|_l^2). \tag{7.2}$$

hold for all $t \in J$ with the constant C independent only on bounds for the coefficient $\alpha(c)$, but not on c itself.

Theorem 7.1 (Error estimate for the velocity and pressure) *Let* ($\mathbf{u}$, *p*, *c*) *be the solution to (2.1)-(2.5), and assume* $p \in L^2(0, T; H^l(T_h))$, $\mathbf{u} \in (L^2(0, T; H^{l-1}(T_h)))^d$ *and* $c \in L^2(0, T; H^k(T_h))$. *We further assume that p,* ∇p, *c and* ∇c *are essentially bounded. If the constant M for the cut-off operator is sufficiently large, then there exists a constant C independent of h such that*

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\text{stab}}^2(t) \leq C(\|c - c_h\|_0^2(t) + h^{2l-2}). \tag{7.3}$$

Proof For the sake of brevity we will assume $\theta = \frac{1}{2}, \delta = 1$ in the following content. Consider the case $\gamma = 1, \beta = 1$. From the second equation of (5.1) and (6.16) we have

$$\begin{aligned}&(\alpha(c)\mathbf{u} - \alpha(\mathcal{M}(c_h))\mathbf{u}_h, \mathbf{v}) - b(p - p_h, \psi) + b(\psi, \mathbf{u} - \mathbf{u}_h) + e(p - p_h, \psi)\\ &-\frac{1}{2}(\mathbf{u}_h + a(\mathcal{M}(c_h))\nabla_h p_h), -\alpha(\mathcal{M}(c_h))\mathbf{v} + \nabla_h \psi)\\ &+\frac{1}{2}(\mathbf{u} + a(c)\nabla_h p, -\alpha(c)\mathbf{v} + \nabla_h \psi) = 0.\end{aligned} \tag{7.4}$$

That is

$$\begin{aligned}&(\alpha(c)(\mathbf{u} - \tilde{\mathbf{u}}), \mathbf{v}) + (\alpha(\mathcal{M}(c_h))(\tilde{\mathbf{u}} - \mathbf{u}_h), \mathbf{v}) + ((\alpha(c) - \alpha(\mathcal{M}(c_h)))\tilde{\mathbf{u}}, \mathbf{v}) - b(p - p_h, \mathbf{v})\\ &+b(\psi, \mathbf{u} - \mathbf{u}_h) + e(p - p_h, \psi) + \frac{1}{2}(\alpha(\mathcal{M}(c_h))\mathbf{u}_h - \alpha(c)\mathbf{u}, \mathbf{v}) + \frac{1}{2}(\mathbf{u} - \mathbf{u}_h, \nabla_h \psi)\\ &-\frac{1}{2}(\nabla_h p - \nabla_h p_h, \mathbf{v}) + \frac{1}{2}(a(c)\nabla_h p - a(\mathcal{M}(c_h))\nabla_h p_h, \nabla_h \psi) = 0.\end{aligned}$$

Choosing $v = \xi_1$, $\psi = \eta_1$ and splitting e_p according $e_p = \eta_1$- η_2, from (7.1) and we obtain

$$\begin{aligned}&\frac{1}{2}(\alpha(\mathcal{M}(c_h))\xi_1, \xi_1) + e(\eta_1, \eta_1) + \frac{1}{2}(a(\mathcal{M}(c_h))\nabla_h \eta_1, \nabla_h \eta_1) = \frac{1}{2}((\alpha(\mathcal{M}(c_h)) - \alpha(c))\tilde{\mathbf{u}}, \xi_1)\\ &-\frac{1}{2}(\alpha(c)\xi_2, \xi_1) + \frac{1}{2}(a(c)\nabla_h \eta_2, \nabla_h \eta_1) - \frac{1}{2}((a(c) - a(\mathcal{M}(c_h)))\nabla_h \tilde{p}, \nabla_h \eta_1)\\ &+\frac{1}{2}(\xi_2, \nabla_h \eta_1) - \frac{1}{2}(\nabla_h \eta_2, \xi_1).\end{aligned} \tag{7.5}$$

Let us first consider the left side of error equation (7.5)

$$\begin{aligned}&\frac{1}{2}(\alpha(\mathcal{M}(c_h))\xi_1, \xi_1) + e(\eta_1, \eta_1) + \frac{1}{2}(a(\mathcal{M}(c_h))\nabla_h \eta_1, \nabla_h \eta_1)\\ &= \frac{1}{2}(\||\alpha^{1/2}(\mathcal{M}(c_h))\xi_1|\|_0^2 + \|a^{1/2}(\mathcal{M}(c_h))\nabla_h \eta_1\|_0^2) + \|[[\eta_1]]\|_{0,\Gamma_h}^2.\end{aligned}$$

We know that (7.2) and quasi-regularity that $\nabla_h \tilde{p}, \tilde{\mathbf{u}}$ are bounded in $L^\infty(\Omega)$. So the right side of the error equation (7.5) can be bounded from below. Noting that $|\alpha(\mathcal{M}(c_h)) - \alpha(c)| \le C|c_h - c|$, we have

$$|(\alpha(\mathcal{M}(c_h)) - \alpha(c))\tilde{\mathbf{u}}, \xi_1)| \le C\|c - c_h\|_0^2 + \varepsilon\||\xi_1|\|_0^2. \tag{7.6}$$

The second and the third terms of the right side of the error equation (7.5) can be bounded using Cauchy-Schwartz inequality and approximation results,

$$|(\alpha(c)\xi_2, \xi_1)| \le \|\alpha(c)\|_{0,\infty}\|\xi_2\|_0\|\xi_1\|_0 \le \varepsilon\||\xi_1|\|_0^2 + Ch^{2l-2}, \tag{7.7}$$

$$|(a(c)\nabla_h \eta_2, \nabla_h \eta_1)| \le \varepsilon\|\nabla_h \eta_1\|_0^2 + Ch^{2l-2}. \tag{7.8}$$

The fourth term can be bounded in a similar way as that for the first term

$$|(a(c) - a(\mathcal{M}(c_h))\nabla_h \tilde{p}, \nabla_h \eta_1)| \le C\|c - c_h\|_0^2 + \varepsilon\|\nabla_h \eta_1\|_0^2. \tag{7.9}$$

The last two terms can be bounded as follows

$$(\xi_2, \nabla_h \eta_1) \le \varepsilon\|\nabla_h \eta_1\|_0^2 + Ch^{2l-2}, \quad (\nabla_h \eta_2, \xi_1) \le \varepsilon\||\xi_1|\|_0^2 + Ch^{2l-2}. \tag{7.10}$$

Substituting all these inequalities into Equation 7.5, we have

$$\begin{aligned}&\frac{1}{2}(\||\alpha^{1/2}(\mathcal{M}(c_h))\xi_1|\|_0^2 + \|a^{1/2}(\mathcal{M}(c_h))\nabla_h\eta_1\|_0^2) + \|a^{1/2}(c)[[\eta_1]]\|_{0,\Gamma_h}^2 \\ &\le \varepsilon(\||\xi_1|\|_0^2 + \|\nabla_h\eta_1\|_0^2) + C(\|c - c_h\|_0^2 + h^{2l-2}).\end{aligned} \tag{7.11}$$

Using the facts $\frac{1}{C}\mathbf{I} \le \alpha(\mathcal{M}(c_h)) \le C\mathbf{I}$, $\frac{1}{C}\mathbf{I} \le \alpha(c) \le C\mathbf{I}$, and $\frac{1}{C}\mathbf{I} \le a(\mathcal{M}(c_h)) \le C\mathbf{I}$, $\frac{1}{C}\mathbf{I} \le a(c) \le C\mathbf{I}$ we have

$$\|(\xi_1, \eta_1)\|_{stab}^2 \le C(\|c - c_h\|_0^2 + h^{2l-2}). \tag{7.12}$$

The theorem follows from the triangle inequality.

Now consider the case $\gamma = 0$, $\beta = 1$. The bilinear form (6.8) from the second equation of (5.1) for $\theta = \frac{1}{2}$ reads

$$A_{\mathrm{BR}}(p_h, \psi; c_h) + \int_\Omega a(\mathcal{M}(c_h))R[[p_h]] \cdot R[[\psi]]\mathrm{d}x = (q, \psi), \tag{7.13}$$

where

$$A_{\mathrm{BR}}(p_h, \psi; c_h) = (a(\mathcal{M}(c_h))(\nabla_h p_h + R[[p_h]]), \nabla_h\psi + R[[\psi]]).$$

Replacing (6.8) with $p_h = p$ and subtracting it from (7.13) we finally obtain

$$a(c)(\nabla_h p, \nabla_h\psi + R[[\psi]]) - A_{\mathrm{BR}}(p_h, \psi; c_h) = 0. \tag{7.14}$$

Choosing $\psi = \eta_1$, we have

$$\begin{aligned}&A_{\mathrm{BR}}(\eta_1, \eta_1; c_h) + \int_\Omega a(\mathcal{M}(c_h))R[[\eta_1]] \cdot R[[\eta_1]]\mathrm{d}x = A_{\mathrm{BR}}(\eta_2, \eta_1) \\ &+ \int_\Omega a(c)R[[\eta_2]] \cdot R[[\eta_1]]\mathrm{d}x + (a(\mathcal{M}(c_h)) - a(c))((\nabla_h\tilde{p} + R[[\tilde{p}]], \nabla_h\eta_1 \\ &+R[[\eta_1]]) + \int_\Omega a(c)R[[\tilde{p}]] \cdot R[[\eta_1]]\mathrm{d}x).\end{aligned} \tag{7.15}$$

Let us first estimate the left side of (7.15). From (6.17) and using the fact $\frac{1}{C}\mathbf{I} \le a(\mathcal{M}(c_h)) \le C\mathbf{I}$, $\frac{1}{C}\mathbf{I} \le a(c) \le C\mathbf{I}$, we have

$$A_{\mathrm{BR}}(\eta_1, \eta_1; c_h) + \int_\Omega a(\mathcal{M}(c_h))R[[\eta_1]] \cdot R[[\eta_1]]\mathrm{d}x \ge C\|\eta\|_{1,h}^2. \tag{7.16}$$

The first and the second terms of the right side of (7.15) can be bounded using Lemma 6.1 and (3.5)

$$\begin{aligned}A_{\mathrm{BR}}(\eta_2, \eta_1) + \int_\Omega a(c)R[[\eta_2]] \cdot R[[\eta_1]]\mathrm{d}x &\le C\|\eta_2\|_{1,h}\|\eta_1\|_{1,h}, \\ &\le \varepsilon\|\eta_1\|_{1,h}^2 + Ch^{2l-2}.\end{aligned} \tag{7.17}$$

Note that $\nabla_h\tilde{p}, \tilde{p}$ are bounded in $L^\infty(\Omega)$ and $|a(\mathcal{M}(c_h)) - a(c)| \le C\|c_h - c\|$, we have

$$\begin{aligned}&(a(\mathcal{M}(c_h)) - a(c))\left\{(\nabla_h\tilde{p} + R[[\tilde{p}]], \nabla_h\eta_1 + R[[\eta_1]]) + \int_\Omega a(c)R[[\tilde{p}]]\cdot R[[\eta_1]]\mathrm{d}x\right\}\\&\le C\|c - c_h\|_0\|\eta_1\|_{1,h} \le C\|c - c_h\|_0^2 + \varepsilon\|\eta_1\|_{1,h}^2.\end{aligned} \tag{7.18}$$

Substituting all these inequalities into the (7.15) and using the triangle inequality we have

$$\|p - p_h\|_{1,h}^2 \le C(\|c - c_h\|_0^2 + h^{2l-2}). \tag{7.19}$$

We easily deduce,using (7.19)

$$\||\alpha^{1/2}(c)(\mathbf{u} - \mathbf{u}_h)|\|_0^2 \le C\|p - p_h\|_{1,h}^2 \le C(\|c - c_h\|_0^2 + h^{2l-2}), \tag{7.20}$$

which completes the proof. □.

From [7], we state two lemmas for the properties of the dispersion-diffusion tensor, which will be used to prove error estimates for the concentration.

Lemma 7.1 [7] (Uniform positive definiteness of $\mathbf{D}(\mathbf{u})$) *Let* $\mathbf{D}(\mathbf{u})$ *defined as in Equation 2.6, where* $\varphi d_m(x) \ge 0$, $d_l(x) \ge 0$ *and* $d_t(x) \ge 0$ *are non-negative functions of* $x \in \Omega$. *Then*

$$\mathbf{D}(u)\nabla_h c\cdot\nabla_h c \ge (\phi d_m + \min(d_l, d_t)|\mathbf{u}|)|\nabla_h c|^2. \tag{7.21}$$

If, in addition, $\phi d_m(x) \ge d_{m,*} > 0$ *uniformly in the domain* Ω, *then* $\mathbf{D}(\mathbf{u})$ *is uniformly positive definite in* Ω:

$$\mathbf{D}(u)\nabla_h c\cdot\nabla_h c \ge d_{m,*}|\nabla_h c|^2. \tag{7.22}$$

Lemma 7.2 [7] (Uniform Lipschitz continuity of $\mathbf{D}(\mathbf{u})$) *Let* $\mathbf{D}(\mathbf{u})$ *defined as in Equation 2.6, where* $d_m(x) \ge 0$, $d_l(x) \ge 0$ *and* $d_t(x) \ge 0$ *are non-negative functions of* $x \in \Omega$, *and the dispersivities* d_l *and* d_t *is uniformly bounded, i.e.,* $d_l(x) \le d_l^*$ *and* $d_t(x) \le d_t^*$. *Then*

$$\|\mathbf{D}(u) - \mathbf{D}(v)\|_{(L^2(\Omega))^{d\times d}} \le k_D\|\mathbf{u} - \mathbf{v}\|_{(L^2(\Omega))^d}. \tag{7.23}$$

where $k_D = (7d_t^* + 6d_l^*)d^{3/2}$*is a fixed number (d = 2 or 3 is the dimension of domain* Ω*).*

Theorem 7.2 (Error estimate for concentration) *Let (*$\mathbf{u}$, p, c*) be the solution to (2.1)-(2.5), and assume* $p \in L^2(0, T; H^l(T_h))$, $\mathbf{u} \in (L^2(0, T; H^{l-1}(T_h)))^d$ *and* $c \in L^2(0, T; H^k(T_h))$. *We further assume that* p, ∇p, c *and* ∇c *are essentially bounded. If the constant M for the cut-off operator is sufficiently large, then there exists a constant C independent of h and* Δt *such that*

$$\begin{aligned}&\|\sqrt{\phi}(c - c_h)\|_{L^\infty(0,T;L^2(\Omega))} + \Big(\sum_{i=1}^N \triangle t(\||\mathbf{D}^{1/2}(\mathbf{u}^{i-1})\nabla_h(c^i - c_h^i)|\|_0^2 + \|[[c^i - c_h^i]]\|_{0,\Gamma_h}^2)\Big)^{1/2}\\&\le C(\triangle t + h^{k-1} + h^{l-1}).\end{aligned} \tag{7.24}$$

Proof The first equation of (5.1) is

$$(\phi\frac{c_h^{n+1} - c_h^n}{\triangle t}, w) + B(c_h^{n+1}, w; \mathbf{u}_M^n) = L(w; \mathbf{u}_M^n, c_h^{n+1}).$$

It can be written as

$$(\phi\frac{\tau_2^{n+1}-\tau_2^n}{\triangle t},w)-(\phi\frac{\tau_1^{n+1}-\tau_1^n}{\triangle t},w)+(\phi\frac{c^{n+1}-c^n}{\triangle t},w)+B(c_h^{n+1},w;\mathbf{u}_M^n)=L(w;\mathbf{u}_M^n,c_h^{n+1}).$$

Subtracting the DG scheme equation from the weak formulation, we have for any $w \in D_{k-1}(T_h)$

$$(\phi c_t,w)-\left(\phi\frac{\tau_2^{n+1}-\tau_2^n}{\triangle t},w\right)+\left(\phi\frac{\tau_1^{n+1}-\tau_1^n}{\triangle t},w\right)-\left(\phi\frac{c^{n+1}-c^n}{\triangle t},w\right)-B(c_h^{n+1},w;\mathbf{u}_M^n)$$
$$+B(c^{n+1},w;\mathbf{u}^n)=L(w;\mathbf{u}^n,c^{n+1})-L(w;\mathbf{u}_M^n,c_h^{n+1}).$$

that is

$$(\phi c_t,w)+\left(\phi\frac{\tau_1^{n+1}-\tau_1^n}{\triangle t},w\right)+B(\tau_1^{n+1},w;\mathbf{u}_M^n)=\left(\phi\frac{\tau_2^{n+1}-\tau_2^n}{\triangle t},w\right)+\left(\phi\frac{c^{n+1}-c^n}{\triangle t},w\right)$$
$$+B(\tau_2^{n+1},w;\mathbf{u}_M^n)+B(c^{n+1},w;\mathbf{u}_M^n)-B(c^{n+1},w;\mathbf{u}^n)$$
$$+L(w;\mathbf{u}^n,c^{n+1})-L(w;\mathbf{u}_M^n,c_h^{n+1}).$$

Choosing $w = {\tau_1}^{n+1}$, we obtain

$$\begin{aligned}&\left(\phi\frac{\tau_1^{n+1}-\tau_1^n}{\triangle t},\tau_1^{n+1}\right)+B(\tau_1^{n+1},\tau_1^{n+1};\mathbf{u}_M^n)=\left(\phi\frac{c^{n+1}-c^n}{\triangle t},\tau_1^{n+1}\right)-(\phi c_t,\tau_1^{n+1})\\&+\left(\phi\frac{\tau_2^{n+1}-\tau_2^n}{\triangle t},\tau_1^{n+1}\right)+B(\tau_2^{n+1},\tau_1^{n+1};\mathbf{u}_M^n)+B(c^{n+1},\tau_1^{n+1};\mathbf{u}_M^n)\\&-B(c^{n+1},\tau_1^{n+1};\mathbf{u}^n)+L(\tau_1^{n+1};\mathbf{u}^n,c^{n+1})-L(\tau_1^{n+1};\mathbf{u}_M^n,c_h^{n+1}).\end{aligned}\tag{7.25}$$

Let us first consider the left side of the error equation (7.25). The first term can be bounded as

$$\left(\phi\frac{\tau_1^{n+1}-\tau_1^n}{\triangle t},\tau_1^{n+1}\right)\geq\frac{\phi}{2\triangle t}((\tau_1^{n+1},\tau_1^{n+1})-(\tau_1^n,\tau_1^n)).\tag{7.26}$$

The second term of Equation 7.25 is

$$B(\tau_1^{n+1},\tau_1^{n+1};\mathbf{u}_M^n)=(\mathbf{D}(\mathbf{u}_M^n)\nabla_h\tau_1^{n+1},\nabla_h\tau_1^{n+1})+(\mathbf{u}_M^n\cdot\nabla_h\tau_1^{n+1},\tau_1^{n+1})$$
$$-\int_\Omega q^-(\tau_1^{n+1})^2\mathrm{d}x+\int_{\Gamma_h}C_{11}[[\tau_1^{n+1}]][[\tau_1^{n+1}]]\mathrm{d}s.$$

The second term of $B(\cdot,\cdot;\cdot)$ can be estimated using the boundedness of $\mathbf{u}_M$ and $(\mathbf{u}_M^n\cdot\nabla_h\tau_1^{n+1},\tau_1^{n+1})\leq\varepsilon|||\mathbf{D}^{1/2}(\mathbf{u}^n)\nabla_h\tau_1^{n+1}|||_0^2+C\|\sqrt{\phi}\tau_1^{n+1}\|_0^2$:

$$(\mathbf{u}_M^n\cdot\nabla_h\tau_1^{n+1},\tau_1^{n+1})\leq\varepsilon|||\mathbf{D}^{1/2}(\mathbf{u}^n)\nabla_h\tau_1^{n+1}|||_0^2+C\|\sqrt{\phi}\tau_1^{n+1}\|_0^2.\tag{7.27}$$

Thus

$$\begin{aligned}&\left(\phi\frac{\tau_1^{n+1}-\tau_1^n}{\triangle t},\tau_1^{n+1}\right)+B(\tau_1^{n+1},\tau_1^{n+1};\mathbf{u}_M^n)\\&\geq\frac{\phi}{2\triangle t}(\|\tau_1^{n+1}\|_0^2-\|\tau_1^n\|_0^2)+|||\mathbf{D}^{1/2}(\mathbf{u}^n)\nabla_h\tau_1^{n+1}|||_0^2\\&+\|[[\tau_1^{n+1}]]\|_{0,\Gamma_h}^2-C\|\sqrt{\phi}\tau_1^{n+1}\|_0^2.\end{aligned}\tag{7.28}$$

Let us bound the right side of the error equation (7.25).

$$L(\tau_1^{n+1}; \mathbf{u}^n, c^{n+1}) - L(\tau_1^{n+1}; \mathbf{u}_M^n, c_h^{n+1}) \le C\|\sqrt{\phi}\tau_1^{n+1}\|_0^2. \tag{7.29}$$

Using Taylor series expansion, we have

$$\left(\phi\frac{c^{n+1}-c^n}{\triangle t}, \tau_1^{n+1}\right) - (\phi c_t, \tau_1^{n+1}) \le \frac{\phi}{2}\triangle t\|D_t^2 c\|_{L^2(t_k,t_{k+1};L^2(\Omega))} + C\|\sqrt{\phi}\tau_1^{n+1}\|_0^2. \tag{7.30}$$

The fourth term in the right side of the error equation (7.25) is

$$\begin{aligned}B(\tau_2^{n+1}, \tau_1^{n+1}; \mathbf{u}_M^n) = &(\mathbf{D}(\mathbf{u}_M^n)\nabla_h\tau_2^{n+1}, \nabla_h\tau_1^{n+1}) + (\mathbf{u}_M^n\cdot\nabla_h\tau_2^{n+1}, \tau_1^{n+1})\\ &-(q^-\tau_2^{n+1}, \tau_1^{n+1}) + \int_{\Gamma_h}\{\{\mathbf{D}(u_M^n)\nabla_h\tau_1^{n+1}\}\}[[\tau_2^{n+1}]]\mathrm{d}s\\ &-\int_{\Gamma_h}\{\{\mathbf{D}(u_M^n)\nabla_h\tau_2^{n+1}\}\}[[\tau_1^{n+1}]]\mathrm{d}s\\ &+\int_{\Gamma_h}C_{11}[[\tau_1^{n+1}]][[\tau_2^{n+1}]]\mathrm{d}s =: \sum_{i=1}^{6}T_i.\end{aligned}$$

Terms T_1 through T_3 can be bounded by using Cauchy-Schwartz inequality and approximation results,

$$\begin{aligned}|T_1| &\le \varepsilon\||\mathbf{D}^{1/2}(\mathbf{u}^n)\nabla_h\tau_1^{n+1}\||_0^2 + C\|\nabla_h(c^{n+1}-\tilde{c}^{n+1})\|_0^2,\\ &\le \varepsilon\||\mathbf{D}^{1/2}(\mathbf{u}^n)\nabla_h\tau_1^{n+1}\||_0^2 + Ch^{2k-2},\end{aligned} \tag{7.31}$$

and

$$|T_2| \le \varepsilon\||\mathbf{D}^{1/2}(\mathbf{u}^n)\nabla_h\tau_1^{n+1}\||_0^2 + Ch^{2k-2}, \quad |T_3| \le C(\|\sqrt{\phi}\tau_1^{n+1}\|_0^2 + h^{2k}). \tag{7.32}$$

Terms T_4 and T_5 can be estimated using inverse inequalities,

$$\begin{aligned}|T_4| &\le \varepsilon h\sum_{K\in\Gamma_h}\||\mathbf{D}^{1/2}(\mathbf{u}^n)\nabla_h\tau_1^{n+1}\||_{0,\partial K} + \frac{C}{h}\sum_{K\in\Gamma_h}\|\tau_2^{n+1}\|_{0,\partial K}^2,\\ &\le \varepsilon\||\mathbf{D}^{1/2}(\mathbf{u}^n)\nabla_h\tau_1^{n+1}\||_0^2 + Ch^{2k-2},\end{aligned} \tag{7.33}$$

and

$$\begin{aligned}|T_5| &\le \varepsilon\|[[\tau_1^{n+1}]]\|_{0,\Gamma_h}^2 + \frac{C}{h}\sum_{K\in\Gamma_h}\|\tau_2^{n+1}\|_{0,\partial K}^2,\\ &\le \varepsilon\|[[\tau_1^{n+1}]]\|_{0,\Gamma_h}^2 + Ch^{2k-2}.\end{aligned} \tag{7.34}$$

Using Cauchy-Schwartz inequality and the trace inequality, we have

$$|T_6| \le \varepsilon\|[[\tau_1^{n+1}]]\|_{0,\Gamma_h}^2 + \frac{C}{h}\sum_{K\in\Gamma_h}\|\tau_2^{n+1}\|_{0,\partial K}^2 \le \varepsilon\|[[\tau_1^{n+1}]]\|_{0,\Gamma_h}^2 + Ch^{2k-2}. \tag{7.35}$$

Noting that $[[c^{n+1}]] = 0$, if the constant M for the cut-off operator is sufficiently large, we write the last two terms in the right side of the error equation (7.25) as

$$B(c^{n+1}, \tau_1^{n+1}; \mathbf{u}_M^n) - B(c^{n+1}, \tau_1^{n+1}; \mathbf{u}^n) = ((\mathbf{D}(\mathbf{u}_M^n) - \mathbf{D}(\mathbf{u}^n))\nabla_h c^{n+1}, \nabla_h \tau_1^{n+1})$$
$$+((\mathbf{u}_M^n - \mathbf{u}^n)\cdot \nabla_h c^{n+1}, \tau_1^{n+1}) - \int_{\Gamma_h} \{\{(\mathbf{D}(\mathbf{u}_M^n) - \mathbf{D}(\mathbf{u}^n))\nabla_h c^{n+1}\}\}[[\tau_1^{n+1}]]\mathrm{d}s$$
$$=: \sum_{i=1}^{3} S_i.$$

Noting that $|\mathbf{u}^n - \mathbf{u}_M^n| = |\mathbf{u}^n - \mathbf{u}_h^n|$ point-wise if the constant M for the cut-off operator is sufficiently large, we can bound term S_1 as

$$\begin{aligned} |S_1| &\le \varepsilon \|| \mathbf{D}^{1/2}(\mathbf{u}^n)\nabla_h \tau_1^{n+1}\||_0^2 + C\|\mathbf{D}(\mathbf{u}^n) - \mathbf{D}(\mathbf{u}_M^n)\|_0^2, \\ &\le \varepsilon \|| \mathbf{D}^{1/2}(\mathbf{u}^n)\nabla_h \tau_1^{n+1}\||_0^2 + C\|\mathbf{u}^n - \mathbf{u}_M^n\|_0^2, \\ &\le \varepsilon \|| \mathbf{D}^{1/2}(\mathbf{u}^n)\nabla_h \tau_1^{n+1}\||_0^2 + C\|\mathbf{u}^n - \mathbf{u}_h^n\|_0^2. \end{aligned} \tag{7.36}$$

Term S_2 can be bounded in a similar way as that for S_1

$$|S_2| \le \varepsilon \|\sqrt{\phi}\tau_1^{n+1}\|_0^2 + C\|\mathbf{u}^n - \mathbf{u}_h^n\|_0^2. \tag{7.37}$$

Term S_3 can be bounded using the penalty term and continuity of dispersion-diffusion tensor

$$\begin{aligned} |S_3| &\le \varepsilon \int_{\Gamma_h} C_{11}[[\tau_1^{n+1}]]^2 ds + \|\mathbf{D}(\mathbf{u}^n) - \mathbf{D}(\mathbf{u}_M^n)\|_0^2, \\ &\le \varepsilon \|[[\tau_1^{n+1}]]\|_{0,\Gamma_h}^2 + C\|\mathbf{u}^n - \mathbf{u}_h^n\|_0^2. \end{aligned} \tag{7.38}$$

Combining all the terms in (7.25), we have

$$\frac{1}{2\Delta t}(\|\sqrt{\phi}\tau_1^{n+1}\|_0^2 - \|\sqrt{\phi}\tau_1^n\|_0^2) + \frac{1}{2}(\||\mathbf{D}^{1/2}(\mathbf{u}^n)\nabla_h \tau_1^{n+1}\||_0^2 + \|[[\tau_1^{n+1}]]\|_{0,\Gamma_h}^2)$$
$$\le C(\|\sqrt{\phi}\tau_1^{n+1}\|_0^2 + \frac{\phi}{2}\Delta t\|D_t^2 c\|_{L^2(t_k,t_{k+1};L^2(\Omega))}^2 + \Delta t\|\partial_t \tau_2\|_{L^2(t_k,t_{k+1};L^2(\Omega))}^2)$$
$$+C(h^{2k-2} + \|\mathbf{u}^n - \mathbf{u}_h^n\|_0^2).$$

Suppose that m is an integer, $0 \le m \le N - 1$. Multiplying by $2\Delta t$, summing from $n = 0$ to $n = m$, we obtain

$$\|\sqrt{\phi}\tau_1^{n+1}\|_0^2 + \Delta t\sum_{n=1}^{m+1}(\||\mathbf{D}^{1/2}(\mathbf{u}^n)\nabla_h \tau_1^{n+1}\||_0^2 + \|[[\tau_1^{n+1}]]\|_{0,\Gamma_h}^2)$$
$$\le C\Delta t(\sum_{n=1}^{m}\|\tau_1^n\|_0^2 + \sum_{n=0}^{m}\|e_{\mathbf{u}}\|_0^2) + C(\Delta t^2\|D_t^2 c\|_{L^2(t_k,t_{k+1};L^2(\Omega))}^2$$
$$+\|\partial_t \tau_2\|_{L^2(t_k,t_{k+1};L^2(\Omega))}^2).$$

The theorem follow from (7.3), the discrete Gronwall's lemma and the triangle inequality. □

Theorem 7.3 (Error estimate for flow in coupled system) *Let (*$\mathbf{u}$*, p, c) be the solution to (2.1)-(2.5), and assume* $p \in L^2(0, T; H^l(T_h))$, $\mathbf{u} \in (L^2(0, T; H^{l-1}(T_h)))^d$ *and* $c \in L^2(0, T; H^k(T_h))$. *We further assume that p,* ∇p*, c and* ∇c *are essentially bounded. If the constant M for the cut-off operator is sufficiently large, then there exists a constant C independent of h and* Δt *such that*

$$\max_{0\le t\le T}\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_{stab}(t) \le C(\Delta t + h^{k-1} + h^{l-1}). \tag{7.39}$$

Proof Taking L^∞ norm with time in (7.3), we have

$$\max_{0\leq t\leq T} \|(\mathbf{u}-\mathbf{u}_h, p-p_h)\|^2_{stab}(t) \leq C(\|c-c_h\|^2_{L\infty(0,T;L^2)} + h^{2l-2}). \tag{7.40}$$

Substituting (7.24) into the above inequality, we obtain (7.39). □

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Author details

[1]School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 610054, PR China [2]School of Mathematics, Sichuan University, Chengdu, Sichuan 610064, PR China

Authors' contributions

YL participated in the design and theoretical analysis of the study, drafted the manuscript. MF conceived the study, and participated in its design and coordination. YX participated in the design and the revision of the study. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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