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Rothe-Galerkin's method for a nonlinear integrodifferential equation

Abderrazek Chaoui* and Assia Guezane-Lakoud

* Correspondence: razwel2004@yahoo.fr
Laboratory of advanced materials, Badji Mokhtar University, Annaba, Algeria

Abstract
In this article we propose approximation schemes for solving nonlinear initial boundary value problem with Volterra operator. Existence, uniqueness of solution as well as some regularity results are obtained via Rothe-Galerkin method.
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1 Introduction

The aim of this work is the solvability of the following equation

$$\partial_t \beta(u) - \partial_t \Delta a(u) - \nabla d(t, x, u, \nabla a(u)) + K(u) = f(t, x, u) \tag{1.1}$$

where $(t, x) \in (0, T) \times \Omega = Q_T$, with the initial condition

$$\beta(u(0, x)) = \beta(u_0(x)), x \in \Omega \tag{1.2}$$

and the boundary condition

$$u(t, x) = 0, (t, x) \in (0, T) \times \partial\Omega. \tag{1.3}$$

The memory operator K is defined by

$$\langle K(t)u, v\rangle = \int_\Omega \int_0^t k(t, s) g(s, x, \nabla u(s, x)) \nabla v(t, x) ds dx. \tag{1.4}$$

Let us denote by (P), the problem generated by Equations (1.1)-(1.3). The problem (P) has relevant interest applications to the porous media equation and to integro-differential equation modeling memory effects. Several problems of thermoelasticity and viscoelasticity can also be reduced to this type of problems. A variety of problems arising in mechanics, elasticity theory, molecular dynamics, and quantum mechanics can be described by doubly nonlinear problems.

The literature on the subject of local in time doubly nonlinear evolution equations is rather wide. Among these contributions, we refer the reader to [1] where the authors studied the convergence of a finite volume scheme for the numerical solution for an elliptic-parabolic equation. Using Rothe method, the author in [2] studied a nonlinear degenerate parabolic equation with a second-order differential Volterra operator. In [3] the solutions of nonlinear and degenerate problems were investigated. In general, existence of solutions for a class of nonlinear evolution equations of second order is proved by studying a full discretization.

The article is organized as follows. In Section 2, we specify some hypotheses, precise sense of the weak solution, then we state the main results and some Lemmas that needed in the sequel. In Section 3, by the Rothe-Galerkin method, we construct approximate solutions to problem (P). Some a priori estimates for the approximations are derived. In Section 4, we prove the main results.

2 Hypothesis and mean results

To solve problem (P), we assume the following hypotheses:

(H_1) The function $\beta : \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing, $\beta(0) = 0$, $\beta(u_0) \in L^2(\Omega)$ and satisfies $|\beta(s)|^2 \leq C_1 B^*(a(s)) + C_2$, $\forall s \in \mathbb{R}$.

(H_2) $a : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing function, $a(0) = 0$ and $a(u_0) \in H_0^1(\Omega)$.

(H_3) $d : (0, T) \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous, elliptic i.e., $\exists d_0 > 0$ such that $d(t, x, z, \xi)\,\xi \geq d_0 |\xi|^p$ for $\xi \in \mathbb{R}^N$ and $p \geq 2$, strongly monotone i.e.,

$(d(t, x, \eta, \xi_1) - d(t, x, \eta, \xi_2))(\xi_1 - \xi_2) \geq d_1 |\xi_1 - \xi_2|^p$ for $\xi_1, \xi_2 \in \mathbb{R}^N$, $d_1 > 0$ and satisfies $|d(t,x,z,\xi)| \leq C\left(1 + |\xi|^{p-1} + (B^*(a(z)))^{\frac{p-1}{p}}\right)$ for any $(t, x) \in (0, T) \times \Omega$, $\forall z \in \mathbb{R}$, $\xi \in \mathbb{R}^N$.

(H_4) $f : (0, T) \times \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous such that

$$|f(t,x,z)| \leq C\left(1 + (B^*(a(z)))^{\frac{p-1}{p}}\right)$$

for any $(t, x) \in (0, T) \times \Omega$, $\forall z \in \mathbb{R}$.

The functions g and k given in (1.4) satisfy the following hypotheses (H_5) and (H_6), respectively:

(H_5) $g : (0, T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous and satisfies $|g(t, x, \xi)| \leq C(1 + |\xi|^{p-1})$ and $|g(t, x, \xi_1) - g(t, x, \xi_2)| \leq d_1 |\xi_1 - \xi_2|^{p-1}$.

(H_6) $k : (0, T) \times (0, T) \to \mathbb{R}$ is weak singular, i.e. $|k(t, s)| \leq |t - s|^{-\gamma}\omega(t, s)$ for $0 \leq \gamma \leq \frac{1}{p}$ and the function $\omega : [0, T] \times [0, T] \to \mathbb{R}$ is continuous.

(H_7) For $p = 2$, we have

$$|d(t,x,\eta_1,\xi_1) - d(t,x,\eta_2,\xi_2)| \leq C(|a(\eta_1) - a(\eta_2)| + |\xi_1 - \xi_2|)$$

and

$$|f(t,x,\eta_1) - (t,x,\eta_2)| \leq C|a(\eta_1) - a(\eta_2)|$$

where $(t, x) \in (0, T) \times \Omega$, $\eta_1, \eta_2 \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathbb{R}^N$.

As in [3] we define the function B^* by

$$B^*(s) := \beta(a^{-1}(s))s - \int_0^s \beta(a^{-1}(z))dz \quad \text{for s} \in \{\gamma \in \mathbb{R} : \gamma = a(z), z \in \mathbb{R}\}.$$

We are concerned with a weak solution in the following sense:

Definition 1 *By a weak solution of the problem (P) we mean a function* $u : Q_T \to \mathbb{R}$ *such that:*

(1) $\beta(u) \in L^2(Q_T)$, $\partial_t(\beta(u) - \Delta a(u)) \in L^q((0,T), W^{-1,q}(\Omega))$, $a(u) \in L^p((0,T), W_0^{1,p}(\Omega))$, $a(u) \in L^\infty((0,T), H_0^1(\Omega))$.

(2) $\forall v \in L^p((0,T), W_0^{1,p}(\Omega))$, $v_t \in L^2((0,T), H_0^1(\Omega))$ *and* $v(T) = 0$ *we have*

$$\begin{aligned}&-\int_{Q_T} \beta(u)\partial_t v dxdt - \int_{Q_T} \nabla a(u)\nabla\partial_t v dxdt \\ &+\int_{Q_T} d(t,x,u,\nabla a(u))\nabla v dxdt + \int_{Q_T} \beta(u_0)v_t dxdt \\ &+\int_{Q_T} \nabla a(u_0)\nabla v_t dxdt + \int_0^T \langle K(u), v\rangle dt = \int_{Q_T} f(t,x,u)v dxdt.\end{aligned} \tag{2.1}$$

The main result of this article is the following theorem.

Theorem 2 *Under hypotheses* (H_1) - (H_6)*, there exists a weak solution u for problem (P) in the sense of Definition 1. In addition, if* (H_7) *is also satisfied, then u is unique.*

The proof of this theorem will be done in the last section. In the sequel, we need the following lemmas:

Lemma 3 [3]*Let* $J : \mathbb{R}^N \to \mathbb{R}^N$ *be continuous and for any* $R > 0$, $(J(x), x) \geq 0$ *for all* $|x| = R$*. Then there exists an* $y \in \mathbb{R}^N$ *such that* $y \neq 0$, $|y| \leq R$ *and* $J(y) = 0$.

Lemma 4 [4]*Assume that* $\partial_t(\beta(u) - \Delta a(u)) \in L^q((0,T), W^{-1,q}(\Omega))$, $a(u) \in L^p(0,T), W_0^{1,p}(\Omega))$, $a(u) \in L^\infty((0,T), H_0^1(\Omega))$, $B^* \in L^\infty((0,T), L^1(\Omega))$, $\beta(u_0) \in L^2(\Omega)$ *and* $a(u_0) \in H_0^1(\Omega)$*. Then for almost all* $t \in (0,T)$*, we have*

$$\begin{aligned}&\int_0^t (\partial_t(\beta(u) - \Delta a(u)), a(u))dt = \int_\Omega B^*(a(u(t)))dx \\ &+\frac{1}{2}\int_\Omega |\nabla a(u(t))|^2 dx - \int_\Omega B^*(a(u_0))dx - \frac{1}{2}\int_\Omega |\nabla a(u_0)|^2 dx.\end{aligned}$$

3 Discretization scheme and *a priori* estimates

To solve problem (P) by Rothe-Galerkin method, we proceed as follows. We divide the interval $I = [0, T]$ into n subintervals of the length $h = \frac{T}{n}$ and denote $u_i = u(t_i)$, with $t_i = ih$, $i = 1, \ldots, n$, then problem (P) is approximated by the following recurrent sequence of time-discretized problems

$$\begin{aligned}\frac{1}{h}(\beta(u_i) - \beta(u_{i-1})) - \frac{1}{h}\Delta(a(u_i) - a(u_{i-1})) - \nabla d(t_i, x, u_{i-1}, \nabla a(u_i)) \\ -f(t_i, x, u_{i-1}) + K(\hat{u}_{i-1}) = 0 \\ u_i(x) = 0 \text{ on } \partial\Omega\end{aligned} \tag{3.1}$$

where $\hat{u}_{i-1} = \begin{cases} u_{j-1}, t \in [t_{j-1}, t_j), j = 1, \ldots, i-1 \\ u_{i-1}, \qquad t \in [t_{i-1}, T] \end{cases}$

Hence, we obtain a system of elliptic problems that can be solved by Galerkin method.

Let $\phi_1, \ldots, \phi_m, \ldots$ be a basis in $W_0^{1,p}(\Omega)$ and let V_m be a subspace of $W_0^{1,p}(\Omega)$ generated by the m first vectors of the basis. We search for each $m \in \mathbb{N}^*$ the functions $\{u_i^m\}_{i=1}^n$ such that $a(u_i^m) = \sum_{k=1}^m a_{ik}^m e_k$ and satisfying

$$\begin{aligned}&\int_\Omega \frac{1}{h}(\beta(u_i^m) - \beta(u_{i-1}^m))\xi\,dx + \int_\Omega \frac{1}{h}(\nabla a(u_i^m) - \nabla a(u_{i-1}^m))\nabla\xi\,dx\\ &+ \int_\Omega d(t_i, x, u_{i-1}^m, \nabla a(u_i^m))\nabla\xi\,dx + \langle K(\hat{u}_{i-1}^m), \xi\rangle - \int_\Omega f(t_i, x, u_{i-1}^m)\xi\,dx = 0\end{aligned} \tag{3.2}$$

Remark 5 *In what follows we denote by C a nonnegative constant not depending on n, m, j and h.*

Theorem 6 *There exists a solution* u_i^m *in* V_m *of the family of discrete Equation (3.2).*

Proof. We proceed by recurrence, suppose that u_0^m is given and that u_{i-1}^m is known. Define the continuous function $J_{hm} : \mathbb{R}^m \to \mathbb{R}^m$ by:

$$\begin{aligned}J_{hm}(r) = &\frac{1}{h}\int_\Omega (\beta(v)e_j + \nabla a(v)\nabla e_j)dx - \frac{1}{h}\int_\Omega (\beta(u_{i-1}^m)e_j\\ &+\nabla a u_{i-1}^m)\nabla e_j dx + \int_\Omega d(t_i, x, u_{i-1}^m, \nabla a(v))\nabla e_j dx + \langle K(\hat{u}_{i-1}^m), \xi\rangle\\ &- \int_\Omega f(t_i, x, u_{i-1}^m)e_j dx\end{aligned} \tag{3.3}$$

where $a(v) = \sum_{j=1}^m r_j e_j$. We shall prove that J_{hm} satisfies the following estimates

$$\begin{aligned}J_{hm}(r)r \geq\ &\frac{1}{h}\int_\Omega \left(B^*(a(v)) + \frac{1}{2}|\nabla a(v)|^2\right)dx\\ &- \frac{1}{h}\int_\Omega \left(B^*(a(u_{i-1}^m)) + \frac{1}{2}|\nabla a(u_{i-1}^m)|^2\right)dx\\ &+ d_0\int_\Omega |\nabla a(v)|^p dx - C\delta\int_\Omega |\nabla a(v)|^p dx\\ &- C(\delta).C(\gamma)\sum_{k=1}^{i} h\int_\Omega |\nabla u_{k-1}^m|^p dx\\ &- C\delta_0\int_\Omega |\nabla a(v)|^p dx - C(\delta_0)\int_\Omega |f(t_i, x, u_{i-1}^m)|^q dx\\ \geq\ & C\int_\Omega |\nabla a(v)|^2 dx + C\int_\Omega |\nabla a(v)|^p dx - C\end{aligned} \tag{3.4}$$

Indeed, from hypothesis (H_1) and the definition of B^* we deduce

$$\frac{1}{h}\int_\Omega (\beta(v) - \beta(u_{i-1}^m))a(v)dx \geq \frac{1}{h}\int_\Omega B^*(a(v))dx - \frac{1}{h}\int_\Omega B^*(a(u_{i-1}^m))dx, \tag{3.5}$$

the hypotheses on a and d imply

$$\int_\Omega d(t_i, x, u_{i-1}^m, \nabla a(v))\nabla a(v)dx \geq d_0 \int_\Omega |\nabla a(v)|^p dx, \tag{3.6}$$

using the identity

$$2(x, x-y) = \| x\|^2 - \| y\|^2 + \| x-y\|^2, \tag{3.7}$$

we obtain

$$\begin{aligned} &\frac{1}{h}\int_\Omega (\nabla a(v) - \nabla a(u_{i-1}^m))\nabla a(v)dx \\ &\geq \frac{1}{2h}\int_\Omega |\nabla a(v)|^2 dx - \frac{1}{2h}\int_\Omega |\nabla a(u_{i-1}^m)|^2 dx, \end{aligned}$$

applying Holder and δ-inequalities to the integral operator, it yields

$$\begin{aligned} \langle K(\hat{u}_{i-1}^m), a(v)\rangle \leq \frac{C}{\delta}\int_\Omega \left(\int_0^{t_i} k(t_i, s)g(s, x, \nabla \hat{u}_{i-1}^m)ds\right)^q \nabla dx \\ +C\delta \int_\Omega |\nabla a(v)|^p dx \end{aligned} \tag{3.8}$$

the first integral in (3.8) can be estimated as

$$\begin{aligned} &\left|\int_0^{t_i} k(t_i, s)g(s, x, \nabla \hat{u}_{i-1}^m)ds\right| \\ &\leq \left(\sum_{k=1}^{i} h|\nabla u_{k-1}^m|^p\right)^{\frac{1}{q}} \left(\sum_{k=1}^{i} h(t_i - t_k)^{-\gamma p}\right)^{\frac{1}{p}} + C. \end{aligned} \tag{3.9}$$

Since $\gamma < \frac{1}{p}$, then

$$\sum_{k=1}^{i} h(t_i - t_k)^{-\gamma p} \leq \frac{1}{1-\gamma p} =: C(\gamma)$$

for the function f we have

$$\int_\Omega f(t_i, x, u_{i-1}^m)a(v)dx \leq C\delta \int_\Omega |\nabla a(v)|^p dx + C(\delta)\int_\Omega B^*(a(u_i^m))dx + C. \tag{3.10}$$

Therefore (3.4) holds. Then for $|r|$ big enough, $J_{hm}(r)\ r \geq 0$. Taking into account that J_{hm} is continuous, Lemma 3 states that J_{hm} has a zero. Since the function a is strictly increasing then there exists $v = u_i^m$ solution of (3.2). ■

Now we derive the following estimates.

Lemma 7 *There exists a constant* $C > 0$ *such that*

$$\max_{1\le i\le n}\int_\Omega B^*(a(u_i^m))dx \le C, \tag{3.11}$$

$$\max_{1\le i\le n}\int_\Omega |\nabla a(u_i^m)|^2 dx \le C, \tag{3.12}$$

$$\sum_{i=1}^n h\int_\Omega |\nabla a(u_i^m)|^p dx \le C. \tag{3.13}$$

Proof. Testing Equation (3.2) with the function $a(u_i^m)$, then summing on i it yields

$$\begin{aligned}&\sum_{i=1}^j\int_\Omega \frac{1}{h}(\beta(u_i^m)-\beta(u_{i-1}^m))a(u_i^m)dx\\&+\sum_{i=1}^j\int_\Omega \frac{1}{h}(\nabla a(u_i^m)-\nabla a(u_{i-1}^m))\nabla a(u_i^m)dx\\&+\sum_{i=1}^j\int_\Omega d(t_i,x,u_{i-1}^m,\nabla a(u_i^m))\nabla a(u_i^m)dx\\&+\sum_{i=1}^j\langle K(\hat{u}_{i-1}^m),a(u_i^m)\rangle-\sum_{i=1}^j\int_\Omega f(t_i,x,u_{i-1}^m)a(u_i^m)dx=0.\end{aligned} \tag{3.14}$$

From the definition of B^* we obtain

$$\sum_{i=1}^j\int_\Omega \frac{1}{h}(\beta(u_i^m)-\beta(u_{i-1}^m))a(u_i^m)dx \ge \int_\Omega B^*(u_j^m)dx-\int_\Omega B^*(u_0^m)dx. \tag{3.15}$$

Using the identity (3.7) for the second integral in (3.14), we get

$$\begin{aligned}&\sum_{i=1}^i\int_\Omega \frac{1}{h}(\nabla a(u_i^m)-\nabla a(u_{i-1}^m))\nabla a(u_i^m)dx\\&\ge \frac{1}{2}\int_\Omega |\nabla a(u_j^m)|^2dx-\frac{1}{2}\int_\Omega |\nabla a(u_0^m)|^2dx.\end{aligned} \tag{3.16}$$

The hypotheses on d imply

$$\sum_{i=1}^j\int_\Omega d(t_i,x,u_{i-1}^m,\nabla a(u_i^m))\nabla a(u_i^m)dx \ge d_0\sum_{i=1}^j\int_\Omega |\nabla a(u_j^m)|^p dx. \tag{3.17}$$

The memory operator can be estimated as

$$\sum_{i=1}^{j}\langle K(\hat{u}_{i-1}^m), a(u_i^m)\rangle \le \frac{C}{\delta}\sum_{i=1}^{j}\int_\Omega\left(\int_0^{t_i} k(t_i,s)g(s,x,\nabla\hat{u}_{i-1}^m)ds\right)^q dx$$
$$+C\delta\sum_{i=1}^{j}\int_\Omega |\nabla a(u_i^m)|^p dx.$$

Using similar steps as in the proof of Theorem 6 we obtain

$$\left|\int_0^{t_i} k(t_i,s)g(s,x,\nabla\hat{u}_{i-1}^m)ds\right|$$
$$\le\left(\sum_{k=1}^{i} h|\nabla u_{k-1}^m|^p\right)^{\frac{1}{q}}\left(\sum_{k=1}^{i} h(t_i-t_k)^{-\gamma p}\right)^{\frac{1}{p}}+C.$$

Applying Poincaré inequality, we get

$$\sum_{i=1}^{j}\int_\Omega f(t_i,x,u_{i-1}^m)a(u_i^m)dx$$
$$\le C(\delta)\sum_{i=1}^{j}\int_\Omega B^*(u_i^m)dx + C\delta\sum_{i=1}^{j}\int_\Omega|\nabla a(u_i^m)|^p dx + C \tag{3.18}$$

Substituting inequalities (3.15)-(3.18) in (3.14) it yields

$$\int_\Omega B^*(u_j^m)dx+\frac{1}{2}\int_\Omega|\nabla a(u_j^m)|^2dx+(d_0-C\delta)\sum_{i=1}^{j}\int_\Omega|\nabla a(u_j^m)|^p dx$$
$$\le\int_\Omega B^*(u_0^m)dx+\frac{1}{2}\int_\Omega|\nabla a(u_0^m)|^2dx+C.C(\gamma)\sum_{i=1}^{j}h\sum_{k=1}^{i}h\int_\Omega|\nabla a(u_j^m)|^p dx \tag{3.19}$$
$$+C(\delta)\sum_{i=1}^{j}h\int_\Omega B^*(u_i^m)dx+C.$$

Choosing δ conveniently and applying the discrete Gronwall inequality, we achieve the proof of Lemma 7. ■

Lemma 8 *There exists a constant C >0 independent on m, n, h, i, and j such that*

$$\sum_{j=1}^{n-k}h\int_\Omega(\beta(u_{j+k}^m)-\beta(u_j^m))(a(u_{j+k}^m)-a(u_j^m))dx\le chk, \tag{3.20}$$

$$\sum_{j=1}^{n-k}h\int_\Omega|\nabla a(u_{j+k}^m)-\nabla a(u_j^m)|^2dx\le chk. \tag{3.21}$$

Proof. Summing Equation (3.2) for $i = j + 1, j + k$, choosing $a(u_{j+k}^m)-a(u_j^m)$ as test function, then summing the resultant equations for $j = 1 \dots, n - k$, we get

$$
\begin{aligned}
&\sum_{j=1}^{n-k}\int_\Omega \frac{1}{h}(\beta(u_{j+k}^m)-\beta(u_j^m))(a(u_{j+k}^m)-a(u_j^m))dx\\
&+\sum_{j=1}^{n-k}\int_\Omega \frac{1}{h}|\nabla a(u_{j+k}^m)-\nabla a(u_j^m)|^2dx\\
&+\sum_{j=1}^{n-k}\sum_{i=j+1}^{j+k}\int_\Omega d(t_i,x,u_{i-1}^m,\nabla a(u_i^m))(\nabla a(u_{j+k}^m)-\nabla a(u_j^m))dx\\
&+\sum_{j=1}^{n-k}\sum_{i=j+1}^{j+k}\langle K(\hat{u}_{i-1}^m),a(u_{j+k}^m)-a(u_j^m)\rangle\\
&-\sum_{j=1}^{n-k}\sum_{i=j+1}^{j+k}\int_\Omega f(t_i,x,u_{i-1}^m)(a(u_{j+k}^m)-a(u_j^m))dx\\
&-\sum_{j=1}^{n-k}\sum_{i=j+1}^{j+k}\int_\Omega f(t_i,x,u_{i-1}^m)(a(u_{j+k}^m)-a(u_j^m))dx=0.
\end{aligned}
\tag{3.22}
$$

The third and fifth integrals in (3.22) can be estimated as

$$
\begin{aligned}
&\sum_{j=1}^{n-k}\sum_{i=j+1}^{j+k}\int_\Omega d(t_i,x,u_{i-1}^m,\nabla a(u_i^m))(\nabla a(u_{j+k}^m)-\nabla a(u_j^m))dx\\
&\qquad\le C\sum_{i=1}^{n}\int_\Omega |d(t_i,x,u_{i-1}^m,\nabla a(u_i^m))|^q dx\\
&\qquad+Ck\int_\Omega (|\nabla a(u_{j+k}^m)|^p+|\nabla a(u_j^m)|^p)dx,
\end{aligned}
\tag{3.23}
$$

$$
\sum_{j=1}^{n-k}\sum_{i=j+1}^{j+k}\int_\Omega f(t_i,x,u_{i-1}^m)(a(u_{:i+k}^m)-a(u_{:i}^m))dx
$$

$$
\le C\sum_{i=1}^{n}\int_\Omega |f(t_i,x,u_{i-1}^m)|^q dx+Ck\int_\Omega (|a(u_{j+k}^m)|^p+|a(u_j^m)|^p)dx. \tag{3.24}
$$

From hypotheses on d and f it yields

$$
\begin{aligned}
&\sum_{i=1}^{n}\int_\Omega |d(t_i,x,u_{i-1}^m,\nabla a(u_i^m))|^q dx\\
&\le C+C\sum_{i=1}^{n}\int_\Omega |\nabla a(u_i^m)|^p dx+C\sum_{i=1}^{n}\int_\Omega B^*(a(u_i^m))dx,
\end{aligned}
\tag{3.25}
$$

$$
\sum_{i=1}^{n}\int_\Omega |f(t_i,x,u_{i-1}^m)|^q dx\le C+C\sum_{i=1}^{n}\int_\Omega B^*(a(u_i^m))dx. \tag{3.26}
$$

The operator K can be estimated as previously. Therefore we get

$$\begin{aligned}
&\sum_{j=1}^{n-k}\int_{\Omega}\frac{1}{h}(\beta(u_{j+k}^m)-\beta(u_j^m))(a(u_{j+k}^m)-a(u_j^m))dx\\
&+\sum_{j=1}^{n-k}\int_{\Omega}\frac{1}{h}|\nabla a(u_{j+k}^m)-\nabla a(u_j^m)|^2dx\le\\
&\sum_{i=1}^{n}\int_{\Omega}|\nabla a(u_i^m)|^pdx+C\sum_{i=1}^{n}\int_{\Omega}B^*(a(u_i^m))dx\\
&+C(\gamma)\sum_{i=1}^{n}\int_{\Omega}|\nabla a(u_i^m)|^pdx\\
&+Ck\sum_{j=1}^{n-k}\int_{\Omega}|a(u_{j+k}^m)|^p+|a(u_j^m)|^p)dx+C
\end{aligned}\tag{3.27}$$

Using the estimates of previous Lemma we obtain the desired results. ■

Notation 9 *Let us introduce the step functions*

$$\begin{cases}\bar{u}_n^m(t,x)=u^m(t_i,x),\ i=\overline{1,n}\\ \bar{u}_n^m(0,x)=u_0^m(x)\end{cases}\quad\begin{cases}\bar{u}_{n,h}^m(t,x)=u_n^m(t-h,x),\ t\in[h,T]\\ \bar{u}_{n,h}^m(t,x)=u_0^m(x),\quad t\in[0,h]\end{cases}$$

$$\begin{cases}d_n(t,x,s,z)=d(t_i,x,s,z),\quad t\in(t_{i-1},t_i],\quad i=\overline{1,n},\\ d_n(0,x,s,z)=d(0,x,s,z)\end{cases}$$

Corollary 10 *There exists a constant C independent of n, m, j and h such that*

$$\begin{aligned}
&\sup_{0\le t\le T}\int_{\Omega}B^*(a(\bar{u}_n^m(t,x)))dx\le C,\ \sup_{0\le t\le T}\int_{\Omega}|\nabla a(\bar{u}_n^m(t,x))|^2dx\le C\\
&\int_{Q_\tau}|\nabla a(\bar{u}_n^m(t,x))|^pdxdt\le C,\\
&\int_0^{T-\tau}\int_{\Omega}|\nabla a(\overline{u}_n^m(t+\tau,x))-\nabla a(\overline{u}_n^m(t,x))|^2dxdt\le C\tau\\
&\int_0^{T-\tau}\int_{\Omega}(\beta(\overline{u}_n^m(t+\tau,x))-\beta(\overline{u}_n^m(t,x)))\\
&\times(a(\overline{u}_n^m(t+\tau,x))-a(\overline{u}_n^m(t,x)))dx\le C\tau
\end{aligned}\tag{3.28}$$

for $k=\overline{0,n-1}$ *and* $\tau\in(kh,(k+1)h)$.

Remark 11 *(1) Corollary 10 and hypothesis* (H_3) *imply*

$$\|d_n(t,x,\bar{u}_{n,h}^m(t,x),\nabla a(\bar{u}_n^m))\|_{L^q(Q_T)^N}\le C$$

(2)From Equation (3.2) we get

$$\|\partial_h(\beta(\bar{u}_n^m)-\Delta a(\bar{u}_n^m))\|_{L^q((0,T),H^{-1,q}(\Omega))}\le C$$

(3) The estimate of B^* in Corollary 10 and hypothesis (H_1) give

$$\| \beta(\bar{u}_n^m)\|_{L^2(Q_T)} \leq C$$

(4) For the memory operator we have

$$\| K(\hat{u}_{n-1}^m)\|_{L^q((0,T),H^{-1,q}(\Omega))} \leq C$$

4 Convergence results and existence

Now we attend to the question of convergence and existence. From Corollary 10, Remark 11 and Kolomogorov compactness criterion, one can cite the following:

Corollary 12 *There exist subsequences with respect to n and m for $(\bar{u}_n^m)$ that we will note again $(\bar{u}_n^m)$ such that*

$$\begin{aligned} a(\bar{u}_n^m) &\rightharpoonup \alpha \text{ in } L^\infty((0,T),H_0^1(\Omega)) \\ a(\bar{u}_n^m) &\rightharpoonup \alpha \text{ in } L^p((0,T),W_0^{1,p}(\Omega)) \\ \beta(\bar{u}_n^m) &\rightharpoonup b \text{ in } L^2(Q_T) \\ \partial_h(\beta(\bar{u}_n^m) - \Delta a(\bar{u}_n^m)) &\rightharpoonup z \text{ in } L^q((0,T),H^{-1,q}(\Omega)) \\ d_n(t,x,\bar{u}_{n,h}^m(t,x),\nabla a(\bar{u}_n^m)) &\rightharpoonup \chi \text{ in } L^q(Q_T)^N \\ K(\hat{u}_{n-1}^m) &\rightharpoonup \mu \text{ in } L^q((0,T),H^{-1,q}(\Omega)) \end{aligned}$$

when $m, n \to \infty$.

Proof of Theorem 2. We have to show that the limit function satisfies all the conditions of Definition 1. Using Corollary 10 (third and fourth inequalities) and Kolmogorov compactness criterion [[5], p. 72] it yields $a(\bar{u}_n^m) \to \alpha$ in $L^2(Q_T)$. Since a is strictly increasing then $u_n^m \to u$ almost everywhere in Q_T. From the continuity of a it yields $a(\bar{u}_n^m) \to a(u)$ almost everywhere in Q_T and $\alpha = a(u)$, consequently $a(\bar{u}_n^m) \to a(u)$ a.e. in $L^2(Q_T)$. Applying Poincaré inequality and the fourth estimate in (3.28) we obtain

$$\| a(\bar{u}_n^m) - a(\bar{u}_{n,h}^m) \|^2_{L^2((0,T),H_0^1(\Omega))} \leq \frac{C}{n}$$

then $\bar{u}_{n,h}^m \to u$ a.e. in Q_T. Analogously $b(\bar{u}_n^m) \to b(u)$ a.e. in $L^2(Q_T)$. According to the hypothesis (H_4) we get $\| f_n(t,x,\bar{u}_{n,h}^m)\|_{L^q(Q_T)} \leq C$ and consequently $f_n(\bar{u}_{n,h}^m) - f(u)$ in $L^q(Q_T)$. For B^* we can easily prove that $B^*(u) \in L^\infty((0,T),L^1(\Omega))$. Based on the foregoing points, Equation (3.2) involves

$$\int_0^T \langle z, v\rangle dt + \int_{Q_T} \chi \nabla v dxdt + \int_0^T \langle \mu, v\rangle dt = \int_0^T \int_\Omega f_n(t,x,u)\, v dxdt. \tag{4.1}$$

Rewriting the discrete derivative with respect to t and taking into account $a(\bar{u}_n^m(0)) = a(u_0^m) \to a(u_0)$ in $H_0^1(\Omega)$ we obtain

$$
\begin{aligned}
&\int_{Q_T} \frac{1}{h}(\beta(\bar{u}_n^m(t)) - \beta(\bar{u}_n^m(t-h)))\, v dxdt \\
&+\int_{Q_T} \frac{1}{h}(\nabla a(\bar{u}_n^m(t)) - \nabla a(\bar{u}_n^m(t-h)))\nabla v dxdt \\
&= -\int_{Q_T} (\beta(\bar{u}_n^m(t))\partial_{-h}v + \nabla a(\bar{u}_n^m(t))\partial_{-h}\nabla v)dxdt \\
&+\int_{\Omega} (\beta(\bar{u}_n^m(0))v(0) + \nabla a(\bar{u}_n^m(0))\nabla v(0))dx \\
&\to -\int_{Q_T} \beta(u)v_t dxdt - \int_{Q_T} \nabla a(u)\nabla v_t dxdt + \int_{Q_T} \beta(u_0)v_t dxdt \\
&+\int_{Q_T} \nabla a(u_0)\nabla v_t dxdt \\
&= \int_0^T \langle z, v\rangle dt
\end{aligned}
\tag{4.2}
$$

$\forall v \in L^p((0, T), W_0^{1,p}(\Omega))$, $v_t \in L^2((0, T), H_0^1(\Omega))$ and $v(T) = 0$. Since v belongs to a dense subspace in $L^p((0, T), W_0^{1,p}(\Omega))$ and using the second estimate in Remark 11 we get

$$z = \partial_t(\beta(u) - \Delta a(u)) \in L^q((0, T), W^{-1,q}(\Omega)).$$

Now we prove that

$$a(\bar{u}_n^m) \to a(u) \text{in } L^p((0, T), W_0^{1,p}(\Omega)).$$

In fact, taking in (3.2) the function $\xi = a(\bar{u}_n^m) - a(\bar{v}_n^m)$ as test function and integrating on the interval $(0, \tau)$, where $a(\bar{v}_n^m) \in L^p((0, T), V_m(\Omega))$ is the approximate of $a(u)$ in $L^p\left((0, T), W_0^{1,p}(\Omega)\right)$, constant on each interval $((k - 1) h, kh)$, we obtain

$$
\begin{aligned}
&\int_{Q_\tau} \partial_h\beta(\bar{u}_n^m)(a(\bar{u}_n^m) - a(\bar{v}_n^m))dxdt \\
&+\int_{Q_\tau} \partial_h\nabla a(\bar{u}_n^m)(\nabla a(\bar{u}_n^m) - \nabla a(\bar{v}_n^m))dxdt \\
&+\int_{Q_\tau} d_n(t, x, \bar{u}_{n,h}^m, \nabla a(\bar{u}_n^m))(\nabla a(\bar{u}_n^m) - \nabla a(\bar{v}_n^m))dxdt \\
&+\int_0^\tau \langle K(\hat{u}_{n-1}^m), a(\bar{u}_n^m) - a(\bar{v}_n^m)\rangle dt \\
&= \int_{Q_\tau} f_n(t, x, \bar{u}_{n,h}^m)(a(\bar{u}_n^m) - a(\bar{v}_n^m))dxdt
\end{aligned}
\tag{4.3}
$$

Lemma 4 implies

$$\begin{aligned}&\int_0^\tau\int_\Omega \partial_h\beta(\bar{u}_n^m)(a(\bar{u}_n^m)-a(\bar{v}_n^m))dxdt\\&+\int_0^\tau\int_\Omega \partial_h\nabla a(\bar{u}_n^m)(\nabla a(\bar{u}_n^m)-\nabla a(\bar{v}_n^m))dxdt\\ \geq\ & \frac{1}{h}\int_{\tau-h}^\tau\int_\Omega B^*(a(\bar{u}_n^m))dxdt+\frac{1}{2h}\int_{\tau-h}^\tau\int_\Omega |\nabla a(\bar{u}_n^m)|^2dxdt\\&-\int_\Omega B^*(a(u(\tau)))dx-\frac{1}{2}\int_\Omega |\nabla a(u(\tau))|^2dx+c\varepsilon\end{aligned}$$

From Fatou Lemma we deduce

$$\begin{aligned}&\liminf_{m;n\to\infty}\frac{1}{h}\int_{\tau-h}^\tau\int_\Omega\left(B^*(a(\bar{u}_n^m))+\frac{1}{2}|\nabla a(\bar{u}_n^m)|^2\right)dxdt\\ \geq\ &\int_\Omega B^*(a(u(\tau)))dx+\frac{1}{2}\int_\Omega |\nabla a(u(\tau))|^2dx,\end{aligned}$$

consequently

$$\begin{aligned}&\liminf_{m;n\to\infty}\int_0^\tau\int_\Omega [\partial_h\beta(\bar{u}_n^m)(a(\bar{u}_n^m)-a(\bar{v}_n^m))\\&+\partial_h\nabla a(\bar{u}_n^m)(\nabla a(\bar{u}_n^m)-\nabla a(\bar{v}_n^m))]dxdt\geq 0\end{aligned}$$

Taking into account the convergence of $a(\bar{u}_n^m)$ to $a(u)$ in $L^2(Q_T)$, the convergence of $a(\bar{v}_n^m)$ to $a(u)$ in $L^p\left((0,\ T),\ W_0^{1,p}(\Omega)\right)$, the continuity of d, the weak convergence of d in $L^q(Q_T)^N$ and the dominated convergence theorem, we obtain

$$d_n(t,\ x,\ \bar{u}_{n,h}^m,\ \nabla a(u))\to d(t,\ x,\ u,\ \nabla a(u))\quad\text{in}\quad L^q(Q_T)^N$$

In addition to monotonicity of d gives

$$\begin{aligned}&\int_0^\tau\int_\Omega d_n\ (t,\ x,\ \bar{u}_{n,h}^m,\ \nabla a\ (\bar{u}_n^m))\ (\nabla a(\bar{u}_n^m)-\nabla a(\bar{v}_n^m))dxdt\\ =\ &\int_0^\tau\int_\Omega (d_n(t,\ x,\ \bar{u}_{n,h}^m,\ \nabla a(\bar{u}_n^m))-d_n(t,\ x,\ \bar{u}_{n,h}^m,\ \nabla a(\bar{v}_n^m)))\\&\times(\nabla a(\bar{u}_n^m)-\nabla a(\bar{v}_n^m))dxdt\\&+\int_0^\tau\int_\Omega d_n\ (t,\ x,\ \bar{u}_{n,h}^m,\ \nabla a\ (\bar{v}_n^m))\ (\nabla a(\bar{u}_n^m)-\nabla a(\bar{v}_n^m))dxdt\\ \geq\ & d_1\int_0^\tau\int_\Omega |\nabla a(\bar{u}_n^m)-\nabla a(\bar{v}_n^m)|^p dxdt-c\varepsilon\end{aligned}$$

as previously using hypotheses (H_5) and (H_6), the operator memory can be estimated as

$$\begin{aligned}&\int_0^\tau \langle K(\hat{u}_{n-1}^m),\ a(\bar{u}_n^m) - a(\bar{v}_n^m)\rangle dt\\ \leq\ & \frac{C}{\delta}\int_0^\tau\int_\Omega\int_0^t |\nabla a(\bar{u}_n^m) - \nabla a(\bar{v}_n^m)|^p dxdt\\ &+C\delta\|a(\bar{u}_n^m) - a(\bar{v}_n^m)\|^p_{L^p((0,T),W_0^{1,p}(\Omega))} + C\varepsilon\end{aligned}$$

For f_n we have

$$\int_0^\tau\int_\Omega f_n(t,\ x,\ \bar{u}_{n,h}^m)(a(\bar{u}_n^m) - a(\bar{v}_n^m))dxdt \leq C\varepsilon,$$

regrouping the estimates of all terms of Equation (4.3) we obtain

$$\begin{aligned}&(d_1 - C\delta)\int_0^\tau\int_\Omega |\nabla a(\bar{u}_n^m) - \nabla a(\bar{v}_n^m)|^p dxdt\\ \leq\ & C\int_0^\tau\int_\Omega\int_0^t |\nabla a(\bar{u}_n^m) - \nabla a(\bar{v}_n^m)|^p dxdt.\end{aligned}$$

Gronwall inequality implies

$$\int_0^\tau\int_\Omega |\nabla a(\bar{u}_n^m) - \nabla a(\bar{v}_n^m)|^p dxdt \leq C\varepsilon,$$

hence we get

$$a(\bar{u}_n^m) \to a(u) \quad \text{in} \quad L^p((0,\ T),\ W_0^{1,p}(\Omega)).$$

Following the Proof of Theorem 2: From the continuity of d and g it yields

$$\begin{aligned}d_n(t,\ x,\ \bar{u}_{n,h}^m,\ \nabla a(\bar{u}_n^m)) &\to d(t,\ x,\ u,\ \nabla a(u)) \quad \text{a.e.} \quad Q_T\\ g_n(t,\ x,\ \nabla\hat{u}_{n-1}^m) &\to g(t,\ x,\ \nabla u) \quad \text{a.e.} \quad Q_T\end{aligned}$$

The weak convergences of $d_n(t, x, \bar{u}_{n,h}^m, \nabla a(\bar{u}_n^m))$ and $K(\hat{u}_{n-1}^m)$ and the almost everywhere convergences imply that $\chi = d(t, x, u, \nabla a(u))$ and $\mu = K(u)$. So u is the weak solution of the problem (P) in the sense of Definition 1.

Now we prove the uniqueness of the weak solution. We assume that the problem (P) has two solutions u^1 and $u^2 \in L^2((0,\ T),\ H_0^1(\Omega))$. Taking into account that $\beta(u_0^1) = \beta(u_0^2)$ and $\nabla a(u_0^1) = \nabla a(u_0^2)$, we get

$$\begin{aligned}
&\int_0^T\int_\Omega ((\beta(u^1)-\beta(u^2))v_t+\nabla(a(u^1)-a(u^2))\nabla v_t)\,dxdt\\
&+\int_0^T\int_\Omega (d(t,\ x,\ u^1,\ \nabla a(u^1))-d(t,\ x,\ u^2,\ \nabla a(u^2)))\nabla v dxdt\\
&+\int_0^T \langle K(u^1)-K(u^2),\ v\rangle\,dt\\
&\quad=\int_0^T\int_\Omega (f(t,\ x,\ u^1)-f(t,\ x,\ u^2))\,vdxdt.
\end{aligned}\tag{4.4}$$

Choosing in (4.4) the test function

$$v_s(t)=\begin{cases}\int_t^s(a(u^1(\tau))-a(u^2(\tau)))d\tau, & t<s\\ 0, & t\geq s\end{cases}$$

and since $v_s(s) = 0$ then integrating by parts it yields

$$\begin{aligned}
&\int_0^s\langle\beta(u^1)-\beta(u^2),\ a(u^1)-a(u^2)\rangle\,dt\\
&+\int_0^s\int_\Omega |\nabla a(u^1)-\nabla a(u^2)|^2\,dxdt\\
\leq\quad &\delta\int_0^s\int_\Omega |\nabla a(u^1)-\nabla a(u^2)|^2\,dxdt+\frac{C}{\delta}\int_0^s\int_\Omega |\nabla v_s|^2\,dtdx.
\end{aligned}$$

On the other hand, we have

$$\int_0^s\int_\Omega |\nabla v_s|^2\,dtdx\leq C\int_0^s\int_0^t\int_\Omega |\nabla a(u^1(x,\ \tau))-\nabla a(u^2(x,\ \tau))|^2\,dxd\tau dt.$$

Applying Gronwall lemma we get

$$\int_0^s\int_\Omega |\nabla a(u^1)-\nabla a(u^2)|^2\,dxdt=0$$

consequently $u^1 \equiv u^2$. This achieves the Proof of Theorem 2.

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Authors' contributions
The authors declare that the work was realized in collaboration with same responsibility. All authors read and approved the final manuscript.

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