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Existence of solutions for a differential inclusion problem with singular coefficients involving the p(x)-Laplacian

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Abstract

Using the non-smooth critical point theory we investigate the existence and multiplicity of solutions for a differential inclusion problem with singular coefficients involving the $p(x)$-Laplacian.

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1 Introduction

In this article, we study the existence and multiplicity of solutions for the differential inclusion problem with singular coefficients involving the $p(x)$-Laplacian of the form

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \in \lambda a_1(x)\partial G_1(x,u) + \mu a_2(x)\partial G_2(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the following conditions are satisfied:

(**P**) Ω is a bounded open domain in $\mathbb{R}^N$, $N \geq 2$, $p \in C(\bar{\Omega})$, $1 < p^- := \inf_\Omega p(x) \leq p^+ := \sup_\Omega p(x) < +\infty$, $\lambda, \mu \in \mathbb{R}$.

(**A**) For $i = 1, 2$, $a_i \in L^{r_i(x)}(\Omega)$, $a_i(x) > 0$ for $x \in \Omega$, $G_i(x, u)$ is measurable with respect to x (for every $u \in \mathbb{R}$) and locally Lipschitz with respect to u (for a.e. $x \in \Omega$), $\partial G_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is the Clarke sub-differential of G_i and $|\xi_i| \leq c_1 + c_2|t|^{q_i(x)-1}$ for $x \in \Omega$, $t \in \mathbb{R}$ and $\xi_i \in \partial G_i$, where c_i is a positive constant, $r_i, q_i \in C(\bar{\Omega})$, $r_i^- > 1$, $q_i^- > 1$, $r_i(x) > q_i(x)$ for all $x \in \Omega$, and

$$q_i(x) < \frac{r_i(x) - q_i(x)}{r_i(x)} p^*(x), \ \forall x \in \overline{\Omega}, \tag{1.2}$$

here

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases} \tag{1.3}$$

($\mathbf{A}_1$) $q_1^+ < p^-$.

($\mathbf{A}_2$) $q_2^- > p^+$.

A typical example of (1.1) is the following problem involving subcritical Sobolev-Hardy exponents of the form

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \in \lambda \frac{1}{|x|^{s_1(x)}}\partial G_1(x,u) + \mu \frac{1}{|x|^{s_2(x)}}\partial G_2(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

and in this case the assumption corresponding to (**A**) is the following

(**A**)*$0 \in \bar{\Omega}$, for $i = 1, 2$, $\partial G_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is the Clarke sub-differential of G_i and $|\xi_i| \le c_1 + c_2|t|^{q_i(x)-1}$ for $x \in \Omega$, $t \in \mathbb{R}$ and $\xi_i \in \partial G_i$, where c_i is a positive constant, s_i, $q_i \in C(\bar{\Omega})$, $0 \le s_i^- \le s_i^+ < N$, $q_i^- > 1$, and

$$q_i(x) < \frac{N - s_i(x)q_i(x)}{N} p^*(x), \forall x \in \bar{\Omega}. \tag{1.5}$$

The operator -div($|\nabla u|^{p(x)-2} \nabla u$) is said to be the $p(x)$-Laplacian, and becomes p-Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian; for example, it is inhomogeneous. The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years. These problems are interesting in applications and raise many difficult mathematical problems. One of the most studied models leading to problem of this type is the model of motion of electro-rheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electro-magnetic field [1,2]. Problems with variable exponent growth conditions also appear in the mathematical modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal baro-tropic gas through a porous medium [3,4]. Another field of application of equations with variable exponent growth conditions is image processing [5]. The variable nonlinearity is used to outline the borders of the true image and to eliminate possible noise. We refer the reader to [6-11] for an overview of and references on this subject, and to [12-21] for the study of the $p(x)$-Laplacian equations and the corresponding variational problems.

Since many free boundary problems and obstacle problems may be reduced to partial differential equations with discontinuous nonlinearities, the existence of multiple solutions for Dirichlet boundary value problems with discontinuous nonlinearities has been widely investigated in recent years. Chang [22] extended the variational methods to a class of non-differentiable functionals, and directly applied the variational methods for non-differentiable functionals to prove some existence theorems for PDE with discontinuous nonlinearities. Later Kourogenis and Papageorgiou [23] obtained some nonsmooth critical point theories and applied these to nonlinear elliptic equations at resonance, involving the p-Laplacian with discontinuous nonlinearities. In the celebrated work [24,25], Ricceri elaborated a Ricceri-type variational principle and a three critical points theorem for the Gâteaux differentiable functional, respectively. Later, Marano and Motreanu [26,27] extended Ricceri's results to a large class of non-differentiable functionals and gave some applications to differential inclusion problems involving the p-Laplacian with discontinuous nonlinearities.

In [21], by means of the critical point theory, Fan obtain the existence and multiplicity of solutions for (1.1) under the condition of $G_i(x,\cdot) \in C^1(\mathbb{R})$ and $g_i = G_i'$ satisfying the Carathéodory condition for i = 1, 2, $x \in \Omega$. The aim of the present article is to generalize the main results of [21] to the case of the functional of problem (1.1) is nonsmooth.

This article is organized as follows: In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces and the generalized gradient of the locally Lipschitz function; In Section 3, we give the variational principle which is needed in the sequel; In Section 4, using the critical point theory, we prove the existence and multiplicity results for problem (1.1).

2 Preliminaries

2.1 Variable exponent Sobolev spaces

Let Ω be a bounded open subset of $\mathbb{R}^N$, denote $L_+^\infty(\Omega) = \{p \in L^\infty(\Omega) : \text{ess}\inf_\Omega p(x) \geq 1\}$.

For $p \in L_+^\infty(\Omega)$, denote

$$p^- = p^-(\Omega) = \text{ess} \inf_{x\in\Omega} p(x), \quad p^+ = p^+(\Omega) = \text{ess} \sup_{x\in\Omega} p(x).$$

On the basic properties of the space $W^{1,p(x)}(\Omega)$ we refer to [7,28-30]. Here we display some facts which will be used later.

Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. For $p \in L_+^\infty(\Omega)$, define the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{ u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

and

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the norm

$$||u||_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Hereafter, we always assume that $p^- > 1$.

Proposition 2.1. [7,31] *The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.2. [7,31] *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p^0(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p^0(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p^0(x)}(\Omega)$, $\int_\Omega |uv| dx \leq 2|u|_{p(x)}|v|_{p^0(x)}$.*

Proposition 2.3. [7,31]*In* $W_0^{1,p(x)}(\Omega)$ *the Poincaré inequality holds, that is, there exists a positive constant c such that*

$$|u|_{L^{p(x)}(\Omega)} \leq c|\nabla u|_{L^{p(x)}(\Omega)}, \forall u \in W_0^{1,p(x)}(\Omega).$$

So $|\nabla u|_{L^{p(x)}(\Omega)}$ *is an equivalent norm in* $W_0^{1,p(x)}(\Omega)$.

Proposition 2.4. [7,28,29,31]*Assume that the boundary of* Ω *possesses the cone property and* $p \in C(\bar{\Omega})$. *If* $q \in C(\bar{\Omega})$ *and* $1 \leq q(x) < p^*(x)$ *for* $x \in \bar{\Omega}$, *then there is a compact embedding* $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$.

Let us now consider the weighted variable exponent Lebesgue space.

Let $a \in \mathbf{S}(\Omega)$ and $a(x) > 0$ for $x \in \Omega$. Define

$$L_{a(x)}^{p(x)}(\Omega) = \left\{ u \in \mathbf{S}(\Omega) : \int_\Omega a(x)|u(x)|^{p(x)} dx < \infty \right\}$$

with the norm

$$|u|_{L_{a(x)}^{p(x)}(\Omega)} = |u|_{(p(x),a(x))} = \inf \left\{ \lambda > 0 : \int_\Omega a(x) \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

then $L_{a(x)}^{p(x)}(\Omega)$ is a Banach space. The following proposition follows easily from the definition of $|u|_{L_{a(x)}^{p(x)}(\Omega)}$.

Proposition 2.5. (see [7,31]) *Set* $\rho(u) = \int_\Omega a(x)|u(x)|^{p(x)}$ dx. For $u, u_k \in L_{a(x)}^{p(x)}(\Omega)$, *we have*

(1) For $u \neq 0$, $|u|_{(p(x),a(x))} = \lambda \Leftrightarrow \rho\left(\frac{u}{\lambda}\right) = 1$.

(2) $|u|_{(p(x),a(x))} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$.

(3) If $|u|_{(p(x),a(x))} > 1$, *then* $|u|_{(p(x),a(x))}^{p^-} \leq \rho(u) \leq |u|_{(p(x),a(x))}^{p^+}$.

(4) If $|u|_{(p(x),a(x))} < 1$, *then* $|u|_{(p(x),a(x)}^{p^+} \leq \rho(u) \leq |u|_{(p(x),a(x))}^{p^-}$.

(5) $\lim_{k\to\infty} |u_k|_{(p(x),a(x))} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = 0$.

(6) $|u_k|_{(p(x),a(x))} \to \infty \Leftrightarrow \rho(u_k) \to \infty$.

Proposition 2.6. (see [21]) *Assume that the boundary of* Ω *possesses the cone property and* $p \in C(\bar{\Omega})$. *Suppose that* $a \in L^{r(x)}(\Omega)$, $a(x) > 0$ *for* $x \in \Omega$, $r \in C(\bar{\Omega})$ *and* $r^- > 1$. *If* $q \in C(\bar{\Omega})$ *and*

$$1 \leq q(x) < \frac{r(x) - 1}{r(x)} p^*(x) := p_{a(x)}^*(x), \forall x \in \bar{\Omega}, \tag{2.1}$$

then there is a compact embedding $W^{1,p(x)}(\Omega) \to L_{a(x)}^{q(x)}(\Omega)$.

The following proposition plays an important role in the present article.

Proposition 2.7. *Assume that the boundary of* Ω *possesses the cone property and* $p \in C(\bar{\Omega})$. *Suppose that* $a \in L^{r(x)}(\Omega)$, $a(x) > 0$ *for* $x \in \Omega$, $r \in C(\bar{\Omega})$ *and* $r(x) > q(x)$ *for all* $x \in \Omega$. *If* $q \in C(\bar{\Omega})$ *and*

$$1 \le q(x) < \frac{r(x) - q(x)}{r(x)} p^*(x), \forall x \in \bar{\Omega}, \tag{2.2}$$

then there is a compact embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}_{(a(x))^{q(x)}}(\Omega)$.

Proof. Set $r_1(x) = \frac{r(x)}{q(x)}$, then $r_1^- > 1$ and $(a(x))^{q(x)} \in L^{r_1(x)}(\Omega)$. Moreover, from (2.2) we can get

$$1 \le q(x) < \frac{r_1(x) - 1}{r_1(x)} p^*(x), \ \forall x \in \bar{\Omega}.$$

Using Proposition 2.6, we see that the embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}_{(a(x))^{q(x)}}(\Omega)$ is compact.

■

2.2 Generalized gradient of the locally Lipschitz function

Let $(X, ||\cdot||)$ be a real Banach space and X^* be its topological dual. A function $f : X \to \mathbb{R}$ is called locally Lipschitz if each point $u \in X$ possesses a neighborhood Ω_u such that $|f(u_1) - f(u_2)| \le L||u_1 - u_2||$ for all $u_1, u_2 \in \Omega_u$, for a constant $L > 0$ depending on Ω_u. The generalized directional derivative of f at the point $u \in X$ in the direction $v \in X$ is

$$f^0(u, v) = \limsup_{w \to u, t \to 0} \frac{1}{t}(f(w + tv) - f(w)).$$

The generalized gradient of f at $u \in X$ is defined by

$$\partial f(u) = \{u^* \in X^* : \langle u^*, \varphi \rangle \le f^0(u; \varphi) \quad \text{for all } \varphi \in X\},$$

which is a non-empty, convex and w^*-compact subset of X, where $\langle \cdot, \cdot \rangle$ is the duality pairing between X^* and X. We say that $u \in X$ is a critical point of f if $0 \in \partial f(u)$. For further details, we refer the reader to Chang [22].

We list some fundamental properties of the generalized directional derivative and gradient that will be used throughout the article.

Proposition 2.8. (see [22,32]) (1) *Let* $j : X \to \mathbb{R}$ *be a continuously differentiable function. Then* $\partial j(u) = \{j'(u)\}$, $j^0(u; z)$ *coincides with* $\langle j'(u), z \rangle_X$ *and* $(f + j)^0(u, z) = f^0(u; z) + \langle j'(u), z \rangle_X$ *for all* $u, z \in X$.

(2) *The set-valued mapping* $u \to \partial f(u)$ *is upper semi-continuous in the sense that for each* $u_0 \in X$, $\varepsilon > 0$, $v \in X$, *there is a* $\delta > 0$, *such that for each* $w \in \partial f(u)$ *with* $||w - u_0|| < \delta$, *there is* $w_0 \in \partial f(u_0)$

$$|\langle w - w_0, v \rangle| < \varepsilon.$$

(3) *(Lebourg's mean value theorem) Let* u *and* v *be two points in* X. *Then there exists a point* w *in the open segment joining* u *and* v *and* $x_w^* \in \partial f(w)$ *such that*

$$f(u) - f(v) = \langle x_w^*, u - v \rangle_X.$$

(4) *The function*

$$m(u) = \min_{w \in \partial f(u)} w_{X^*}$$

exists, and is lower semi continuous; i.e., $\liminf_{u \to u_0} m(u) \geq m(u_0)$.

In the following we need the nonsmooth version of Palais-Smale condition.

Definition 2.1. We say that ϕ satisfies the $(\mathrm{PS})_c$-condition if any sequence $\{u_n\} \subset X$ such that $\phi(u_n) \to c$ and $m(u_n) \to 0$, as $n \to +\infty$, has a strongly convergent subsequence, where $m(u_n) = \inf\{||u^*||_{X^*} : u^* \in \partial\phi\ (u_n)\}$.

In what follows we write the $(\mathrm{PS})_c$-condition as simply the PS-condition if it holds for every level $c \in \mathbb{R}$ for the Palais-Smale condition at level c.

3 Variational principle

In this section we assume that Ω and $p(x)$ satisfy the assumption (**P**). For simplicity we write $X = W_0^{1,p(x)}(\Omega)$ and $||u|| = |\nabla u|_{p(x)}$ for $u \in X$. Denote by $u_n \rightharpoonup u$ and $u_n \to u$ the weak convergence and strong convergence of sequence $\{u_n\}$ in X, respectively, denote by c and c_i the generic positive constants, $B_\rho = \{u \in X : ||u|| < \rho\}$, $S_\rho = \{u \in X : ||u|| = \rho\}$.

Set

$$F(x, t) = \lambda a_1(x) G_1(x, t) + \mu a_2(x) G_2(x, t), \tag{3.1}$$

where a_i and G_i $(i = 1, 2)$ are as in (**A**).

Define the integral functional

$$\varphi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \int_\Omega F(x, u) dx, \forall u \in X. \tag{3.2}$$

We write

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx, \ \Psi(u) = \int_\Omega F(x, u) dx,$$

then it is easy to see that $J \in C^1(X, \mathbb{R})$ and $\phi = J - \Psi$.

Below we give several propositions that will be used later.

Proposition 3.1. (see [19]) *The functional* $J : X \to \mathbb{R}$ *is convex. The mapping* $J' : X \to X^*$ *is a strictly monotone, bounded homeomorphism, and is of* (S_+) *type, namely*

$$u_n \rightharpoonup u \text{ and } \overline{\lim}_{n\to\infty} J'(u_n)(u_n - u) \leq 0 \text{ implies } u_n \to u.$$

Proposition 3.2. Ψ *is weakly-strongly continuous, i.e.,* $u_n \rightharpoonup u$ *implies* $\Psi(u_n) \to \Psi(u)$.

Proof. Define $\Upsilon_1 = \int_\Omega G_1(x, u)\, dx$ and $\Upsilon_2 = \int_\Omega G_2(x, u)\, dx$. In order to prove Ψ is weakly-strongly continuous, we only need to prove Υ_1 and Υ_2 are weakly-strongly continuous. Since the proofs of Υ_1 and Υ_2 are identical, we will just prove Υ_1.

We assume $u_n \rightharpoonup u$ in X. Then by Proposition 2.8.3, we have

$$\begin{aligned}\Upsilon_1(u_n) - \Upsilon_1(u) &= \int_\Omega (G_1(x, u_n) - G_1(x, u)) dx \\ &= \int_\Omega \xi_n(x)(u_n - u) dx,\end{aligned}$$

where $\xi_n \in \partial G_1(,\tau_n(x))$ for some $\tau_n(x)$ in the open segment joining u and u_n. From Chang [22] we know that $\xi_n \in L^{q_1^0(x)}(\Omega)$. So using Proposition 2.5, we have

$$\Upsilon_1(u_n) - \Upsilon_1(u) \to 0.$$

■

Proposition 3.3. *Assume* (**A**) *holds and F satisfies the following condition:*

(**B**) $F(x,u) \le \theta\lambda a_1(x)\langle\xi_1,u\rangle + \theta\mu a_2(x)\langle\xi_2,u\rangle + b(x) + \sum_{i=1}^m d_i(x)|u|^{k_i(x)}$ *for a.e.* $x \in \Omega$, *all* $u \in X$ *and* $\xi_1 \in \partial G_1$, $\xi_2 \in \partial G_2$, *where* θ *is a constant,* $h_i, k_i \in C(\bar\Omega)$, $k_i(x) < \frac{h_i(x)-1}{h_i(x)} p^*(x)$ *for* $x \in \overline{\Omega}$, $k_i^+ < p^-$,

$h_i, k_i \in C(\bar\Omega)$, $k_i(x) < \frac{h_i(x)-1}{h_i(x)} p^*(x)$ *for* $x \in \overline{\Omega}$, $k_i^+ < p^-$.

Then ϕ *satisfies the nonsmooth* (PS) *condition on X.*

Proof. Let $\{u_n\}$ be a nonsmooth (PS) sequence, then by (B) we have

$$\begin{aligned}
c + 1 + \|u_n\| &\ge \varphi(u_n) - \theta\langle\omega, u_n\rangle \\
&= \int_\Omega \left(\frac{1}{p(x)} - \theta\right)|\nabla u_n|^{p(x)}dx \\
&\quad - \int_\Omega (F(x,u_n) - \theta\lambda a_1(x)\langle\xi_1,u_n\rangle - \theta\mu a_2(x)\langle\xi_2,u_n\rangle)dx \\
&\ge \left(\frac{1}{p^+} - \theta\right)||u_n||^{p^-} - c_1 - \int_\Omega \left(b(x) + \sum_{i=1}^m d_i(x)|u_n|^{k_i(x)}\right)dx \\
&\ge \left(\frac{1}{p^+} - \theta\right)||u_n||^{p^-} - c_2 - \sum_{i=1}^m |u_n|_{(k_i(x),d_i(x))}^{k_i^+} \\
&\ge \left(\frac{1}{p^+} - \theta\right)||u_n||^{p^-} - c_2 - c_3\sum_{i=1}^m ||u_n||^{k_i^+},
\end{aligned}$$

and consequently $\{u_n\}$ is bounded.

Thus by passing to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ in X as $n \to \infty$. We have

$$\langle J'(u_n), u_n - u\rangle - \int_\Omega \lambda\xi_{1n}(x)a_1(x)(u_n - u) - \int_\Omega \mu\xi_{2n}(x)a_2(x)(u_n - u)dx \le \varepsilon_n||u_n - u||$$

with $\varepsilon_n \downarrow 0$, where $\xi_{in}(x) \in \partial G_i(x, u_n)$ for a.e. $x \in \Omega$, $i = 1, 2$. From Chang [22] or Theorem 1.3.10 of [33], we know that $\xi_{in}(x) \in L^{q_1^0(x)}$, $i = 1, 2$. Since X is embedded compactly in $L^{q_i(x)}_{(a_i(x))^{q_i(x)}}(\Omega)$, we have that $u_n \to u$ as $n \to \infty$ in $L^{q_i(x)}_{(a_i(x))^{q_i(x)}}(\Omega)$, $i = 1, 2$. So using Proposition 2.2, we have

$$\int_\Omega \xi_{in}(x)a_i(x)(u_n - u)dx \to 0 \text{ as } n \to \infty \quad , i = 1, 2.$$

Therefore we obtain $\limsup_{n\to\infty}\langle J\prime(u_n), u_n - u\rangle \le 0$. But we know that J' is a mapping of type (S_+). Thus we have

$u_n \to u \, \text{in} \, X.$

Remark 3.1. Note that our condition (1.2) is stronger than (1.2) of [21]. Because Ψ' is weakly-strongly continuous in [21], to verify that ϕ satisfies (PS) condition on X, it is enough to verify that any (PS) sequence is bounded. However, in this paper we do not know whether $\xi(u)$ is weakly-strongly continuous, where $\xi(u) \in \neg\Psi$. Therefore, it will be very useful to consider this problem.

Below we denote

$$F_1(x,t) = \lambda a_1(x) G_1(x,t), \; F_2(x,t) = \mu a_2(x) G_2(x,t).$$

We shall use the following conditions.

($\mathbf{B}_1$) $\exists\, c_0 > 0$ such that $G_2(x, t) \geq -c_0$ for $x \in \Omega$ and $t \in \mathbb{R}$.

($\mathbf{B}_2$) $\exists \theta \in \left(0, \frac{1}{p^+}\right)$ and $M > 0$ such that $0 < G_2(x, u) \leq \theta \langle u, \xi_2 \rangle$ for $x \in \Omega$, $u \in X$ and $|u| \geq M$, $\xi_2 \in \neg G_2$.

Corollary 3.1. *Assume* (**P**), (**A**) *and* ($\mathbf{A}_1$) *hold. Then* ϕ *satisfies nonsmooth* (PS) *condition on X provided either one of the following conditions is satisfied.*

(1). $\lambda \in \mathbb{R}$ and $\mu = 0$.
(2). $\lambda \in \mathbb{R}$, $\mu = 0$ and ($\mathbf{B}_1$) holds.
(3). $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$ and ($\mathbf{B}_2$) holds.

Proof. In case (1) or (2), we have, for $x \in \Omega$ and $t \in \mathbb{R}$,

$$F(x,t) \leq F_1(x,t) + |\mu| c_0 a_2(x) \leq (c_1 a_1(x) + |\mu| c_0 a_2(x)) + c_2 a_1(x) |t|^{q_1(x)},$$

which shows that the condition (**B**) with $\theta = 0$ is satisfied.

In case (3), noting that ($\mathbf{B}_2$) and (**A**) imply ($\mathbf{B}_1$), by the conclusion (1) and (2) we know ϕ satisfies (PS) condition if $\mu \leq 0$. Below assume $\mu > 0$. The conditions ($\mathbf{B}_2$) and (**A**) imply that, for $x \in \Omega$ and $u \in X$,

$$G_2(x,u) \leq \theta \langle u, \xi_2 \rangle + c_3, \; \text{and} \, F_2(x,u) \leq \theta \mu \mathbf{a}_2(x) \langle u, \xi_2 \rangle + c_3 \mu a_2(x),$$

so we have

$$\begin{aligned} F(x,u) - \theta \lambda a_1(x) \langle \xi_1, u \rangle - \theta \mu a_2(x) \langle \xi_2, u \rangle &= (F_1(x,u) - \theta \lambda a_1(x) \langle \xi_1, u \rangle) \\ &\quad + (F_2(x,u) - \theta \mu a_2(x) \langle \xi_2, u \rangle) \\ &\leq c_1 a_1(x) + c_2 a_1(x) |u|^{q_1(x)} + c_3 \mu a_2(x), \end{aligned}$$

which shows (**B**) holds. The proof is complete. ■

As X is a separable and reflexive Banach space, there exist (see [[34], Section 17]) $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{f_n\}_{n=1}^{\infty} \subset X^*$ such that

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1 & \text{if} \, n = m \\ 0 & \text{if} \, n \neq m, \end{cases}$$

$$X = \overline{\operatorname{span}}\{e_n : n = 1, 2, \ldots, \}, \; X^* = \overline{\operatorname{span}}^{W*}\{f_n : n = 1, 2, \ldots, \}.$$

For $k = 1, 2, \ldots,$ denote

$$X_k = \operatorname{span}\{ e_k\}, \; Y_k = \oplus_{j=1}^{k} X_j, \; Z_k = \overline{\oplus_{j=k}^{\infty} X_j}. \tag{3.3}$$

Proposition 3.5. [35]*Assume that* $\Psi : X \to \mathbb{R}$ *is weakly-strongly continuous and* $\Psi(0) = 0$. *Let* $\gamma > 0$ *be given. Set*

$$\beta_k = \beta_k(\gamma) = \sup_{u \in Z_k, \|u\| \le \gamma} |\Psi(u)|.$$

Then $\beta_k \to 0$ *as* $k \to \infty$.

Proposition 3.6. (Nonsmooth Mountain pass theorem, see [23,33]) *If X is a reflexive Banach space,* $\phi : X \to \mathbb{R}$ *is a locally Lipschitz function which satisfies the nonsmooth* $(\mathrm{PS})_c$*-condition, and for some* $r > 0$ *and* $e_1 \in X$ *with* $||e_1|| > r$, $\max\{\phi(0), \phi(e_1)\} \le \inf\{\phi(u) : ||u|| = r\}$. *Then* ϕ *has a nontrivial critical* $u \in X$ *such that the critical value* $c = \phi(u)$ *is characterized by the following minimax principle*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)$$

where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e_1\}$.

Proposition 3.7. (Nonsmooth Fountain theorem, see [36]) *Assume* (F_1) *X is a Banach space,* $\phi : X \to \mathbb{R}$ *be an invariant locally Lipschitz functional, the subspaces* X_k, Y_k *and* Z_k *are defined by* (3.3).

If, for every $k \in \mathbb{N}$, *there exist* $\rho_k > r_k > 0$ *such that*

(F_2) $a_k := \inf_{\substack{u \in Z_k \\ ||u|| = r_k}} \varphi(u) \to \infty, \; k \to \infty,$

(F_3) $b_k := \max_{\substack{u \in Y_k \\ ||u|| = \rho_k}} \varphi(u) \le 0,$

(F_4) ϕ *satisfies the nonsmooth* $(\mathrm{PS})_c$ *condition for every* $c > 0$, *then* ϕ *has an unbounded sequence of critical values.*

Proposition 3.8. (Nonsmooth dual Fountain theorem, see [37]) *Assume* (F_1) *is satisfied and there is a* $k_0 > 0$ *such that, for each* $k \ge k_0$, *there exists* $\rho_k > \gamma_k > 0$ *such that*

(D_1) $a_k := \inf_{\substack{u \in Z_k \\ ||u|| = \rho_k}} \varphi(u) \ge 0,$

(D_2) $b_k := \max_{\substack{u \in Y_k \\ ||u|| = r_k}} \varphi(u) < 0,$

(D_3) $d_k := \inf_{\substack{u \in Z_k \\ ||u|| \le \rho_k}} \varphi(u) \to 0, \; k \to \infty,$

(D_4) ϕ *satisfies the nonsmooth* $(\mathrm{PS})_c^*$ *condition for every* $c \in [d_{k_0}, 0)$, *then* ϕ *has a sequence of negative critical values converging to 0.*

Remark 3.2. We say ϕ that satisfies the nonsmooth $(\mathrm{PS})_c^*$ condition at level $c \in \mathbb{R}$ (with respect to (Y_n)) if any sequence $\{u_n\} \subset X$ such that

$$n_j \to \infty, u_{n_j} \in Y_{n_j}, \varphi(u_{n_j}) \to c, m|_{Y_{n_j}}(u_n) \to 0$$

contains a subsequence converging to a critical point of ϕ.

4 Existence and multiplicity of solutions

In this section, using the critical point theory, we give the existence and multiplicity results for problem (1.1). We shall use the following assumptions:

($\mathbf{O}_1$) $\exists\delta_1 > 0$, $c_3 > 0$ and $q_3 \in C(\bar{\Omega})$ with $q_3(x) < p^*_{a_1(x)}(x)$ for $x \in \bar{\Omega}$ and $q_3^+ < p^-$, such that

$$G_1(x,t) \geq c_3 t^{q_3(x)},\ \forall x \in \Omega,\ \forall t \in (0, \delta_1].$$

($\mathbf{O}_2$) $\exists\delta_2 > 0$, $c_4 > 0$ and $q_4 \in C(\bar{\Omega})$ with $q_4(x) < p^*_{a_2(x)}(x)$ for $x \in \bar{\Omega}$ and $q_4^- > p^+$, such that

$$|G_2(x,t)| \leq c_4 |t|^{q_4(x)},\ \forall x \in \Omega,\ \forall |t| \leq \delta_2.$$

(S) For $i = 1, 2$, $G_i(x, -t) = G_i(x, t)$, $\forall x \in \Omega$, $\forall t \in \mathbb{R}$.

Remark 4.1.

(1) It follows from (**A**), ($\mathbf{A}_2$) and ($\mathbf{O}_2$) that

$$\left|G_2(x,t)\right| \leq c_4 |t|^{q_4(x)} + c_5 |t|^{q_2(x)},\ \forall x \in \Omega, \quad \forall t \in \mathbb{R}.$$

(2) It follows from (**A**) and ($\mathbf{B}_2$) that (see [33, p. 298])

$$G(x,t) \geq c_6 |t|^{1/\theta} - c_7, \forall x \in \Omega, \forall t \in \mathbb{R}.$$

The following is the main result of this article.

Theorem 4.1. *Assume* (**P**), (**A**), ($\mathbf{A}_1$) *hold.*

(1) If ($\mathbf{B}_1$) *holds, then for every* $\lambda \in \mathbb{R}$ *and* $\mu \leq 0$, *problem* (1.1) *has a solution which is a minimizer of the corresponding functional* ϕ.

(2) If ($\mathbf{B}_1$), ($\mathbf{A}_2$), ($\mathbf{O}_1$), ($\mathbf{O}_2$) *hold, then for every* $\lambda > 0$ *and* $\mu \leq 0$, *problem* (1.1) *has a nontrivial solution* v_1 *such that* v_1 *is a minimizer of* ϕ *and* $\phi(v_1) < 0$.

(3) If ($\mathbf{A}_2$), ($\mathbf{B}_2$), ($\mathbf{O}_2$) *hold, then for every* $\mu > 0$, *there exists* $\lambda_0(\mu) > 0$ *such that when* $|\lambda| \leq \lambda_0(\mu)$, *problem* (1.1) *has a nontrivial solution* u_1 *such that* $\phi(u_1) > 0$.

(4) If ($\mathbf{A}_2$), ($\mathbf{B}_2$), ($\mathbf{O}_1$), ($\mathbf{O}_2$) *holds, then for every* $\mu > 0$, *there exists* $\lambda_0(\mu) > 0$ *such that when* $0 < \lambda \leq \lambda_0(\mu)$, *problem* (1.1) *has two nontrivial solutions* u_1 *and* v_1 *such that* $\phi(u_1) > 0$ *and* $\phi(v_1) < 0$.

(5) If ($\mathbf{A}_2$), ($\mathbf{B}_2$), ($\mathbf{O}_1$), ($\mathbf{O}_2$) *and* (**S**) *holds, then for every* $\mu > 0$ *and* $\lambda \in \mathbb{R}$, *problem* (1.1) *has a sequence of solutions* $\{\pm u_k\}$ *such that* $\phi(\pm u_k) \to \infty$ *as* $k \to \infty$.

(6) If ($\mathbf{A}_2$), ($\mathbf{B}_2$), ($\mathbf{O}_1$), ($\mathbf{O}_2$) *and* (**S**) *holds, then for every* $\lambda > 0$ *and* $\mu \in \mathbb{R}$, *problem* (1.1) *has a sequence of solutions* $\{\pm v_k\}$ *such that* $\phi(\pm v_k) < 0$ *and* $\phi(\pm v_k) \to 0$ *as* $k \to \infty$.

Proof. We will use c, c' and c_i as a generic positive constant. By Corollary 3.1, under the assumptions of Theorem 4.1, ϕ satisfies nonsmooth (PS) condition. We write

$$\Psi_1(u) = \lambda \int_\Omega a_1(x) G_1(x,u)\,dx,\ \Psi_2(u) = \mu \int_\Omega a_2(x) G_2(x,u)\,dx,$$

then $\Psi = \Psi_1 + \Psi_2$, $\phi(u) = J(u) - \Psi(u) = J(u) - \Psi_1(u) - \Psi_2(u)$. Firstly, we use $\widehat{\Psi}_i$ to denote its extension to $L^{q_i(x)}(\Omega)$, where $i = 1, 2$. From (**A**) and Theorem 1.3.10 of [33] (or Chang [22]), we see that $\widehat{\Psi}_i(u)$ is locally Lipschitz on $L^{q_i(x)}(\Omega)$ and

$\partial \widehat{\Psi}_i(u) \subseteq \{\xi_i(x) \in L^{q_i^0}(\Omega) : \xi_i(u) \in \partial G_i(x,u)\}$ for a.e. $x \in \Omega$ and $i = 1, 2$. In view of Proposition 2.4 and Theorem 2.2 of [22], we have that $\Psi_i = \widehat{\Psi}_i|_X$ is also locally Lipschitz, and $\partial\Psi_1(u) \subseteq \lambda \int_\Omega a_1(x)\, \partial G_1(x, u)\, dx$, $\partial\Psi_2(u) \subseteq \mu \int_\Omega a_2(x)\, \partial G_1(x, u)\, dx$, (see [38]), where $\widehat{\Psi}_i|_X$ stands for the restriction of $\widehat{\Psi}_i$ to X for $i = 1, 2$. Therefore, ϕ is a locally Lipschitz functional on X.

(1) Let $\lambda \in \mathbb{R}$ and $\mu \leq 0$. By (**A**),

$$|\Psi_1(u)| \leq c_1 \int_\Omega a_1(x)|u|^{q_1(x)} dx + c_2 \leq c_1 \left(|u|_{(q_1(x),a_1(x))}\right)^{q_1^+} + c_3 \leq c_4 ||u||^{q_1^+} + c_3.$$

By (**B**$_1$), $\Psi_2(u) \leq -\mu c_0 \int_\Omega a_2(x)\, dx = c_5$. Hence $\varphi(u) \geq \frac{1}{p^+}||u||^{p^-} - c_4||u||^{q_1^+} - c_6$. By (**A**$_1$), $q_1^+ < p^-$, so ϕ is coercive, that is, $\phi(u) \to \infty$ as $||u|| \to \infty$. Thus ϕ has a minimize which is a solution of (1.1).

(2) Let $\lambda > 0$, $\mu \leq 0$ and the assumptions of (2) hold. By the above conclusion (1), ϕ has a minimize v_1. Take $v_0 \in C_0^\infty(\Omega)$ such that $0 \leq v_0(x) \leq \min\{\delta_1, \delta_2\}$, $\int_\Omega a_1(x)v_0(x)^{q_3(x)} dx = d_1 > 0$ and $\int_\Omega a_2(x)v_0(x)^{q_4(x)} dx = d_2 > 0$. By (**O**$_1$) and (**O**$_2$) we have, for $t \in (0, 1)$ small enough,

$$\begin{aligned}
\varphi(tv_0) &= \int_\Omega \frac{1}{p(x)}|t\nabla v_0|^{p(x)} dx - \lambda \int_\Omega a_1(x)G_1(x, tv_0(x))dx - \mu \int_\Omega a_2(x)G_2(x, tv_0(x))dx \\
&\leq t^{p^-} \int_\Omega \frac{1}{p(x)}|\nabla v_0|^{p(x)} dx - \lambda \int_\Omega a_1(x)c_3(tv_0(x))^{q_3(x)} dx \\
&- \mu \int_\Omega a_2(x)c_4(tv_0(x))^{q_4(x)} dx \\
&\leq t^{p^-} \int_\Omega \frac{1}{p(x)}|\nabla v_0|^{p(x)} dx - t^{q_3^+}\lambda c_3 d_1 - t^{q_4^-}\mu c_4 d_2.
\end{aligned}$$

Since $q_3^+ < p^- < q_4^-$, we can find $t_0 \in (0, 1)$ such that $\phi(t_0 v_0) < 0$, and this shows $\phi(v_1) = \inf_{u \in X} \phi(u) < 0$. So $v_1 \neq 0$ because $\phi(0) = 0$. The conclusion (2) is proved.

(3) Let $\mu > 0$ and the assumptions of (3) hold. By Remark 4.1.(1), for sufficiently small $||u||$

$$\begin{aligned}
\Psi_2(u) &\leq \mu \int_\Omega a_2(x) \left(c_4|u|^{q_4(x)} + c_5|u|^{q_2(x)}\right) dx \\
&\leq \mu c_4 \left(|u|_{(q_4(x),a_2(x))}\right)^{q_4^-} + \mu c_5 \left(|u|_{(q_2(x),a_2(x))}\right)^{q_2^-} \\
&\leq \mu c_8 \left(||u||^{q_4^-} + ||u||^{q_2^-}\right).
\end{aligned}$$

Since $p^+ < q_2^-$ and $p^+ < q_4^-$, there exists $\gamma > 0$ and $\alpha > 0$ such that $J(u) - \Psi_2(u) \geq \alpha$ for $u \in S_\gamma$. We can find $\lambda_0(\mu) > 0$ such that when $|\lambda| \leq \lambda_0(\mu)$, $\Psi_1(u) \leq \alpha/2$ for $u \in S_\gamma$. So when $|\lambda| \leq \lambda_0(\mu)$, $\phi(u) \geq \alpha/2 > 0$ for $u \in S_\gamma$. By Remark 4.1.(2), noting that $1/\theta > p^+ > q_1^+$, we can find a $u_0 \in X$ such that $||u_0|| > \gamma$ and $\phi(u_0) < 0$. By Proposition 3.6 problem (1.1) has a nontrivial solution u_1 such that $\phi(u_1) > 0$.

(4) Let $\mu > 0$ and the assumptions of (4) hold. By the conclusion (3), we know that, there exists $\lambda_0(\mu) > 0$ such that when $0 < \lambda \leq \lambda_0(\mu)$, problem (1.1) has a nontrivial

solution u_1 such that $\phi(u_1)$ >0. Let γ and α be as in the proof of (3), that is, $\phi(u) \geq \alpha/2$ > 0 for $u \in S_\gamma$. By ($\mathbf{O}_1$), ($\mathbf{O}_2$) and the proof of (2), there exists $w \in X$ such that $||w|| < \gamma$ and $\phi(w) < 0$. It is clear that there is $v_1 \in B_\gamma$, a minimizer of ϕ on B_γ. Thus v_1 is a nontrivial solution of (1.1) and $\phi(v_1) < 0$.

(5) Let μ >0, $\lambda \in \mathbb{R}$ and the assumptions of (5) hold. By (**S**), we can use the nonsmooth version Fountain theorem with the antipodal action of $\mathbb{Z}_2$ to prove (5) (see Proposition 3.7). Denote

$$\Psi(u) = \int_\Omega F(x,u)dx = \lambda \int_\Omega a_1(x)G_1(x,u)dx + \mu \int_\Omega a_2(x)G_2(x,u)dx.$$

Let $\beta_k(\gamma)$ be as in Proposition 3.5. By Proposition 3.5, for each positive integer n, there exists a positive integer $k_0(n)$ such that $\beta_k(n) \leq 1$ for all $k \geq k_0(n)$. We may assume $k_0(n) < k_0(n+1)$ for each n. We define $\{\gamma_k : k = 1, 2, . . . , \}$ by

$$\gamma_k = \begin{cases} n \text{ if } k_0(n) \leq k < k_0(n+1) \\ 1 \text{ if } 1 \leq k < k_0(1). \end{cases}$$

Note that $\gamma_k \to \infty$ as $k \to \infty$. Then for $u \in Z_k$ with $||u|| = \gamma_k$ we have

$$\varphi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \Psi(u) \geq \frac{1}{p^+}(\gamma_k)^{p^-} - 1$$

and consequently

$$\inf_{u\in Z_k,\ \|u\|=\gamma_k} \varphi(u) \to \infty \text{ as } k \to \infty,$$

i.e., the condition (F_2) of Proposition 3.7 is satisfied.

By (**A**), ($\mathbf{A}_1$), ($\mathbf{B}_2$) and Remark 4.1.(2), we have

$$\varphi(u) \leq \frac{1}{p^-}||u||^{p^+} + c_1|\lambda|\left(|u|_{(q_1(x),a_1(x))}\right)^{q_1^+} - c_6\mu\left(|u|_{(1/\theta,a_2(x))}\right)^{1/\theta} + c_9.$$

Noting that $1/\theta > p^+ > q_1^+$ and all norms on a finite dimensional vector space are equivalent each other, we can see that, for each Y_k, $\phi(u) \to -\infty$ as $u \in Y_k$ and $||u|| \to \infty$. Thus for each k there exists $\rho_k > \gamma_k$ such that $\phi(u)$ <0 for $u \in Y_k \cap S_{\rho k}$, so the condition ($F_3$) of Proposition 3.7 is satisfied. As was noted earlier, ϕ satisfies nonsmooth (PS) condition. By Proposition 3.7 the conclusion (5) is true.

(6) Let λ >0, $\mu \in \mathbb{R}$ and the assumptions of (5) hold. Let us verify the conditions of the Nonsmooth dual Fountain theorem (see Proposition 3.8). By (**S**), ϕ is invariant on the antipodal action of $\mathbb{Z}_2$. For $\Psi(u) = \int_\Omega F(x, u)dx = \Psi_1(u)+ \Psi_2(u)$ let $\beta_k(1)$ be as in Proposition 3.5, that is

$$\beta_k(1) = \sup_{u\in Z_k,||u||\leq 1} |\Psi(u)|.$$

By Proposition 3.5, there exists a positive integer k_0 such that $\beta_k(1) \leq \frac{1}{2p^+}$ for all $k \geq k_0$. Setting $\rho_k = 1$, then for $k \geq k_0$ and $u \in Z_k \cap S_1$, we have

$$\varphi(u) \geq \frac{1}{p^+} - \frac{1}{2p^+} = \frac{1}{2p^+} > 0,$$

which shows that the condition (D_1) of Proposition 3.8 is satisfied.

Since $X = W_0^{1,p(x)}$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, we may choose $\{Y_k : k = 1, 2, . . . , \}$, a sequence of finite dimensional vector subspaces of X defined by (3.5), such that $Y_k \subset C_0^\infty(\Omega)$ for all k. For each Y_k, because all norms on Y_k are equivalent each other, there is $\varepsilon \in (0, 1)$ such that for every $u \in Y_k \cap B_\varepsilon$, $|u|_\infty \le \min\{\delta_1, \delta_2\}$, $|u|_{(q_3(x),a_1(x))} \le 1$ and $|u|_{(q_4(x),a_2(x))} \le 1$ By ($\mathbf{O}_1$) and ($\mathbf{O}_2$), for $u \in Y_k \cap B_\varepsilon$ we have

$$\begin{aligned}\varphi(u) &\le \frac{1}{p^-}||u||^{p^-} - \lambda c_3 \int_\Omega a_1(x)|u|^{q_3(x)}dx + |\mu|c_4 \int_\Omega a_2(x)|u|^{q_4(x)}dx \\ &\le \frac{1}{p^-}||u||^{p^-} - \lambda c_3\left(|u|_{(q_3(x),a_1(x))}\right)^{q_3^+} + |\mu|c_4\left(|u|_{(q_4(x),a_2(x))}\right)^{q_4^-}.\end{aligned}$$

Because $q_3^+ < p^- < q_4^-$ there exists $\gamma_k \in (0, \varepsilon)$ such that

$$b_k := \max_{u\in Y_k, ||u||=\gamma_k} \varphi(u) < 0,$$

thus the condition (D_2) of Proposition 3.8 is satisfied.

Because $Y_k \cap Z_k \neq \varnothing$ and $\gamma_k < \rho_k$, we have

$$d_k := \inf_{u\in Z_k, ||u||\le\rho_k} \varphi(u) \le b_k := \max_{u\in Y_k, ||u||=r_k} \varphi(u) < 0.$$

On the other hand, for any $u \in Z_k$ with $||u|| \le 1 = \rho_k$, we have $\phi(u) = J(u) - \Psi(u) \ge -\Psi(u) \ge -\beta_k(1)$. Noting that $\beta_k \to 0$ as $k \to \infty$, we obtain $d_k \to 0$, i.e., (D_3) of Proposition 3.8 is satisfied.

Finally let us prove that ϕ satisfies nonsmooth $(\mathrm{PS})_c^*$ condition for every $c \in R$. Suppose $\{u_{n_j}\} \subset X$, $n_j \to \infty, u_{n_j} \in Y_{n_j}, \varphi\left(u_{n_j}\right) \to c$ and $m|_{Y_{n_j}}\left(u_{n_j}\right) \to 0$. Similar to the process of verifying the (PS) condition in the proof of Proposition 3.3, we can get $u_{n_j} \to u$ in X. Let us prove $0 \in \partial\phi(u)$ below. Notice that

$$0 \le m(u) = m(u) - m(u_{n_j}) + m(u_{n_j}) = m(u) - m(u_{n_j}) + m|_{Y_{n_j}}(u_{n_j}).$$

Using Proposition 2.8.4, Going to limit in the right side of above equation, we have

$$m(u) \le 0,$$

so $m(u) \equiv 0$, i.e., $0 \in \partial\phi(u)$, this shows that ϕ satisfies the nonsmooth $(\mathrm{PS})_c^*$ condition for every $c \in \mathbb{R}$. So all conditions of Proposition 3.8 are satisfied and the conclusion (6) follows from Proposition 3.8. The proof of Theorem 4.1 is complete. ■

Remark 4.2

Theorem 4.1 includes several important special cases. In particular, in the case of the problem (1.4), i.e., in the case that

$$a_1(x) = \frac{1}{|x|^{s_1(x)}},\ a_2(x) = \frac{1}{|x|^{s_2(x)}},$$

all conditions of Theorem 4.1 are satisfied provided ($\mathbf{P}$), ($\mathbf{A}^*$), ($\mathbf{A}_1$), and ($\mathbf{A}_2$) hold.

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Authors' contributions
GD conceived of the study, and participated in its design and coordination and helped to draft the manuscript. RM participated in the design of the study. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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