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Existence and multiplicity of solutions for nonlocal $p(x)$-Laplacian equations with nonlinear Neumann boundary conditions

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Abstract

In this article, we study the nonlocal $p(x)$-Laplacian problem of the following form

$$\begin{cases} a(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx)(-div(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u) \\ \qquad = b(\int_\Omega F(x,u)\mathrm{d}x)f(x,u) \text{ in } \Omega \\ a\left(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\right)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = g(x,u) \text{ on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain and ν is the outward normal vector on the boundary $\partial\Omega$, and $F(x,u) = \int_0^u f(x,t)dt$. By using the variational method and the theory of the variable exponent Sobolev space, under appropriate assumptions on *f*, *g*, *a* and *b*, we obtain some results on existence and multiplicity of solutions of the problem.

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1 Introduction

In this article, we consider the following problem

$$(P)\begin{cases} a\left(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\right)(-div(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u) \\ \qquad = b\left(\int_\Omega F(x,\ u)\mathrm{d}x\right)f(x,\ u) \text{ in } \Omega \\ a\left(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\right)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = g(x,u) \text{ on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in R^N, $p \in C(\bar{\Omega})$ with $1 < p^- := \inf_\Omega p(x) \le p(x) \le p^+ := \sup_\Omega p(x) < N$, $a(t)$ is a continuous real-valued function, $f : \Omega \times R \to R$, $g : \partial\Omega \times R \to R$ satisfy the Caratheodory condition, and $F(x,u) = \int_0^u f(x,t)dt$. Since the equation contains an integral related to the unknown u over Ω, it is no longer an identity pointwise, and therefore is often called nonlocal problem.

Kirchhoff [1] has investigated an equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

which is called the Kirchhoff equation. Various equations of Kirchhoff type have been studied by many authors, especially after the work of Lions [2], where a functional analysis framework for the problem was proposed; see e.g. [3-6] for some interesting results and further references. In the following, a key work on nonlocal elliptic problems is the article by Chipot and Rodrigues [7]. They studied nonlocal boundary value problems and unilateral problems with several applications. And now the study of nonlocal elliptic problem has already been extended to the case involving the p-Laplacian; see e.g. [8,9]. Recently, Autuori, Pucci and Salvatori [10] have investigated the Kirchhoff type equation involving the p(x)-Laplacian of the form

$$u_{tt} - M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \Delta_{p(x)} u + Q(t, x, u, u_t) + f(x, u) = 0.$$

The study of the stationary version of Kirchhoff type problems has received considerable attention in recent years; see e.g. [5,11-16].

The operator $\Delta_{p(x)}\mathrm{u} = div(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, which becomes p-Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated nonlinearities than p-Laplacian. The study of various mathematical problems with variable exponent are interesting in applications and raise many difficult mathematical problems. We refer the readers to [17-23] for the study of $p(x)$-Laplacian equations and the corresponding variational problems.

Corrêa and Figueiredo [13] presented several sufficient conditions for the existence of positive solutions to a class of nonlocal boundary value problems of the p-Kirchhoff type equation. Fan and Zhang [20] studied $p(x)$-Laplacian equation with the nonlinearity f satisfying Ambrosetti-Rabinowitz condition. The p(x)-Kirchhoff type equations with Dirichlet boundary value problems have been studied by Dai and Hao [24], and much weaker conditions have been given by Fan [25]. The elliptic problems with nonlinear boundary conditions have attracted expensive interest in recent years, for example, for the Laplacian with nonlinear boundary conditions see [26-30], for elliptic systems with nonlinear boundary conditions see [31,32], for the p-Laplacian with nonlinear boundary conditions of different type see [33-37], and for the $p(x)$-Laplacian with nonlinear boundary conditions see [38-40]. Motivated by above, we focus the case of nonlocal $p(x)$-Laplacian problems with nonlinear Neumann boundary conditions. This is a new topics even when $p(x) \equiv p$ is a constant.

This rest of the article is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. In Section 3, we consider the case where the energy functional associated with problem (P) is coercive. And in Section 4, we consider the case where the energy functional possesses the Mountain Pass geometry.

2 Preliminaries

In order to discuss problem (P), we need some theories on variable exponent Sobolev space $W^{1,p(x)}(\Omega)$. For ease of exposition we state some basic properties of space $W^{1,p(x)}$ (Ω) (for details, see [22,41,42]).

Let Ω be a bounded domain of R^N, denote

$$C_+(\bar{\Omega}) = \{p | p \in C(\bar{\Omega}), p(x) > 1, \ \forall x \in \bar{\Omega}\},$$
$$p^+ = \max_{x\in\bar{\Omega}} p(x), p^- = \min_{x\in\bar{\Omega}} p(x), \forall p \in C(\bar{\Omega}),$$
$$L^{p(x)}(\Omega) = \left\{ u | u \text{ is a measurable real - valued function on } \Omega, \int_\Omega |u|^{p(x)} \mathrm{d}x < \infty \right\},$$

we can introduce the norm on $L^{p(x)}$ (Ω) by

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \le 1 \right\}$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space, we call it the variable exponent Lebesgue space.

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) | |\nabla u| \in L^{p(x)}(\Omega)\},$$

and it can be equipped with the norm

$$||u|| = |u|_{p(x)} + |\nabla u|_{p(x)},$$

where $|\nabla u|_{p(x)} = ||\nabla u||_{p(x)}$; and we denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, $p^* = \frac{Np(x)}{N-p(x)}$, $p_* = \frac{(N-1)p(x)}{N-p(x)}$, when $p(x) < N$, and $p^* = p_* = \infty$, when $p(x) > N$.

Proposition 2.1 [22,41]. (1) If $p \in C_+(\overline{\Omega})$, the space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, uniform convex Banach space, and its dual space is $L^{q(x)}$ (Ω), where $1/q(x) + 1/p(x) = 1$. For any $u \in L^{p(x)}$ (Ω) and $v \in L^{q(x)}$ (Ω), we have

$$\left| \int_\Omega uv \mathrm{d}x \right| \le \left(\frac{1}{p-} + \frac{1}{q-}\right) |u|_{p(x)} |v|_{q(x)};$$

(2) If $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$, for any $x \in \Omega$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, and the imbedding is continuous.

Proposition 2.2 [22]. If $f : \Omega \times R \to R$ is a Caratheodory function and satisfies

$$|f(x, s)| \le d(x) + e|s|^{\frac{p_1(x)}{p_2(x)}}, \quad \textit{for any } x \in \Omega, \ s \in R,$$

where $p_1, p_2 \in C_+(\overline{\Omega})$, $d \in L^{p_2(x)}(\Omega)$, $d(x) \ge 0$ and $e \ge 0$ is a constant, then the superposition operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $(N_f(u))(x) = f(x, u(x))$ is a continuous and bounded operator.

Proposition 2.3 [22]. If we denote

$$\rho(u) = \int_\Omega |u|^{p(x)} \mathrm{d}x, \quad \forall u \in L^{p(x)}(\Omega),$$

then for $u, u_n \in L^{p(x)}$ (Ω)

(1) $|u(x)|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$;

(2) $$|u(x)|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+};$$
$$|u(x)|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^-} \ge \rho(u) \ge |u|_{p(x)}^{p^+};$$

(3) $$|u_n(x)|_{p(x)} \to 0 \Leftrightarrow \rho(u_n) \to 0 \text{ as } n \to \infty;$$
$$|u_n(x)|_{p(x)} \to \infty \Leftrightarrow \rho(u_n) \to \infty \text{ as } n \to \infty.$$

Proposition 2.4 [22]. If u, $u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, \ldots$, then the following statements are equivalent to each other

(1) $\lim_{k \to \infty} |u_k - u|_{p(x)} = 0$;

(2) $\lim_{k \to \infty} \rho\, |u_k - u| = 0$;

(3) $u_k \to u$ in measure in Ω and $\lim_{k \to \infty} \rho\,(u_k) = \rho\,(u)$.

Proposition 2.5 [22]. (1) If $p \in C_+(\overline{\Omega})$, then $W_0^{1,p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces;

(2) if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous;

(3) if $q \in C_+(\overline{\Omega})$ and $q(x) < p_*(x)$ for any $x \in \overline{\Omega}$, then the trace imbedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\partial\Omega)$is compact and continuous;

(4) (Poincare inequality) There is a constant $C > 0$, such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

So, $|\nabla u|_{p(x)}$ is a norm equivalent to the norm $\|u\|$ in the space $W_0^{1,p(x)}(\Omega)$.

3 Coercive functionals

In this and the next sections we consider the nonlocal $p(x)$-Laplacian-Neumann problem (P), where a and b are two real functions satisfying the following conditions

(a_1) $a : (0, +\infty) \to (0, +\infty)$ is continuous and $a \in L^1(0, t)$ for any $t > 0$.

(b_1) $b : R \to R$ is continuous.

Notice that the function a satisfies (a_1) may be singular at $t = 0$. And f, g satisfying

(f_1) $f : \Omega \times R \to R$ satisfies the Caratheodory condition and there exist two constants $C_1 \geq 0$, $C_2 \geq 0$ such that

$$|f(x, t)| \leq C_1 + C_2|t|^{q_1(x)-1}, \ \forall (x, t) \in \Omega \times R,$$

where $q_1 \in C_+(\overline{\Omega})$ and $q_1(x) < p^*(x)$, $\forall x \in \overline{\Omega}$.

(g_1) $g : \partial\Omega \times R \to R$ satisfies the Caratheodory condition and there exist two constants $C_1' \geq 0, C_2' \geq 0$ such that

$$|g(x, t)| \leq C_1' + C_2'|t|^{q_2(x)-1}, \ \forall (x, t) \in \partial\Omega \times R,$$

where $q_2 \in C_+(\partial\Omega)$ and $q_2(x) < p_*(x)$, $\forall x \in \partial\Omega$. For simplicity we write $X = W^{1,p(x)}(\Omega)$, denote by C the general positive constant (the exact value may change from line to line).

Define

$$\widehat{a}(t) = \int_0^t a(s)ds, \widehat{b}(t) = \int_0^t b(s)ds, \ \forall t \in R,$$

$$I_1(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx, \ I_2(u) = \int_\Omega F(x, u)\, dx, \ \forall u \in X,$$

$$J(u) = \widehat{a}(I_1(u)) = \widehat{a}\left(\int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx\right),$$

$$\Phi(u) = \widehat{b}(I_2(u)) = \widehat{b}\left(\int_\Omega F(x, u)\, \mathrm{d}x\right) \text{ and } \Psi(u) = \int_{\partial\Omega} G(x, u)\, \mathrm{d}\sigma, \ \forall u \in X,$$

$$E(u) = J(u) - \Phi(u) - \Psi(u), \ \forall u \in X,$$

where $F(x, u) = \int_0^u f(x, t)\, dt$, $G(x, u) = \int_0^u g(x, t)\, \mathrm{d}t$.

Lemma 3.1. Let (f_1), (g_1) (a_1) and (b_1) hold. Then the following statements hold true:

(1) $\widehat{a} \in C^0([0, \infty)) \cap C^1((0, \infty))$, $\widehat{a}(0) = 0$, $\widehat{a}'(t) = a(t) > 0$; $\widehat{b} \in C^1(R)$, $\widehat{b}(0) = 0$.

(2) J, Φ, Ψ and $E \in C^0(X)$, $J(0) = \Phi(0) = \Psi(0) = E(0) = 0$. Furthermore $J \in C^1(X\backslash\{0\})$, $\Phi, \Psi \in C^1(X)$, $E \in C^1(X\backslash\{0\})$. And for every $u \in X\backslash\{0\}$, $v \in X$, we have

$$E'(u)v = a\left(\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx\right)\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv\right)\mathrm{d}x$$
$$- b\left(\int_\Omega F(x, u)\mathrm{d}x\right)\int_\Omega f(x, u)v\,\mathrm{d}x - \int_{\partial\Omega} g(x, u)\,v\mathrm{d}\sigma.$$

Thus $u \in X\backslash\{0\}$ is a (weak) solution of (P) if and only if u is a critical point of E.

(3) The functional $J : X \to R$ is sequentially weakly lower semi-continuous, Φ, Ψ: $X \to R$ are sequentially weakly continuous, and thus E is sequentially weakly lower semi-continuous.

(4) The mappings Φ' and Ψ' are sequentially weakly-strongly continuous, namely, $u_n \rightharpoonup u$ in X implies $\Phi'(u_n) \to \Phi'(u)$ in X^*. For any open set $D \subset X\backslash\{0\}$ with $\overline{D} \subset X\backslash\{0\}$, The mappings J' and $E' : \overline{D} \to X^*$ are bounded, and are of type (S_+), namely,

$$u_n \rightharpoonup u \text{ and } \overline{\lim_{n\to\infty}} J'(u_n)(u_n - u) \le 0, \text{ implies } u_n \to u.$$

Definition 3.1. Let $c \in$ R, a C^1-functional $E : X \to R$ satisfies $(\mathrm{P.S})_c$ condition if and only if every sequence $\{u_j\}$ in X such that $\lim_j E(u_j) = c$, and $\lim_j E'(u_j) = 0$ in X^* has a convergent subsequence.

Lemma 3.2. Let (f_1), (g_1), (a_1), (b_1) hold. Then for any $c \neq 0$, every bounded $(\mathrm{P.\ S})_c$ sequence for E, i.e., a bounded sequence $\{u_n\} \subset X\backslash\{0\}$ such that $E(u_n) \to c$ and $E'(u_n) \to 0$, has a strongly convergent subsequence.

The proof of these two lemmas can be obtained easily from [25,40], we omitted them here.

Theorem 3.1. Let (f_1), (g_1), (a_1), (b_1) and the following conditions hold true:

(a_2) There are positive constants α_1, M, and C such that $\widehat{a}(t) \ge Ct^{\alpha_1}$ for $t \ge M$.

(b_2) There are positive constants β_1 and C such that $|\widehat{b}(t)| \le C + C|t|^{\beta_1}$ for $t \in R$.

(H_1) $\beta_1 q_{1+} < \alpha_1 p_-$, $q_{2+} < \alpha_1 p_-$.

Then the functional E is coercive and attains its infimum in X at some $u_0 \in X$. Therefore, u_0 is a solution of (P) if E is differentiable at u_0.

Proof. For $\|u\|$ large enough, by (f_1), (g_1), (a_2), (b_2) and (H_1), we have that

$$J(u) = \widehat{a}(I_1(u)) = \widehat{a}\left(\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx\right)$$
$$\ge \widehat{a}(C_1\|u\|^{p_-}) \ge C_2\|u\|^{\alpha_1 p_-},$$
$$\left|\int_\Omega F(x, u)\,\mathrm{d}x\right| \le C_3\|u\|^{q_{1+}},$$
$$\Phi(u) = \widehat{b}(I_2(u)) = \widehat{b}\left(\int_\Omega F(x, u)\,\mathrm{d}x\right) \le C_4\|u\|^{\beta_1 q_{1+}} + \widetilde{C_4},$$
$$\Psi(u) = \left|\int_{\partial\Omega} G(x, u)\,\mathrm{d}\sigma\right| \le C_5\|u\|^{q_{2+}} + \widetilde{C_5},$$
$$E(u) = J(u) - \Phi(u) - \Psi(u) \ge C_2\|u\|^{\alpha_1 p_-} - C_4\|u\|^{\beta_1 q_{1+}} - C_5\|u\|^{q_{2+}} - +\widetilde{C_6},$$

and hence E is coercive. Since E is sequentially weakly lower semi-continuous and X is reflexive, E attains its infimum in X at some $u_0 \in X$. In this case E is differentiable at u_0, then u_0 is a solution of (P).

Theorem 3.2. Let (f_1), (g_1), (a_1), (b_1), (a_2), (b_2), (H_1) and the following conditions hold true:

(a_3) There is a positive constant α_2 such that $\limsup_{t\to 0+} \frac{\hat{a}(t)}{t^{\alpha_2}} < +\infty$.

(b_3) There is a positive constant β_2 such that $\liminf_{t\to 0} \frac{\widehat{b}(t)}{|t|^{\beta_2}} > 0$.

(f_2) There exist an open subset Ω_0 of Ω and $r_1 > 0$ such that $\liminf_{t\to 0} \frac{F(x,t)}{|t|^{r_2}} > 0$ uniformly for $x \in \Omega_0$.

(g_2) There exists $r_2 > 0$ such that $\liminf_{t\to 0} \frac{G(x,t)}{|t|^{r_2}} > 0$ uniformly for $x \in \partial\Omega$.

(H_2) $\beta_2 r_1 < \alpha_2 p_-$, $r_2 < \alpha_2 p_-$.

Then (P) has at least one nontrivial solution which is a global minimizer of the energy functional E.

Proof. From Theorem 3.1 we know that E has a global minimizer u_0. It is clear that $\widehat{b}(0) = 0, \widehat{b}(0) = 0, F(x, 0)$ and consequently $E(0) = 0$. Take $w \in C_0^\infty(\Omega_0)\backslash\{0\}$. Then, by (f_2), (g_2) (a_3), (b_3) and (H_2), for sufficiently small $\lambda > 0$ we have that

$$
\begin{aligned}
E(\lambda w) &= \widehat{a}\left(\int_\Omega \frac{\lambda^{p(x)}}{p(x)} \left(|\nabla w|^{p(x)} + |w|^{p(x)}\right) dx\right) \\
&\quad - \widehat{b}\left(\int_\Omega F(x, \lambda w)\, dx\right) - \int_{\partial\Omega} G(x, \lambda w)\, d\sigma \\
&\le C_1\left(\int_\Omega \frac{\lambda^{p(x)}}{p(x)} \left(|\nabla w|^{p(x)} + |w|^{p(x)}\right) dx\right)^{\alpha_2} \\
&\quad - C_2\left(\int_{\Omega_0} F(x, \lambda w)\, dx)\right)^{\beta_2} - C_3 \int_{\partial\Omega} |\lambda w|^{r_2} d\sigma \\
&\le C_4 \lambda^{\alpha_2 p-} - C_5 \lambda^{\beta_2 r_1} - C_6 \lambda^{r_2} < 0.
\end{aligned}
$$

Hence $E(u_0) < 0$ and $u_0 \neq 0$.

By the genus theorem, similarly in the proof of Theorem 4.3 in [18], we have the following:

Theorem 3.3. Let the hypotheses of Theorem 3.2 hold, and let, in addition, f and g satisfy the following conditions:

(f_3) $f(x, -t) = -f(x, t)$ for $x \in \Omega$ and $t \in R$.

(g_3) $g(x, -t) = -g(x, t)$ for $x \in \partial\Omega$ and $t \in R$.

Then (P) has a sequence of solutions $\{u_n\}$ such that $E(u_n) < 0$.

Theorem 3.4. Let (f_1), (g_1), (a_1), (b_1), (a_2), (b_2), (a_3), (b_3), (H_1), (H_2) and the following conditions hold true:

(b_+) $b(t) \ge 0$ for $t \ge 0$.

(f_+) $f(x, t) \ge 0$ for $x \in \Omega$ and $t \ge 0$.

(g_+) $g(x, t) \ge 0$ for $x \in \partial\Omega$ and $t \ge 0$.

$(f_2)_+$There exist an open subset Ω_0 of Ω and $r_1 > 0$ such that $\liminf_{t\to 0+} \frac{F(x,t)}{t^{r_1}} > 0$ uniformly for $x \in \Omega_0$.

$(g_2)_+$ There exists $r_2 > 0$ such that $\liminf_{t\to 0^+} \frac{G(x,t)}{t^{r_2}} > 0$ uniformly for $x \in \partial\Omega$.

Then (*P*) has at least one nontrivial nonnegative solution with negative energy.

Proof. Define

$$\tilde{f}(x,t) = \begin{cases} f(x,t) \text{ if } t \geq 0, \\ f(x,0) \text{ if } t < 0, \end{cases} \quad \tilde{g}(x,t) = \begin{cases} g(x,t) \text{ if } t \geq 0, \\ g(x,0) \text{ if } t < 0, \end{cases}$$

$$\tilde{F}(x,t) = \int_0^t \tilde{f}(x,s)ds, \forall x \in \Omega, t \in R,$$
$$\tilde{G}(x,t) = \int_0^t \tilde{g}(x,s)ds, \forall x \in \partial\Omega, t \in R,$$

$$\tilde{b}(t) = \begin{cases} b(t) \text{ if } t \geq 0, \\ b(0) \text{ if } t < 0, \end{cases} \widehat{\tilde{b}}(t) = \int_0^t \tilde{b}(s)ds, \forall t \in R,$$

$$\tilde{E}(u) = \hat{a}\left(\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx\right) - \widehat{\tilde{b}}(\int_\Omega \tilde{F}(x,u)\mathrm{d}x) - \int_{\partial\Omega} \tilde{G}(x,u)\mathrm{d}\sigma, \forall u \in X.$$

Then, using truncation functions above, similarly in the proof of Theorem 3.4 in [25], we can prove that $\tilde{E}$ has a nontrivial global minimizer u_0 and u_0 is a nontrivial nonnegative solution of (*P*).

4 The Mountain Pass theorem

In this section we will find the Mountain Pass type critical points of the energy functional *E* associated with problem (*P*).

Lemma 4.1. Let (f_1), (g_1), (a_1), (b_1) and the following conditions hold true:

$(a_2)'\exists \alpha_1 > 0$, $M > 0$, and $C > 0$ such that

$\hat{a}(t) \geq Ct^{\alpha_1}$ *for all* $t \geq M$

with $\alpha_1 p_- > 1$.

(a_4) $\exists\ \lambda > 0$, $M > 0$ such that

$\lambda\hat{a}(t) \geq a(t)t$ *for all* $t \geq M$

(b_4) $\exists \theta > 0$, $M > 0$ such that:

$0 \leq \theta\widehat{b}(t) \leq b(t)t$, *for all* $t \geq M$.

(f_4) $\exists \mu > 0$, $M > 0$ such that:

$0 \leq \mu F(x,t) \leq f(x,t)t$, *for* $|t| \geq M$ *and* $x \in \Omega$.

(g_4) $\exists \kappa > \theta\mu > 0$, $M > 0$ such that:

$0 \leq \kappa G(x,t) \leq g(x,t)t$, $|t| \geq M$ *and* $x \in \partial\Omega$.

(H_3) $\lambda p_+ < \theta\mu$.

Then E satisfies condition $(P.S)c$ for any $c \neq 0$.

Proof. By (a_4), for $||u||$ large enough,

$$\begin{aligned}\lambda p + J(u) &= \lambda p + \hat{a}(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx) \\ &\geq p + a\left(\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx\right)\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx \\ &\geq a\left(\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx\right)\int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx = J'(u)u.\end{aligned}$$

From (f_4) and (g_4) we can see that there exists C_1 >0 and C_2 >0 such that

$$-C_1 \leq \mu \int_\Omega F(x,u)dx \leq \int_\Omega f(x,u)udx + C_1, \forall u \in X,$$
$$-C_2 \leq \kappa \int_{\partial\Omega} G(x,u)d\sigma \leq \int_{\partial\Omega} g(x,u)ud\sigma + C_2, \forall u \in X,$$

and thus, given any $\varepsilon \in (0, \mu)$, there exists $M_\varepsilon \geq M$ >0 and $M'_\varepsilon \geq M > 0$ such that

$$(\mu-\varepsilon)\int_\Omega F(x,u)dx \leq \int_\Omega f(x,u)udx, \text{if} \int_\Omega F(x,u)dx \geq M_\varepsilon,$$
$$\theta(\mu-\varepsilon)\int_{\partial\Omega} G(x,u)d\sigma \leq \int_{\partial\Omega} g(x,u)ud\sigma, \text{if} \int_{\partial\Omega} G(x,u)d\sigma \geq M'_\varepsilon.$$

We may assume $M_\varepsilon > \frac{c_1}{\mu}$ and $M_\varepsilon' > \frac{c_2}{\theta\mu}$. Note that in this case the inequalities $\int_\Omega F(x,u)dx \geq M_\varepsilon$ and $\int_{\partial\Omega} G(x,u)d\sigma \geq M'_\varepsilon$ are equivalent to $|\int_\Omega F(x,u)dx| \geq M_\varepsilon$and $|\int_{\partial\Omega} G(x,u)d\sigma| \geq M'_\varepsilon$, because $\int_\Omega F(x,u)dx \geq -\frac{C_1}{\mu}$ and $\int_{\partial\Omega} G(x,u)d\sigma \geq -\frac{c_2}{\theta\mu}$ for all $u \in X$. We claim that there exist $C_\varepsilon > 0$ and $C'_\varepsilon > 0$ such that

$$\Phi'(u)u - \theta(\mu-\varepsilon)\Phi(u) \geq -C_\varepsilon \text{ for } u \in X,$$
$$\Psi'(u)u - \theta(\mu-\varepsilon)\Psi(u) \geq -C'_\varepsilon \text{ for } u \in X.$$

Indeed, when $|\int_\Omega F(x,u)dx| \leq M_\varepsilon$ and $|\int_{\partial\Omega} G(x,u)d\sigma| \leq M'_\varepsilon$, the validity is obvious. When $|\int_\Omega F(x,u)dx| \geq M_\varepsilon$ and $|\int_{\partial\Omega} G(x,u)d\sigma| \geq M'_\varepsilon$, i.e., $\int_\Omega F(x,u)dx \geq M_\varepsilon$ and $\int_{\partial\Omega} G(x,u)d\sigma \geq M'_\varepsilon$, we have that

$$\begin{aligned}\theta(\mu-\varepsilon)\Phi(u) &= \theta(\mu-\varepsilon)\hat{b}\left(\int_\Omega F(x,u)\mathrm{d}x\right)\\ &\leq (\mu-\varepsilon)b\left(\int_\Omega F(x,u)\mathrm{d}x\right)\int_\Omega F(x,u)\mathrm{d}x\\ &\leq b\left(\int_\Omega F(x,u)\mathrm{d}x\right)\int_\Omega f(x,u)u\mathrm{d}x = \Phi'(u)u,\end{aligned}$$

and

$$\begin{aligned}\theta(\mu-\varepsilon)\Psi(u) &= \theta(\mu-\varepsilon)\int_{\partial\Omega} G(x,u)d\sigma\\ &\leq \int_{\partial\Omega} g(x,u)ud\sigma = \Psi'(u)u.\end{aligned}$$

Now let $\{u_n\} \subset X\backslash\{0\}$, $E(u_n) \to c \neq 0$ and $E'(u_n) \to 0$. By (H_3), there exists ε >0 small enough such that $\lambda p_+ < \theta(\mu - \varepsilon)$. Then, since $\{u_n\}$ is a $(P.S)_c$ sequence, for sufficiently large n, we have

$$\begin{aligned}&\theta(\mu-\varepsilon)c + 1 + \| u_n \|\\ \geq\ &\theta(\mu-\varepsilon)E(u_n) - E'(u_n)u_n\\ \geq\ &(\theta(\mu-\varepsilon)-\lambda p_+)J(u_n) + (\lambda p_+ J(u_n) - J'(u_n)u_n) + (\Phi'(u_n)u_n - \theta(\mu-\varepsilon)\Phi(u_n))\\ &+ (\Psi'(u_n)u_n - \theta(\mu-\varepsilon)\Psi(u_n))\\ \geq\ &C_3 \| u_n\|^{\alpha_1 p-} - C_4 - C_\varepsilon - C'_\varepsilon\end{aligned}$$

Since $\alpha_1 p_-$ >1, we have that $\{||u_n||\}$ is bounded. By Lemma 3.2, E satisfies condition $(P.S)_c$ for $c \neq 0$.

Theorem 4.1. Under the hypotheses of Lemma 4.1, and let the following conditions hold:

(a_5) There is a positive constant α_3 such that $\limsup_{t\to 0^+} \frac{\hat{a}(t)}{t^{\alpha}3} > 0$.

(b_5) There is a positive constant β_3 such that $\liminf_{t\to 0} \frac{\hat{b}(t)}{|t|^{\beta_3}} < +\infty$.

(f_5) There exists $r_1 \in C^0(\overline{\Omega})$ such that $1 < r_1(x) < p^*(x)$ for $x \in \overline{\Omega}$ and $\liminf_{t\to 0} \frac{|F(x,t)|}{|t|^{r_1(x)}} < +\infty$ uniformly for $x \in \Omega$.

(g_5) There exists such $r_2 \in C^0(\overline{\Omega})$ such that $1 < r_2(x) < p_*(x)$ for $x \in \partial\Omega$ and $\liminf_{t\to 0} \frac{|G(x,t)|}{|t|^{r_2(x)}} < +\infty$ uniformly for $x \in \partial\Omega$.

(H_4) $\alpha_3 p_+ < \beta_3 r_{1-}$, $\alpha_3 p_+ < r_{2-}$, $\lambda p_+ < \theta\mu$.

Then (P) has a nontrivial solution with positive energy.

Proof. Let us prove this conclusion by the Mountain Pass lemma. E satisfies condition $(P.S)_c$ for $c \neq 0$ has been proved in Lemma 4.1.

For $||u||$ small enough, from (a_5) we can obtain easily that $J(u) \geq C_1 \parallel u\parallel^{\alpha_3 p_+}$, from ($b_5$), ($f_1$) and ($f_5$) we have$|\Phi(u)| \leq C_2 \parallel u\parallel^{\beta_3 r_{1-}}$, and in the similar way from(g_1) and (g_5) we have $|\Psi(u)| \leq C_2 \parallel u\parallel^{r_{2-}}$. Thus by ($H_4$), we conclude that there exist positive constants ρ and δ such that $E(u) \geq$ for $||u|| = \rho$.

Let $w \in X\backslash\{0\}$ be given. From (a_4) for sufficiently large $t > 0$ we have $\hat{a}(t) \leq C_1 t^{\lambda}$, which follows that $J(sw) \leq d_1 s^{\lambda p_+}$ for s large enough, where d_1 is a positive constant depending on w. From (f_4) and (f_1) for $|t|$ large enough we have $\int_\Omega F(x, sw)\mathrm{d}x \geq d_2 s^{\mu}$ for s large enough, where d_2 is a positive constant depending on w. From (b_4) for t large enough we have $\Phi(sw) = \hat{b}(\int_\Omega F(x, sw)\mathrm{d}x) \geq d_3 s^{\theta\mu}$ for s large enough, where d_3 is a positive constant depending on w. From (g_4) and (g_1) for $|t|$ large enough we have $\Psi(sw) = \int_{\partial\Omega} G(x, sw)d\sigma \geq d_4 s^{\theta\mu}$. Hence for any $w \in X\backslash\{0\}$ and s large enough, $E(sw) \leq d_1 s^{\lambda p_+} - d_3 s^{\theta\mu} - d_4 s^{\theta\mu}$, thus by ($H_3$), We conclude that $E(sw) \to -\infty$ as $s \to +\infty$.

So by the Mountain Pass lemma this theorem is proved.

By the symmetric Mountain Pass lemma, similarly in the proof of Theorem 4.8 in [40], we have the following:

Theorem 4.2. Under the hypotheses of Theorem 4.1, if, in addition, (f_3) and (g_3) are satisfied, then (P) has a sequence of solutions $\{\pm u_n\}$ such that $E(\pm u_n) \to +\infty$ as $n \to \infty$.

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Authors' contributions
EG and PZ contributed to each part of this work equally. All the authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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