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On symmetric positive homoclinic solutions of semilinear p-Laplacian differential equations

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Abstract

In this paper we study the existence of even positive homoclinic solutions for p-Laplacian ordinary differential equations (ODEs) of the type $(u'|u'|^{p-2})' - a(x)u|u|^{p-2} + \lambda b(x)u|u|^{q-2} = 0$, where $2 \leq p < q$, $\lambda > 0$ and the functions a and b are strictly positive and even. First, we prove a result on symmetry of positive solutions of p-Laplacian ODEs. Then, using the mountain-pass theorem, we prove the existence of symmetric positive homoclinic solutions of the considered equations. Some examples and additional comments are given.

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1 Introduction and main results

In this paper we prove the existence of positive homoclinic solutions for p-Laplacian ODEs of the type

$$\left(u'\left|u'\right|^{p-2}\right)' - a(x)u|u|^{p-2} + \lambda b(x)u|u|^{q-2} = 0, \quad x \in \mathbb{R}, \tag{1}$$

where $2 \leq p < q$ and $\lambda > 0$. We assume that

(H) the functions $a(x)$ are $b(x)$ are continuously differentiable, strictly positive, $0 < a \leq a(x) \leq A$ and $0 < b \leq b(x) \leq B$. Let, moreover, $a(x)$ and $b(x)$ be even functions on $\mathbb{R}$, $xa'(x) > 0$ and $xb'(x) < 0$ for $x \neq 0$.

By a solution of (1), we mean a function $u : \mathbb{R} \to \mathbb{R}$ such that $u \in C^1(\mathbb{R})$, $(u'|u'|^{p-2})' \in C(\mathbb{R})$ and Eq. (1) holds for every $x \in \mathbb{R}$. We are looking for positive solutions of (1) which are homoclinic, *i.e.*, $u(x) \to 0$ and $u'(x) \to 0$ as $|x| \to \infty$.

In the case $p = 2$, $q = 4$ and $\lambda = 1$, similar problems are considered in [1–3] using variational methods. Note that in [2] and [3] the following second-order differential equations are considered:

$$u'' - a(x)u - b(x)u^2 + c(x)u^3 = 0$$

and

$$u'' + a(x)u - b(x)u^2 + c(x)u^3 = 0,$$

where a, b and c are periodic, bounded functions and a and c are positive. These equations come from a biomathematics model suggested by Austin [4] and Cronin [5]. Further results and the phase plane analysis of these equations with constant coefficients are given in [6]. Note that the periodic and homoclinic solutions of p-Laplacian ODEs are considered in [7, 8].

The present work is an extension of these studies to p-Laplacian ODEs. Let $X_T := W_0^{1,p}(-T,T)$ be the Sobolev space of p-integrable absolutely continuous functions $u : [-T,T] \to \mathbb{R}$ such that

$$\|u\|^p = \int_{-T}^{T} \left(|u'(x)|^p + |u(x)|^p\right) dx < \infty$$

and $u(-T) = u(T) = 0$.

We use a variational treatment of the problem considering the functional $J_T : X_T \to \mathbb{R}$

$$J_T(u) = \int_{-T}^{T} \left(\frac{1}{p}\left(|u'(x)|^p + a(x)|u(x)|^p\right) - \frac{\lambda}{q} b(x)\left(u^+(x)\right)^q\right) dx,$$

where $u^+(x) = \max\{u(x), 0\}$.

Using the well-known mountain-pass theorem, we conclude that the functional J_T has a nontrivial critical point $u_{T,\lambda} \in X_T$, which is a solution of the restricted problem

$$\begin{aligned} &\left(u'|u'|^{p-2}\right)' - a(x)u|u|^{p-2} + \lambda b(x)u|u|^{q-2} = 0, \quad x \in (-T,T), \\ &u(-T) = u(T) = 0. \end{aligned} \tag{2}$$

Further, we obtain uniform estimates for the solutions $u_{T,\lambda}$, extended by 0 outside $[-T,T]$. Then, a positive homoclinic solution u_λ of (1) is found as a limit of $u_{T,\lambda}$, as $T \to \infty$ in $C^1_{\text{loc}}(\mathbb{R})$. The function u_λ is also an even function.

To obtain the property, we extend the symmetry lemma of Korman and Ouyang [9] to the p-Laplacian equations. The result is formulated and proved in Section 2.

Our main result is:

Theorem 1 *Suppose that* $2 \le p < q$, $\lambda > 0$ *and assumptions* (H) *hold. Then Eq.* (1) *has a positive solution* u_λ *such that* $u_\lambda(x) \to 0$ *and* $u'_\lambda(x) \to 0$ *as* $|x| \to \infty$. *Moreover, the solution* u_λ *is an even function*, $\max\{u_\lambda(x) : x \in \mathbb{R}\} = u_\lambda(0) \to +\infty$ *as* $\lambda \to 0$ *and* $u'_\lambda(x) < 0$ *for* $x > 0$.

Theorem 1 is proved in Section 3. From its proof we have

$$\max\{u_\lambda(x) : x \in \mathbb{R}\} = u_\lambda(0) \ge \left(\frac{a(0)}{\lambda b(0)}\right)^{1/(q-p)} > 0,$$

from which it follows that $u_\lambda(0) \to +\infty$ as $\lambda \to 0$. Observe that if $\lambda = 0$, the problem

$$\begin{aligned} &\left(u'|u'|^{p-2}\right)' - a(x)u|u|^{p-2} = 0, \quad x \in \mathbb{R}, \\ &u(\pm\infty) = u'(\pm\infty) = 0 \end{aligned}$$

has a unique solution $u = 0$. Indeed, multiplying the equation by u and integrating by parts over $\mathbb{R}$, we obtain

$$\int_{-\infty}^{\infty} \left(\left|u'(x)\right|^p + a(x)\left|u(x)\right|^p\right) dx = 0,$$

which implies that $u \equiv 0$.

A simplified method can be applied to the equations

$$u'' - a(x)u|u|^{p-2} + \lambda b(x)u|u|^{q-2} = 0, \quad x \in \mathbb{R}, \tag{3}$$

under assumptions (H) and $2 \le p < q$, $\lambda > 0$. Note that in this case, the even homoclinic solution u_λ of Eq. (3) satisfies

$$\max\left\{u_\lambda(x) : x \in \mathbb{R}\right\} = u_\lambda(0) \ge \left(\frac{a(0)}{\lambda b(0)}\right)^{1/(q-p)},$$

and again $u_\lambda(0) \to +\infty$ as $\lambda \to 0$. If a and b are constants, Eq. (3) is a conservative system and one can plot the phase curves $(\frac{v}{2})^2 - a\frac{|u|^p}{p} + \lambda b\frac{|u|^q}{q} = C$ in the phase plane $(u, v) = (u, u')$. An example is given at the end of Section 3.

2 Preliminary results

Let $\varphi_p(t) = t|t|^{p-2}$, $p \ge 2$ and $\Phi_p(t) = \frac{|t|^p}{p}$. It is clear that $\Phi_p(t)$ is a differentiable function and $\Phi_p'(t) = \varphi_p(t)$. Moreover, $\varphi_p'(t)$ exists and $\varphi_p'(t) = (p-1)|t|^{p-2}$ for $p \ge 2$.

Let $L^p(a,b)$, $1 < p < \infty$ be the space of Lebesgue measurable functions $u : (a,b) \to \mathbb{R}$ such that the norm $|u|_p^p = \int_a^b |u(x)|^p\, dx < \infty$.

The dual space of $L^p(a,b)$ is $L^{p'}(a,b)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\langle\cdot,\cdot\rangle$ be the duality pairing between $L^{p'}(a,b)$ and $L^p(a,b)$. By the Hölder inequality, $|\langle v,u\rangle| \le |v|_{p'}|u|_p$ for any $v \in L^{p'}(a,b)$ and $u \in L^p(a,b)$. We will use the following lemmata in further considerations.

Lemma 2 *For any $u, v \in L^p(a,b)$, the following inequality holds*:

$$\left\langle \varphi_p(u) - \varphi_p(v), u - v\right\rangle \ge \left(|u|_p^{p-1} - |v|_p^{p-1}\right)\left(|u|_p - |v|_p\right).$$

Proof of Lemma 2. Note that for $u \in L^p(a,b)$, $\varphi_p(u) \in L^{p'}(a,b)$. From the Hölder inequality, we have

$$\begin{aligned}
&\left\langle \varphi_p(u) - \varphi_p(v), u - v\right\rangle \\
&\quad = |u|_p^p + |v|_p^p - \left\langle \varphi_p(u), v\right\rangle - \left\langle \varphi_p(v), u\right\rangle \\
&\quad \ge |u|_p^p + |v|_p^p - |u|_p^{p-1}|v|_p - |v|_p^{p-1}|u|_p \\
&\quad = \left(|u|_p^{p-1} - |v|_p^{p-1}\right)\left(|u|_p - |v|_p\right).
\end{aligned}$$
□

Lemma 3 *Let $p \ge 2$, $u \in C^1([a,b])$ and $(u'|u'|^{p-2})' \in C([a,b])$. Then*

$$\int_a^b \left(u'\left|u'\right|^{p-2}\right)' u'\, dx = \frac{p-1}{p}\left(\left|u'(b)\right|^p - \left|u'(a)\right|^p\right).$$

The statement of Lemma 3 follows simply from the identity

$$\left(|u'|^p\right)' = \frac{p}{p-1}\left(u'|u'|^{p-2}\right)'u'.$$

The one-dimensional p-Laplacian operator L_p for a differentiable function u on the interval I is introduced as $L_p(u) := (\varphi_p(u'))'$. Let us consider the problem

$$\begin{cases} L_p(u) + f(x,u) = 0, & x \in (-T,T), \\ u(-T) = u(T) = 0, \end{cases} \tag{4}$$

where $f \in C^1([-T,T] \times \mathbb{R}^+)$ and satisfies

$$\begin{aligned} & f(-x,u) = f(x,u), \quad x \in (-T,T), u > 0, \\ & xf_x(x,u) < 0, \quad x \in (-T,T)\backslash\{0\}, u > 0. \end{aligned} \tag{5}$$

A function $u : [-T,T] \to \mathbb{R}$ is said to be a solution of the problem (4) if $u \in C^1([-T,T])$ with $u(-T) = u(T) = 0$ is such that $u'|u'|^{p-2}$ is absolutely continuous and $L_pu(x) + f(x,u(x)) = 0$ holds a.e. in $(-T,T)$.

We formulate an extension of Lemma 1 of [9] for p-Laplacian nonlinear equations. The result of Korman and Ouyang is *one-dimensional* analogue of the result of Gidas, Ni and Nirenberg [10] for symmetry of positive solutions of semilinear Laplace equations. In the case of p-Laplacian equations, the symmetry of solutions in higher dimensions is discussed by Reihel and Walter [11].

Theorem 4 *Assume that* $f \in C^1([-T,T] \times \mathbb{R}^+)$ *satisfies* (5). *Then any positive solution u of* (4) *is an even function such that* $\max\{u(x) : -T \leq x \leq T\} = u(0)$ *and* $u'(x) < 0$ *for* $x \in (0,T]$.

Remark 1 Let us note that if the function f satisfies (5), but u is not a positive solution of (4), then u is not necessarily an even function. A simple counter example in the case $p = 2$ is the problem

$$\begin{cases} u'' + u - x^2 + \pi^2 - 2 = 0, & -\pi < x < \pi, \\ u(-\pi) = u(\pi) = 0. \end{cases}$$

The term $f(x,u) = u - x^2 + \pi^2 - 2$ satisfies (5) in the interval $(-\pi,\pi)$, but the solution of the problem $u(x) = x^2 - \pi^2 + \sin x$ is negative in $(-\pi,\pi)$ and not an even function. Its graph is presented in Figure 1. It would be more interesting to show an example for the case $p > 2$ and f satisfying the additional assumption $f(x,0) = 0$.

Sketch of Proof of Theorem 4

Suppose that the function u has only one global maximum on $[-T,T]$.

Assume that the function $u(x)$ has a finite number of local minima in the interval $[0,T]$, and let x_1 be the largest local minimum. Let $\bar{x} \in [x_1,T]$ be the local maximum and $\tilde{x} \in [\bar{x},T]$ be such that $u(x_1) = u(\tilde{x})$. Denote $u_1 = u(x_1) = u(\tilde{x})$ and $u_2 = u(\bar{x})$, and let $x = \alpha(u)$ and $x = \beta(u)$ be the inverse functions of the function $u = u(x)$ in the intervals $[x_1,\bar{x}]$ and $[\bar{x},T]$,

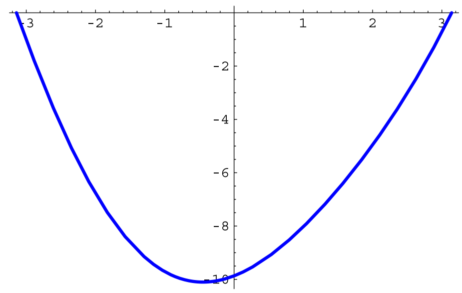

Figure 1 Graph of the functions $u(x) = x^2 - \pi^2 + \sin x$.

respectively. Multiplying the equation in (4) by u' and integrating in $[x_1, \tilde{x}]$, we obtain by Lemma 3 and (5):

$$\begin{aligned} 0 &= \int_{x_1}^{\tilde{x}} \left(L_p(u)u' + f(x,u)u'\right) dx \\ &= \frac{p-1}{p}\left|u'\right|^p(\tilde{x}) + \int_{x_1}^{\bar{x}} f(x,u)u'\,dx + \int_{\bar{x}}^{\tilde{x}} f(x,u)u'\,dx \\ &= \frac{p-1}{p}\left|u'\right|^p(\tilde{x}) + \int_{u_1}^{u_2} \left(f\left(\alpha(u),u\right) - f\left(\beta(u),u\right)\right) du \\ &> 0, \end{aligned}$$

which leads to contradiction. One can prove the last fact using other arguments; see, for instance, Theorem 2.1 of [12]. Suppose now that u has infinitely many local minima in $[-T, x^*]$. Further, we can follow the steps of the proof of Lemma 1 of [9] with corresponding modifications based on Lemma 3. □

3 Proof of the main result

Let $X_T = W_0^{1,p}(-T,T)$ be the Sobolev space of p-integrable absolutely continuous functions $u : [-T,T] \to \mathbb{R}$ such that

$$\|u\|_T^p = \int_{-T}^{T} \left(\left|u'(x)\right|^p + \left|u(x)\right|^p\right) dx < \infty$$

and $u(-T) = u(T) = 0$. Note that if $a(x)$ is strictly positive and bounded, *i.e.*, there exist a and A such that $0 < a \le a(x) \le A$, then $\|u\|_{a,T}^p = \int_{-T}^{T}(|u'(x)|^p + a(x)|u(x)|^p)\,dx$ is an equivalent norm in X_T.

We need an extension to the p-case of the following proposition by Rabinowitz [13].

Proposition 5 *Let $u \in W_{\mathrm{loc}}^{1,p}(\mathbb{R})$. Then*:

(i) *If $T \ge 1$, for $x \in [T - 1/2, T + 1/2]$,*

$$\max_{x \in [T-1/2, T+1/2]} \left|u(x)\right| \le 2^{\frac{p-1}{p}} \left(\int_{T-1/2}^{T+1/2} \left(\left|u'(t)\right|^p + \left|u(t)\right|^p\right) dt\right)^{1/p}. \qquad (6)$$

(ii) *For every* $u \in W_0^{1,p}(-T,T)$,

$$\|u\|_{L^\infty(-T,T)} \le 2^{\frac{p-1}{p}} \|u\|_T. \tag{7}$$

Proof of Proposition 5 Let $x, t \in [T-1/2, T+1/2]$. It follows

$$|u(x)| \le |u(t)| + \int_{T-1/2}^{T+1/2} |u'(s)|\, ds.$$

Integrating with respect to $t \in [T-1/2, T+1/2]$ and using the Hölder and Jensen inequalities, we obtain

$$\begin{aligned}
|u(x)| &\le \int_{T-1/2}^{T+1/2} |u(t)|\, dt + \int_{T-1/2}^{T+1/2} |u'(s)|\, ds \\
&\le \left(\int_{T-1/2}^{T+1/2} |u(t)|^p\, dt\right)^{1/p} + \left(\int_{T-1/2}^{T+1/2} |u'(t)|^p\, dt\right)^{1/p} \\
&\le 2^{\frac{p-1}{p}} \left(\int_{T-1/2}^{T+1/2} \left(|u'(t)|^p + |u(t)|^p\right) dt\right)^{1/p}.
\end{aligned}$$

(ii) Take $u \in W_0^{1,p}(-T,T)$. Since $W_0^{1,p}(-T,T) \subset C[-T,T]$, there exists $\tau \in [-T,T]$ such that by (i)

$$\begin{aligned}
\|u\|_{L^\infty(-T,T)} = \|u\|_{C[\tau-1/2,\tau+1/2]} &\le 2^{\frac{p-1}{p}} \left(\int_{\tau-1/2}^{\tau+1/2} \left(|u'(t)|^p + |u(t)|^p\right) dt\right)^{1/p} \\
&\le 2\|u\|.
\end{aligned}$$

□

We are looking for positive solutions of (1), which are homoclinic, *i.e.*, $u(x) \to 0$ and $u'(x) \to 0$ as $|x| \to \infty$. Firstly, we look for positive solutions of the problem

$$\begin{cases} (\varphi_p(u'))' - a(x)\varphi_p(u) + \lambda b(x)\varphi_q(u) = 0, & x \in (-T,T), \\ u(-T) = u(T) = 0. \end{cases} \tag{P_T}$$

A function $u : [-T,T] \to \mathbb{R}$ is said to be a solution of the problem (P_T) if $u \in C^1([-T,T])$ with $u(-T) = u(T) = 0$ is such that $\varphi_p(u')$ is absolutely continuous and $(\varphi_p(u'))'(x) - a(x)\varphi_p(u)(x) + \lambda b(x)\varphi_q(u)(x) = 0$ holds a.e. in $(-T,T)$.

A function $u : [-T,T] \to \mathbb{R}$ is said to be a weak solution of the problem (P_T) if

$$\int_{-T}^{T} \left(\left(\varphi_p(u')\right)' v'\, dx + a(x)\varphi_p(u)v - \lambda b(x)\varphi_q(u)v\right) dx = 0, \quad \forall v \in W_0^{1,p}\left((-T,T)\right).$$

Standard arguments show that a weak solution of the problem (P_T) is a solution of (P_T) (see [14] and [15]). Consider the modified problem

$$\begin{cases} (\varphi_p(u'))' - a(x)\varphi_p(u) + \lambda b(x)(u^+)^{q-1} = 0, & x \in (-T,T), \\ u(-T) = u(T) = 0, \end{cases} \tag{P_T^+}$$

where $u^+ = \max(u, 0)$. It is easy to see that solutions of the problem (P_T^+) are positive solutions of the problem (P_T). Indeed, if $u(x)$ is a solution of (P_T^+) and $u(x)$ has negative minimum at $x_0 \in (-T, T)$, since for $p \geq 2$, $(\varphi_p(u'))'(x_0) \geq 0$, by the equation $(\varphi_p(u'))' - a(x)\varphi_p(u) + \lambda b(x)(u^+)^{q-1} = 0$, we reach a contradiction

$$0 = \bigl(\varphi_p\bigl(u'\bigr)\bigr)'(x_0) + a(x_0)\bigl(-u(x_0)\bigr)^{p-1} > 0.$$

Then $u(x) \geq 0$ and u is a solution of (P_T). We use a variational treatment of the problem (P_T^+), considering the functional $J_T : X_T \to \mathbb{R}$

$$J_T(u) = \int_{-T}^{T} \left(\frac{1}{p}\bigl(\bigl|u'(x)\bigr|^p + a(x)\bigl|u(x)\bigr|^p\bigr) - \frac{\lambda}{q} b(x)\bigl(u^+(x)\bigr)^q \right) dx.$$

Critical points of J_T are weak solutions of (P_T^+), *i.e.*,

$$\int_{-T}^{T} \bigl(\varphi_p\bigl(u'\bigr)v' + a(x)\varphi_p(u)v - \lambda b(x)\bigl(u^+\bigr)^{q-1}v\bigr)\, dx, \quad \forall v \in W_0^{1,p}(-T, T)$$

and, by a standard way, they are solutions of (P_T^+). We show that J_T satisfies the assumptions of the mountain-pass theorem of Ambrosetti and Rabinowitz [16].

Theorem 6 (Mountain-pass theorem) *Let X be a Banach space with norm* $\|\cdot\|$, $I \in C^1(X, \mathbf{R})$, $I(0) = 0$ *and I satisfy the* (*PS*) *condition. Suppose that there exist* $r > 0$, $\alpha > 0$ *and* $e \in X$ *such that* $\|e\| > r$

(i) $I(x) \geq \alpha$ *if* $\|x\| = r$,

(ii) $I(e) < 0$. *Let* $c = \inf_{\gamma\in\Gamma}\{\max_{0\leq t\leq 1} I(\gamma(t))\} \geq \alpha$, *where*

$$\Gamma = \bigl\{\gamma \in C\bigl([0,1], X\bigr) : \gamma(0) = 0, \gamma(1) = e\bigr\}.$$

Then c is a critical value of I, i.e., there exists x_0 *such that* $I(x_0) = c$ *and* $I'(x_0) = 0$.

Next, denote by C_j several positive constants.

Lemma 7 *Let* $2 \leq p < q$, $\lambda > 0$ *and assumptions* (H) *hold. Then for every* $T > 0$, *the problem* (P_T) *has a positive solution* $u_{T,\lambda}$. *Moreover, there is a constant* $K > 0$, *independent of T, such that*

$$\|u_{T,\lambda}\|_T \leq K. \tag{8}$$

Proof Step 1. J_T *satisfies the (PS) condition.*

Let $(u_k)_k \subset X_T$ be a sequence, and suppose there exist C_1 and k_0 such that for $k \geq k_0$

$$\bigl|J_T(u_k)\bigr| = \left| \int_{-T}^{T} \left(\frac{1}{p}\bigl(\bigl|u_k'(x)\bigr|^p + a(x)\bigl|u_k(x)\bigr|^p\bigr) - \frac{\lambda}{q} b(x)\bigl(u_k^+(x)\bigr)^q \right) dx \right| \leq \frac{C_1}{p}, \tag{9}$$

and

$$\bigl|\bigl\langle J_T(u_k), u_k\bigr\rangle\bigr| = \left| \int_{-T}^{T} \bigl(\bigl|u_k'(x)\bigr|^p + a(x)\bigl|u_k(x)\bigr|^p - \lambda b(x)\bigl(u_k^+(x)\bigr)^q\bigr)\, dx \right| \leq \|u_k\|_T. \tag{10}$$

Let us denote $\hat{a} = \min(1, a)$. From (9) and (10), it follows that

$$C_1 \geq \int_{-T}^{T} \left(\left(|u_k'(x)|^p + a(x)|u_k(x)|^p \right) - \frac{\lambda p}{q} b(x) \left(u_k^+(x) \right)^q \right) dx \geq -C_1$$

and

$$\|u_k\|_T \geq \int_{-T}^{T} \left(-|u_k'(x)|^p - a(x)|u_k(x)|^p + \lambda b(x) \left(u_k^+(x) \right)^q \right) dx \geq -\|u_k\|_T$$

Then

$$C_1 + \|u_k\|_T \geq \lambda \frac{(q-p)b}{p} \int_{-T}^{T} \left(u_k^+(x) \right)^q dx,$$

and

$$\begin{aligned} \hat{a}\|u_k\|_T^p - C_1 &\leq \int_{-T}^{T} \left(|u_k'(x)|^p + a(x)|u_k(x)|^p \right) dx - C_1 \\ &\leq \frac{\lambda p}{q} \int_{-T}^{T} b(x) \left(u_k^+(x) \right)^q dx \leq \frac{\lambda p B}{q} \int_{-T}^{T} \left(u_k^+(x) \right)^q dx. \end{aligned}$$

We have

$$\hat{a}\|u_k\|_T^p - C_1 \leq \frac{B}{q(q-p)b} \left(C_1 + \|u_k\|_T \right),$$

which implies that the sequence $(u_k)_k$ is bounded in X_T. By the compact embedding $X_T \subset C([-T, T])$, there exist $u \in X_T$ and the subsequence of $(u_k)_k$, still denoted by $(u_k)_k$, such that $u_k \rightharpoonup u$ weakly in X_T and $u_k \to u$ strongly in $C([-T, T])$. We will show that $u_k \to u$ strongly in X_T using Lemma 2. By uniform convergence of u_k to u in $C([-T, T])$, it follows that

$$\begin{aligned} &\langle J_T'(u_k) - J_T'(u), u_k - u \rangle \\ &\quad = \langle \varphi_p(u_k') - \varphi_p(u'), u_k' - u' \rangle + \langle \varphi_p(u_k) - \varphi_p(u), a(x)(u_k - u) \rangle \\ &\qquad - \langle \varphi_q(u_k) - \varphi_q(u), b(x)(u_k - u) \rangle \to 0, \quad k \to \infty, \end{aligned}$$

and

$$\langle \varphi_p(u_k) - \varphi_p(u), a(x)(u_k - u) \rangle - \langle \varphi_q(u_k) - \varphi_q(u), b(x)(u_k - u) \rangle \to 0, \quad k \to \infty.$$

Then

$$\langle \varphi_p(u_k') - \varphi_p(u'), u_k' - u' \rangle \to 0, \quad k \to \infty,$$

and by Lemma 2,

$$\langle \varphi_p(u_k') - \varphi_p(u'), u_k' - u' \rangle \geq \left(|u_k'|_p^{p-1} - |u'|_p^{p-1} \right) \left(|u_k'|_p - |u'|_p \right) \geq 0,$$

which implies that $|u_k'|_p \to |u'|_p$. Then $\|u_k\|_T \to \|u\|_T$ and by the uniform convexity of the space X_T, it follows that $\|u_k - u\|_T \to 0$, as $k \to \infty$.

Step 2. Geometric conditions.

Obviously, $J_T(0) = 0$. By assumption (H) it follows

$$J_T(u) \geq \frac{\hat{a}}{2p}\|u\|_T^p + \int_{-T}^{T}\left(\frac{a(x)}{2p}|u(x)|^p - \frac{\lambda p}{q}b(x)\big(u^+(x)\big)^q\right)dx$$
$$\geq \frac{\hat{a}}{2p}\|u\|_T^p + \int_{-T}^{T}|u(x)|^p\left(\frac{a}{2p} - \frac{\lambda p}{q}b(x)|u(x)|^{q-p}\right)dx > 0$$

if $\|u\|_T = \rho := (\frac{aq}{2\lambda p^2})^{1/(q-p)} > 0$. Then $J_T(u) \geq \frac{\hat{a}\rho^p}{2p} > 0$.

Let $u_0(x) \in X_T$ be such that $u_0(x) > 0$ if $x \in (-T, T)$ and also $u_0(-T) = u_0(T) = 0$. Consider the function

$$\hat{u}_0(x) = \begin{cases} \mu u_0(x), & \text{if } x \in [-1,1], \\ 0, & \text{if } x \in [-T,T]\backslash[-1,1]. \end{cases}$$

Then

$$J_T(\hat{u}_0) = \mu^p \int_{-T}^{T}\frac{1}{p}\big(|u_0'(x)|^p + a(x)|u_0(x)|^p\big)\,dx - \mu^q\int_{-T}^{T}\frac{\lambda}{q}b(x)\big(u_0(x)\big)^q\,dx < 0,$$

for μ large enough.

By the mountain-pass theorem, there exists a solution $u_{T,\lambda} \in X_T$ such that

$$c_T = J_T(u_{T,\lambda}) = \inf_{\gamma\in\Gamma_T}\max_{t\in[0,1]} J_T\big(\gamma(t)\big), \qquad J_T'(u_{T,\lambda}) = 0, \tag{11}$$

where

$$\Gamma_T = \big\{\gamma(t) \in C\big([0,1], X_T\big) : \gamma(0) = 0, \gamma(1) = \hat{u}_0(x)\big\}.$$

Moreover, using the variational characterization (11), we have

$$c_T \geq \frac{\hat{a}\rho^p}{2p} > 0.$$

Therefore, $u_{T,\lambda}$ is a nontrivial and positive solution of (P_T). By Theorem 4, $\max\{u_{T,\lambda} : -T \leq x \leq T\} = u_{T,\lambda}(0)$ and $u_{T,\lambda}'(x) < 0$ for $x \in (0, T]$.

Step 3. Uniform estimates.

Let $T_1 \geq T \geq 1$. By continuation with zero of a function $u \in X_T$ to $[-T_1, T_1]$, we have $X_T \subset X_{T_1}$and $\Gamma_T \subset \Gamma_{T_1}$. Using the variational characterization (11), we infer that $c_{T_1} \leq c_T \leq c_1$ and then

$$\int_{-T}^{T}\left(\frac{1}{p}\big(|u_{T,\lambda}'(x)|^p + a(x)u_{T,\lambda}^p(x)\big) - \frac{\lambda}{q}b(x)u_{T,\lambda}^q(x)\right)dx \leq c_1. \tag{12}$$

Multiplying the equation of (P_T) by u_T and integrating by parts, we have

$$\int_{-T}^{T}\big(|u_{T,\lambda}'|^p + a(x)u_{T,\lambda}^p\big)\,dx = \int_{-T}^{T}\lambda b(x)u_{T,\lambda}^q\,dx.$$

Then by (12),

$$c_1 \geq \int_{-T}^{T} \left(\frac{1}{p} \left(|u'_{T,\lambda}|^p + a(x) u_{T,\lambda}^p \right) - \frac{\lambda}{q} \lambda b(x) u_{T,\lambda}^q \right) dx$$
$$\geq \left(\frac{1}{p} - \frac{1}{q} \right) \int_{-T}^{T} \left(|u'_{T,\lambda}|^p + a(x) u_{T,\lambda}^p \right) dx \geq \frac{\hat{a}(q-p)}{pq} \|u_{T,\lambda}\|_T^p.$$

We get (8) with $K = \frac{pqc_1}{\hat{a}(q-p)}$, which completes the proof. □

Proof of Theorem 1 Take $T_n \to \infty$ and let u_n be the solution of the problem (P_{T_n}) given by Lemma 2. Consider the extension of u_n to $\mathbb{R}$ with zero outside $[-T_n, T_n]$ and denote it by the same symbol.

Claim 1. The sequence of functions $(u_n)_n$ is uniformly bounded and equicontinuous.

By (8) and the embedding of X_{T_n} in $C([-T_n, T_n])$, there is K_1 such that $\|u_n\|_{L^\infty([-T_n,T_n])} \leq K_1$. Then by the equation of (P_{T_n}), it follows that

$$\left\| \left(\varphi_p(u'_n)\right)' \right\|_{L^\infty([-T_n,T_n])} \leq K_2. \tag{13}$$

By the mean value theorem for every natural n and every $t \in \mathbb{R}$, there exists $\xi_n \in [t-1, t]$ such that

$$u_n(t) - u_n(t-1) = u'_n(\xi_k).$$

Then, as a consequence of (13), we obtain

$$\begin{aligned} \left|\varphi_p\left(u'_n(t)\right)\right| &= \left| \int_{\xi_k}^{t} \left(\varphi_p\left(u'_n(s)\right)\right)' ds + \varphi_p\left(u'_n(\xi_k)\right) \right| \\ &\leq \int_{t-1}^{t} \left| \left(\varphi_p\left(u'_n(s)\right)\right)' \right| ds + \left|u'_n(\xi_k)\right|^{p-1} \\ &\leq K_2 + \left(\left|u_n(t)\right| + \left|u_n(t-1)\right|\right)^{p-1} \\ &\leq K_2 + (2K_1)^{p-1} =: K_3^{(p-1)/p}, \quad \forall t \in \mathbb{R}, \end{aligned} \tag{14}$$

from which it follows $\|u'_n\|_{L^\infty([-T_n,T_n])} \leq K_3$ and the sequence of functions (u_n) is equicontinuous. Further, we claim that the sequence $(u'_n)_n$ is also equicontinuous.

Claim 2. The sequence of functions $(u'_n)_n$ is equicontinuous.

To prove this statement, we follow the method given by Tang and Xiao [7]. For completeness, we present it in details.

Suppose that $(u'_n)_n$ is not an equicontinuous sequence in $C_{\text{loc}}(\mathbb{R})$. Then there exist an ε_0 and sequences (t_k^1) and (t_k^2) such that $0 < t_k^1 - t_k^2 < \frac{1}{k}$ and

$$\left|u'_n(t_k^1) - u'_n(t_k^2)\right| \geq \varepsilon_0. \tag{15}$$

By (14), there are numbers w^1 and w^2 and the subsequence (u'_{n_k}) such that $u'_{n_k}(t_k^1) \to w^1$ and $u'_{n_k}(t_k^2) \to w^2$ as $k \to \infty$. By (15), $|w^1 - w^2| \geq \varepsilon_0$. On the other hand, by (13) we have

$$\left|\varphi_p\left(u'_{n_k}(t_k^2)\right) - \varphi_p\left(u'_{n_k}(t_k^1)\right)\right| \leq \int_{t_k^1}^{t_k^2} \left|\varphi_p\left(u'_{n_k}(s)\right)'\right| ds \leq \frac{K_2}{k}.$$

Then passing to a limit as $k \to \infty$, we obtain $\varphi_p(w^1) = \varphi_p(w^2)$. Hence, $w^1 = w^2$ which contradicts $|w^1 - w^2| \geq \varepsilon_0$. Thus, the sequence $(u'_n)_n$ is equicontinuous.

Let $T > 0$. By Claim 1 and Claim 2 and the Arzelà-Ascoli theorem, there is a subsequence of $(u_n)_n$, still denoted by $(u_n)_n$, and functions $u_{\lambda 1}$ and $v_{\lambda 1}$ of $C([-T,T])$ such that $\|u_n - u_{\lambda 1}\|_{C([-T,T])} \to 0$ and $\|u'_n - v_{\lambda 1}\|_{C([-T,T])} \to 0$. Trivially, it follows that $u_{\lambda 1} \in C^1([-T,T])$, $u'_{\lambda 1} = v_{\lambda 1}$ and $\|u_n - v_{\lambda 1}\|_{C^1([-T,T])} \to 0$. Repeating this procedure as in [7], we obtain that there is a subsequence of $(u_n)_n$, still denoted by $(u_n)_n$, and u_λ such that $u_n \to u_\lambda$ in $C^1_{\mathrm{loc}}(\mathbb{R})$. The function u_λ satisfies Eq. (1). Indeed, let $[x_1,x_2]$ be an interval of $\mathbb{R}$ and $T_n > 0$ such that $[x_1,x_2] \subset [-T_n,T_n]$. By the above considerations, taking a limit as $n \to \infty$ in the equation

$$\left(u'_n\left|u'_n\right|^{p-2}\right)' - a(x)u_n^{p-1} + \lambda b(x)u_n^{q-1} = 0, \quad x \in [x_1,x_2],$$

equivalent to

$$\begin{aligned} u'_n\left|u'_n\right|^{p-2}(x) &= u'_n\left|u'_n\right|^{p-2}(x_1) + \int_{x_1}^{x}\left(a(t)u_n^{p-1}(t) - \lambda b(t)u_n^{q-1}(t)\right)dt \\ &= 0, \quad x \in [x_1,x_2], \end{aligned}$$

we obtain

$$\begin{aligned} u'_\lambda\left|u'_\lambda\right|^{p-2}(x) &= u'_\lambda\left|u'_\lambda\right|^{p-2}(x_1) + \int_{x_1}^{x}\left(a(t)u_\lambda^{p-1}(t) - \lambda b(t)u_\lambda^{q-1}(t)\right)dt \\ &= 0, \quad x \in [x_1,x_2], \end{aligned}$$

and hence

$$\left(u'_\lambda\left|u'_\lambda\right|^{p-2}\right)' - a(x)u_\lambda^{p-1} + \lambda b(x)u_\lambda^{q-1} = 0, \quad x \in [x_1,x_2].$$

Since x_1 and x_2 are arbitrary, u_λ is a solution of (1). Moreover, we have

$$\int_{-\infty}^{\infty}\left(\left|u'_\lambda(x)\right|^p + a(x)\left|u_\lambda(x)\right|^p\right)dx < \infty. \tag{16}$$

It remains to show that u_λ is nonzero and $u_\lambda(\pm\infty) = 0$ and $u'_\lambda(\pm\infty) = 0$.

By Theorem 4, u_n is an even function and attains its maximum at 0. Then by Eq. (1),

$$u_n^{p-1}(0)\left(-a(0) + \lambda b(0)u_n^{q-p}(0)\right) \geq 0.$$

By assumption (H)

$$u_n(0) \geq \left(\frac{a(0)}{\lambda b(0)}\right)^{1/(q-p)} \geq \left(\frac{a}{\lambda B}\right)^{1/(q-p)} = C_3 > 0,$$

independently of n. Hence, passing to a limit as $n \to \infty$, we obtain

$$u_\lambda(0) \geq \left(\frac{a}{\lambda B}\right)^{1/(q-p)} > 0.$$

Note, that this implies $\max\{u_\lambda(x) : x \in R\} = u_\lambda(0) \to +\infty$ as $\lambda \to 0$.

From (16) and Proposition 5, it follows

$$\lim_{T_n\to\pm\infty}\max_{x\in[T_n-1/2,T_n+1/2]}\left|u_\lambda(x)\right| \\ \leq \lim_{T_n\to\pm\infty} 2^{(p-1)/p}\int_{T_n+1/2}^{T_n-1/2}\left(\left|u_n'(x)\right|^p + a(x)\left|u_n(x)\right|^p\right)dx = 0, \tag{17}$$

so $u_\lambda(\pm\infty) = 0$.

Now, we will show that $u_\lambda'(\infty) = 0$. The arguments for $u_\lambda'(-\infty) = 0$ are similar.

If $u_\lambda'(\infty) \neq 0$, there exist $\varepsilon_1 > 0$ and a monotone increasing sequence $x_k \to \infty$ such that $|u_\lambda'(x_k)| \geq (2\varepsilon_1)^{1/(p-1)}$. Then for $x \in [x_k, x_k + \frac{\varepsilon_1}{K_2}]$,

$$\begin{aligned}\left|u_\lambda'(x)\right|^{p-1} &= \left|\varphi_p\left(u_\lambda'(x_k)\right) + \int_{x_k}^{x}\varphi_p\left(u_\lambda'(t)\right)'\,dt\right| \\ &\geq \left|u_\lambda'(x_k)\right|^{p-1} - \int_{x_k}^{x_k+\frac{\varepsilon_1}{K_2}}\left|\varphi_p\left(u_\lambda'(t)\right)'\right|dt \\ &\geq 2\varepsilon_1 - \frac{\varepsilon_1}{K_2}\cdot K_2 = \varepsilon_1,\end{aligned}$$

which contradicts (16).

Moreover, u is an even function that attains its only maximum at 0, since the same holds for the functions u_n. Arguing as in the proof of Theorem 4, we easily obtain that $u'(x) < 0$ if $x > 0$. □

Remark 2 A simplified method can be applied to the equations

$$u'' - a(x)u|u|^{p-2} + \lambda b(x)u|u|^{q-2} = 0, \quad x \in \mathbb{R},$$

under assumptions (H) and $2 \leq p < q$, $\lambda > 0$. Namely, first one looks for the even positive solutions $u_{T,\lambda}$ of the problem

$$\begin{cases} u'' - a(x)\varphi_p(u) + \lambda b(x)\varphi_q(u) = 0, & x \in (-T, T), \\ u(-T) = u(T) = 0, \end{cases}$$

considering the functional $I_T : H_0^1(-T,T) \to \mathbb{R}$

$$I_T(u) = \int_{-T}^{T}\left(\frac{1}{2}u'(x)^2 + \frac{1}{p}a(x)\left|u(x)\right|^p - \frac{\lambda}{q}b(x)\left(u^+(x)\right)^q\right)dx,$$

where $H_0^1(-T,T)$ is the Sobolev space of square integrable functions such that

$$\|u\|^2 = \int_{-T}^{T} u'(x)^2\,dx < \infty.$$

Since $H_0^1(-T,T)$ is a Hilbert space, compactly embedded in $C([-T,T])$ the proof of the (PS)-condition is easier. Similar considerations are made in [1] and [3]. Then, the even

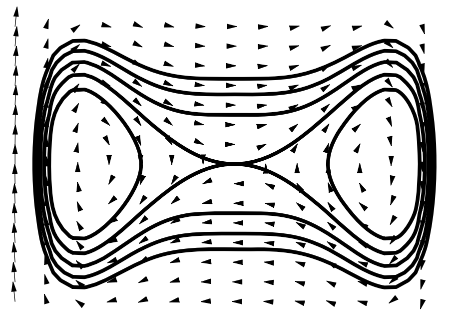

Figure 2 Phase portrait of $u'' - u^3 + 0.5u^5 = 0$, in $[-2, 2] \times [-1.25, 1.25]$.

homoclinic solution u_λ is obtained as a C^1_{loc} limit of the sequence $u_{T,\lambda}$. Note that in this case, the even homoclinic solution u_λ of Eq. (3) satisfies

$$\max\{u_\lambda(x) : x \in \mathbb{R}\} = u_\lambda(0) \geq \left(\frac{a(0)}{\lambda b(0)}\right)^{1/(q-p)},$$

and again $u_\lambda(0) \to +\infty$ as $\lambda \to 0$. If a and b are constants, Eq. (3) is a conservative system and one can plot the phase curves $(\frac{v}{2})^2 - a\frac{|u|^p}{p} + \lambda b\frac{|u|^q}{q} = C$ in the phase plane $(u, v) = (u, u')$. Consider the equation $u'' - u^3 + \lambda u^5 = 0$. The phase portrait in a (u, v) plane, for $\lambda = 0.5$ in the rectangle $\{(u, v) : -2 \leq u \leq 2, -1.25 \leq v \leq 1.25\}$, is plotted on Figure 2.

Competing interests
The author declares that he has no competing interests.

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