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Positive solution for a class of singular semipositone fractional differential equations with integral boundary conditions

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Abstract

In this article, by employing a fixed point theorem in cones, we investigate the existence of a positive solution for a class of singular semipositone fractional differential equations with integral boundary conditions. We also obtain some relations between the solution and Green's function.

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Keywords: fractional differential equations; integral boundary value problem; positive solution; semipositone; cone

1 Introduction

In this article, we consider the existence of a positive solution for the following singular semipositone fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, & u(1) = \lambda \int_0^{\eta} u(s) ds, \end{cases} \quad (1)$$

where $f \in C[(0, 1) \times [0, +\infty), (-\infty, +\infty)]$, $3 < \alpha \leq 4$, $0 < \eta \leq 1$, $0 \leq \frac{\lambda \eta^{\alpha}}{\alpha} < 1$, D_{0+}^{α} is the standard Riemann-Liouville derivative, $f(t, u)$ may be singular at $t = 0$ and/or $t = 1$. Since the nonlinearity $f(t, x)$ may change sign, the problem studied in this paper is called the semipositone problem in the literature which arises naturally in chemical reactor theory. Up to now, much attention has been attached to the existence of positive solutions for semipositone differential equations and the system of differential equations; see [1–11] and references therein to name a few.

Boundary value problems with integral boundary conditions for ordinary differential equations arise in different fields of applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics. Moreover, boundary value problems with integral conditions constitute a very interesting and important class of problems. They include two-point, three-point, multi-point, and nonlocal boundary value problems as special cases, which have received much attention from many authors. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers by Gallardo [12], Karakostas and Tsamatos [13], Lomtatidze and Malaguti [14], and the references therein.

On the other hand, fractional differential equations have been of great interest for many researchers recently. This is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various fields of science and engineering such as control, porous media, electromagnetic, and other fields. For an extensive collection of such results, we refer the readers to the monographs by Samko *et al.* [15], Podlubny [16] and Kilbas *et al.* [17]. For the case where α is an integer, a lot of work has been done dealing with local and nonlocal boundary value problems. For example, in [18] Webb studied the n th-order nonlocal BVP

$$\begin{cases} u^{(n)}(t) + a(t)f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, & u(1) = \int_0^1 u(s) \, dA(s), \end{cases}$$

where $a(t)$ can have singularities, and the nonlinearity f satisfies Carathéodory conditions. Under weak assumptions, Webb obtained sharp results on the existence of positive solutions under a suitable condition on f . In [19] Hao *et al.* consider the n th-order singular nonlocal BVP

$$\begin{cases} u^{(n)}(t) + \lambda a(t)f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, & u(1) = \int_0^1 u(s) \, dA(s), \end{cases}$$

where $\lambda > 0$ is a parameter, a may be singular at $t = 0$ and/or $t = 1$, $f(t, x)$ may also have singularity at $x = 0$.

In two recent papers [20] and [21], by means of the fixed point theory and fixed point index theory, the authors investigated the existence and multiplicity of positive solutions for the following two kinds of fractional differential equations with integral boundary value problems:

$$\begin{cases} D_{0+}^\alpha u(t) + q(t)f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, & u(1) = \int_0^1 u(s) \, dA(s), \end{cases}$$

and

$$\begin{cases} {}^C D^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, & u(1) = \lambda \int_0^1 u(s) \, ds, \end{cases}$$

where $0 < \lambda < 2$, D_{0+}^α and ${}^C D^\alpha$ are the standard Riemann-Liouville derivative and the Caputo fractional derivative, respectively.

To the author's knowledge, there are few papers in the literature to consider fractional differential equations with integral boundary value conditions. Motivated by above papers, the purpose of this article is to investigate the existence of positive solutions for the more general fractional differential equations BVP (1). Firstly, we derive corresponding Green's function known as fractional Green's function and argue its positivity. Then a fixed point theorem is used to obtain the existence of positive solutions for BVP (1). We also obtain some relations between the solution and Green's function. From the example given in Section 4, we know that λ in this article may be greater than 2 and η may take

the value 1. Therefore, compared with that in [21], BVP (1) considered in this article has a more general form.

The rest of this article is organized as follows. In Section 2, we give some preliminaries and lemmas. The main result is formulated in Section 3, and an example is worked out in Section 4 to illustrate how to use our main result.

2 Preliminaries and several lemmas

Let $E = C[0, 1]$, $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$, then $(E, \|\cdot\|)$ is a Banach space. Denote $I = [0, 1]$, $J = (0, 1)$, $R^+ = [0, +\infty)$.

For the reader's convenience, we present some necessary definitions from fractional calculus theory and lemmas. They can be found in the recent literature; see [14–17].

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \rightarrow R$ is given by

$$I_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}y(s) \, ds$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \rightarrow R$ is given by

$$D_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}y(s) \, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α , provided that the right-hand side is pointwise defined on $(0, \infty)$.

From the definition of the Riemann-Liouville derivative, we can obtain the statement.

Lemma 2.1 ([17]) *Let $\alpha > 0$. If we assume $u \in C(0, 1) \cap L(0, 1)$, then the fractional differential equation*

$$D_{0+}^{\alpha}u(t) = 0$$

has $u(t) = C_1t^{\alpha-1} + C_2t^{\alpha-2} + \dots + C_Nt^{\alpha-N}$, $C_i \in R$, $i = 1, 2, \dots, N$, as unique solutions, where N is the smallest integer greater than or equal to α .

Lemma 2.2 ([17]) *Assume that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L(0, 1)$.*

Then

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2} + \dots + C_Nt^{\alpha-N},$$

for some $C_i \in R$, $i = 1, 2, \dots, N$, where N is the smallest integer greater than or equal to α .

In the following, we present Green's function of the fractional differential equation boundary value problem.

Lemma 2.3 Given $y \in C[0, 1]$, the problem

$$\begin{cases} D_{0+}^\alpha u(t) + y(t) = 0, \\ u(0) = u'(0) = u''(0) = 0, \quad u(1) = \lambda \int_0^\eta u(s) \, ds, \end{cases} \quad (2)$$

where $0 < t < 1$, $3 < \alpha \leq 4$, $0 < \eta \leq 1$, $0 \leq \frac{\lambda\eta^\alpha}{\alpha} < 1$, is equivalent to

$$u(t) = \int_0^1 G(t, s)y(s) \, ds,$$

where

$$G(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1} - (1 - \frac{\lambda}{\alpha}\eta^\alpha)(t-s)^{\alpha-1}}{p(0)\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, s \leq \eta; \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (1 - \frac{\lambda}{\alpha}\eta^\alpha)(t-s)^{\alpha-1}}{p(0)\Gamma(\alpha)}, & 0 \leq \eta \leq s \leq t \leq 1; \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1}}{p(0)\Gamma(\alpha)}, & 0 \leq t \leq s \leq \eta \leq 1; \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{p(0)\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \eta \leq s. \end{cases} \quad (3)$$

Here, $p(s) := 1 - \frac{\lambda\eta^\alpha}{\alpha}(1-s)$, $G(t, s)$ is called the Green function of BVP (2). Obviously, $G(t, s)$ is continuous on $[0, 1] \times [0, 1]$.

Proof We may apply Lemma 2.2 to reduce (2) to an equivalent integral equation

$$u(t) = -I_{0+}^\alpha y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3} + C_4 t^{\alpha-4},$$

for some $C_1, C_2, C_3, C_4 \in \mathbb{R}$. Consequently, the general solution of (2) is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \, ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3} + C_4 t^{\alpha-4}.$$

By $u(0) = u'(0) = u''(0) = 0$, one gets that $C_2 = C_3 = C_4 = 0$. On the other hand, $u(1) = \lambda \int_0^\eta u(s) \, ds$ combining with

$$\begin{aligned} u(1) &= -\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds + C_1, \\ \int_0^\eta u(s) \, ds &= -\frac{1}{\Gamma(\alpha)} \int_0^\eta \int_0^x (x-s)^{\alpha-1} y(s) \, ds \, dx + C_1 \int_0^\eta s^{\alpha-1} \, ds \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^\eta \int_s^\eta (x-s)^{\alpha-1} y(s) \, dx \, ds + C_1 \int_0^\eta s^{\alpha-1} \, ds \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^\eta \frac{(\eta-s)^\alpha}{\alpha} y(s) \, ds + \frac{C_1 \eta^\alpha}{\alpha}, \end{aligned}$$

yields

$$C_1 = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1 - \frac{\lambda\eta^\alpha}{\alpha})} y(s) \, ds - \lambda \int_0^\eta \frac{(\eta-s)^\alpha}{\alpha \Gamma(\alpha)(1 - \frac{\lambda\eta^\alpha}{\alpha})} y(s) \, ds.$$

Therefore, the unique solution of the problem (2) is

$$u(t) = - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds + \frac{1}{(1-\frac{\lambda}{\alpha}\eta^\alpha)} \int_0^1 \frac{(1-s)^{\alpha-1} t^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds - \frac{1}{(1-\frac{\lambda}{\alpha}\eta^\alpha)} \int_0^\eta \frac{\frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds.$$

For $t \leq \eta$, one has

$$\begin{aligned} u(t) &= - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds + \frac{1}{(1-\frac{\lambda}{\alpha}\eta^\alpha)} \left[\left(\int_0^t + \int_t^\eta + \int_\eta^1 \right) \frac{(1-s)^{\alpha-1} t^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds \right] \\ &\quad - \frac{\lambda}{(1-\frac{\lambda}{\alpha}\eta^\alpha)} \left[\left(\int_0^t + \int_t^\eta \right) \frac{\frac{1}{\alpha}(\eta-s)^\alpha t^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds \right] \\ &= \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-1} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1} - (1-\frac{\lambda}{\alpha}\eta^\alpha)(t-s)^{\alpha-1}}{(1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} y(s) \, ds \\ &\quad + \int_t^\eta \frac{t^{\alpha-1}(1-s)^{\alpha-1} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1}}{(1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} y(s) \, ds + \int_\eta^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} y(s) \, ds \\ &= \int_0^1 G(t,s)y(s) \, ds. \end{aligned}$$

For $t \geq \eta$, one has

$$\begin{aligned} u(t) &= - \left(\int_0^\eta + \int_\eta^t \right) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds \\ &\quad + \frac{1}{(1-\frac{\lambda}{\alpha}\eta^\alpha)} \left[\left(\int_0^\eta + \int_\eta^t + \int_t^1 \right) \frac{(1-s)^{\alpha-1} t^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds \right] \\ &\quad - \frac{\lambda}{(1-\frac{\lambda}{\alpha}\eta^\alpha)} \int_0^\eta \frac{\frac{1}{\alpha}(\eta-s)^\alpha t^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds \\ &= \int_0^\eta \frac{t^{\alpha-1}(1-s)^{\alpha-1} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1} - (1-\frac{\lambda}{\alpha}\eta^\alpha)(t-s)^{\alpha-1}}{(1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} y(s) \, ds \\ &\quad + \int_\eta^t \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (1-\frac{\lambda}{\alpha}\eta^\alpha)(t-s)^{\alpha-1}}{(1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} y(s) \, ds + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} y(s) \, ds \\ &= \int_0^1 G(t,s)y(s) \, ds. \end{aligned}$$

The proof is complete. □

Lemma 2.4 *The function $G(t,s)$ defined by (3) satisfies*

- (a1) $G(t,s) \geq m_1 e(t)s(1-s)^{\alpha-1}, \forall t,s \in [0,1];$
- (a2) $G(t,s) \leq M_1 e(t)(1-s)^{\alpha-1}, \forall t,s \in [0,1];$
- (a3) $G(t,s) \leq M_1 s(1-s)^{\alpha-1}, \forall t,s \in [0,1];$
- (a4) $p(s) > 0$, and $p(s)$ is not decreasing on $[0,1];$
- (a5) $G(t,s) > 0, \forall t,s \in (0,1),$

where $m_1 = \frac{1-p(0)}{\Gamma(\alpha)p(0)}, M_1 = \frac{\alpha-1}{\Gamma(\alpha)} + \frac{4\frac{\lambda}{\alpha}\eta^{\alpha-1}}{p(0)\Gamma(\alpha)}, e(t) = t^{\alpha-1}.$

Proof For $s \leq t, s \leq \eta$,

$$\begin{aligned}
 G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-s)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
 &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} \eta^\alpha \left(1 - \frac{s}{\eta}\right)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
 &\geq \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} \eta^\alpha (1-s)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
 &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} \left[1 - \frac{\lambda}{\alpha} \eta^\alpha (1-s)\right] - p(0)(t-s)^{\alpha-1} \right\} \\
 &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} p(s) - p(0)(t-s)^{\alpha-1} \right\} \\
 &= \frac{1}{\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1} \right\} + \frac{p(s)-p(0)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
 &= \frac{1}{\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-2} t(1-s) - (t-s)^{\alpha-2} (t-s) \right\} + \frac{p(s)-p(0)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
 &\geq \frac{1}{\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-2} [t(1-s) - (t-s)] \right\} + \frac{p(s)-p(0)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
 &\geq \frac{1}{\Gamma(\alpha)} [t(1-s)]^{\alpha-1} s(1-t) + \frac{1-p(0)}{p(0)\Gamma(\alpha)} s [t(1-s)]^{\alpha-1} \\
 &\geq \frac{1-p(0)}{p(0)\Gamma(\alpha)} t^{\alpha-1} s(1-s)^{\alpha-1}, \\
 G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-s)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
 &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ \left(1 - \frac{\lambda}{\alpha} \eta^\alpha + \frac{\lambda}{\alpha} \eta^\alpha\right) [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-s)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
 &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0) [t^{\alpha-1} (1-s)^{\alpha-1} - (t-s)^{\alpha-1}] \right. \\
 &\quad \left. + \frac{\lambda}{\alpha} \eta^\alpha \left[t^{\alpha-1} (1-s)^{\alpha-1} - \left(1 - \frac{s}{\eta}\right)^\alpha t^{\alpha-1} \right] \right\} \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) \int_{t-s}^{t(1-s)} x^{\alpha-2} dx \right. \\
 &\quad \left. + \frac{\lambda}{\alpha} \eta^\alpha \left[t^{\alpha-1} (1-s)^{\alpha-1} - \left(1 - \frac{s}{\eta}\right)^\alpha t^{\alpha-1} (1-s)^{\alpha-1} \right] \right\} \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-2} (1-s)^{\alpha-2} s(1-t) \right. \\
 &\quad \left. + \frac{\lambda}{\alpha} \eta^\alpha t^{\alpha-1} (1-s)^{\alpha-1} \left[1 - \left(1 - \frac{s}{\eta}\right)^\alpha\right] \right\} \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-2} (1-s)^{\alpha-2} s(1-t) \right. \\
 &\quad \left. + \frac{\lambda}{\alpha} \eta^\alpha t^{\alpha-1} (1-s)^{\alpha-1} \left[1 - \left(1 - \frac{s}{\eta}\right)^\alpha\right] \right\} \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-1} (1-s)^{\alpha-1} \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\lambda}{\alpha} \eta^\alpha t^{\alpha-1} (1-s)^{\alpha-1} \left[1 - \left(1 - \frac{s}{\eta} \right) \right] \left[1 + \left(1 - \frac{s}{\eta} \right) \right] \left[1 + \left(1 - \frac{s}{\eta} \right)^2 \right] \Big\} \\
 & \leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1)t^{\alpha-1}(1-s)^{\alpha-1} + 4\frac{\lambda}{\alpha}\eta^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1} \right\} \\
 & = \left[\frac{\alpha-1}{\Gamma(\alpha)} + \frac{4\frac{\lambda}{\alpha}\eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] t^{\alpha-1}(1-s)^{\alpha-1}, \\
 G(t,s) & = \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
 & = \frac{1}{p(0)\Gamma(\alpha)} \left\{ \left(1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^\alpha \right) [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
 & = \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)[t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}] \right. \\
 & \quad \left. + \frac{\lambda}{\alpha}\eta^\alpha \left[t^{\alpha-1}(1-s)^{\alpha-1} - \left(1 - \frac{s}{\eta} \right)^\alpha t^{\alpha-1} \right] \right\} \\
 & \leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) \int_{t-s}^{t(1-s)} x^{\alpha-2} dx \right. \\
 & \quad \left. + \frac{\lambda}{\alpha}\eta^\alpha \left[t^{\alpha-1}(1-s)^{\alpha-1} - \left(1 - \frac{s}{\eta} \right)^\alpha t^{\alpha-1}(1-s)^{\alpha-1} \right] \right\} \\
 & \leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}s(1-t) \right. \\
 & \quad \left. + \frac{\lambda}{\alpha}\eta^\alpha t^{\alpha-1}(1-s)^{\alpha-1} \left[1 - \left(1 - \frac{s}{\eta} \right)^\alpha \right] \right\} \\
 & \leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}s(1-t) \right. \\
 & \quad \left. + \frac{\lambda}{\alpha}\eta^\alpha t^{\alpha-1}(1-s)^{\alpha-1} \left[1 - \left(1 - \frac{s}{\eta} \right)^4 \right] \right\} \\
 & \leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}s(1-t) \right. \\
 & \quad \left. + \frac{\lambda}{\alpha}\eta^\alpha t^{\alpha-1}(1-s)^{\alpha-1} \left[1 - \left(1 - \frac{s}{\eta} \right) \right] \left[1 + \left(1 - \frac{s}{\eta} \right) \right] \left[1 + \left(1 - \frac{s}{\eta} \right)^2 \right] \right\} \\
 & \leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1)(1-s)^{\alpha-1}s + 4\frac{\lambda}{\alpha}\eta^{\alpha-1}s(1-s)^{\alpha-1} \right\} \\
 & = \left[\frac{\alpha-1}{\Gamma(\alpha)} + \frac{4\frac{\lambda}{\alpha}\eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] s(1-s)^{\alpha-1}.
 \end{aligned}$$

For $\eta \leq s \leq t$,

$$\begin{aligned}
 G(t,s) & = \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
 & \geq \frac{p(s)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} - \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \\
 & = \frac{1}{\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-2} t(1-s) - (t-s)^{\alpha-2} (t-s) \right\} + \frac{p(s)-p(0)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1}
 \end{aligned}$$

$$\begin{aligned}
 &\geq \frac{1}{\Gamma(\alpha)} [t(1-s)]^{\alpha-1} s(1-t) + \frac{1-p(0)}{p(0)\Gamma(\alpha)} s [t(1-s)]^{\alpha-1} \\
 &\geq \frac{1-p(0)}{p(0)\Gamma(\alpha)} t^{\alpha-1} s(1-s)^{\alpha-1}, \\
 G(t,s) &= \frac{1}{p(0)\Gamma(\alpha)} \{ [t(1-s)]^{\alpha-1} - p(0)(t-s)^{\alpha-1} \} \\
 &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ \left(1 - \frac{\lambda}{\alpha} \eta^\alpha + \frac{\lambda}{\alpha} \eta^\alpha \right) [t(1-s)]^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
 &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0) [t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}] + \frac{\lambda}{\alpha} \eta^\alpha [t(1-s)]^{\alpha-1} \right\} \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) \int_{t-s}^{t(1-s)} x^{\alpha-2} dx + \frac{\lambda}{\alpha} \eta^{\alpha-1} s t^{\alpha-1} (1-s)^{\alpha-1} \right\} \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-2} (1-s)^{\alpha-2} s(1-t) + \frac{\lambda}{\alpha} \eta^{\alpha-1} s t^{\alpha-1} (1-s)^{\alpha-1} \right\} \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-1} (1-s)^{\alpha-1} + \frac{\lambda}{\alpha} \eta^{\alpha-1} t^{\alpha-1} (1-s)^{\alpha-1} \right\} \\
 &\leq \left[\frac{\alpha-1}{\Gamma(\alpha)} + \frac{4\frac{\lambda}{\alpha} \eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] t^{\alpha-1} (1-s)^{\alpha-1}, \\
 G(t,s) &= \frac{1}{p(0)\Gamma(\alpha)} \{ [t(1-s)]^{\alpha-1} - p(0)(t-s)^{\alpha-1} \} \\
 &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ \left(1 - \frac{\lambda}{\alpha} \eta^\alpha + \frac{\lambda}{\alpha} \eta^\alpha \right) [t(1-s)]^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
 &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0) [t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}] + \frac{\lambda}{\alpha} \eta^\alpha [t(1-s)]^{\alpha-1} \right\} \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) \int_{t-s}^{t(1-s)} x^{\alpha-2} dx + \frac{\lambda}{\alpha} \eta^{\alpha-1} s t^{\alpha-1} (1-s)^{\alpha-1} \right\} \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-2} (1-s)^{\alpha-2} s(1-t) + \frac{\lambda}{\alpha} \eta^{\alpha-1} s t^{\alpha-1} (1-s)^{\alpha-1} \right\} \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) (1-s)^{\alpha-1} s + \frac{\lambda}{\alpha} \eta^{\alpha-1} s (1-s)^{\alpha-1} \right\} \\
 &\leq \left[\frac{\alpha-1}{\Gamma(\alpha)} + \frac{4\frac{\lambda}{\alpha} \eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] s (1-s)^{\alpha-1}.
 \end{aligned}$$

For $t \leq s \leq \eta$,

$$\begin{aligned}
 G(t,s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-s)^\alpha t^{\alpha-1} \right\} \\
 &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} \eta^\alpha \left(1 - \frac{s}{\eta} \right)^\alpha t^{\alpha-1} \right\} \\
 &\geq \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} \eta^\alpha (1-s)^\alpha t^{\alpha-1} \right\} \\
 &= \frac{1}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \left[1 - \frac{\lambda}{\alpha} \eta^\alpha (1-s) \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{p(0) + p(s) - p(0)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
 &= \frac{1}{\Gamma(\alpha)} [t(1-s)]^{\alpha-1} + \frac{p(s) - p(0)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
 &\geq \frac{1-p(0)}{p(0)\Gamma(\alpha)} t^{\alpha-1} s(1-s)^{\alpha-1}, \\
 G(t,s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-s)^\alpha t^{\alpha-1} \right\} \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
 &\leq \left[\frac{\alpha-1}{\Gamma(\alpha)} + \frac{4\frac{\lambda}{\alpha}\eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] t^{\alpha-1} (1-s)^{\alpha-1}, \\
 G(t,s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-s)^\alpha t^{\alpha-1} \right\} \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
 &\leq \left[\frac{\alpha-1}{\Gamma(\alpha)} + \frac{4\frac{\lambda}{\alpha}\eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] s(1-s)^{\alpha-1}.
 \end{aligned}$$

For $\eta \leq s, t \leq s$,

$$\begin{aligned}
 G(t,s) &= \frac{1}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
 &\geq \frac{1-p(0)}{p(0)\Gamma(\alpha)} t^{\alpha-1} s(1-s)^{\alpha-1}, \\
 G(t,s) &= \frac{1}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
 &\leq \left[\frac{\alpha-1}{\Gamma(\alpha)} + \frac{4\frac{\lambda}{\alpha}\eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] t^{\alpha-1} (1-s)^{\alpha-1}, \\
 G(t,s) &= \frac{1}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
 &\leq \left[\frac{\alpha-1}{\Gamma(\alpha)} + \frac{4\frac{\lambda}{\alpha}\eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] s(1-s)^{\alpha-1}.
 \end{aligned}$$

From above, (a1), (a2), (a3), (a5) are complete. Clearly, (a4) is true. The proof is complete. \square

Throughout this article, we adopt the following conditions.

(H₁) $f \in C[(0,1) \times [0, +\infty), (-\infty, +\infty)]$ and there exist $a, b \in C[J, R^+]$, $h \in C[R^+, R^+]$, $g(t, u) \in [J \times R^+, R^+]$ such that

$$g(t, u) - b(t) \leq f(t, u) \leq a(t)h(u), \quad \forall t \in J, u \in R^+;$$

(H₂) There exists $[a, b] \subset I$ such that $\lim_{u \rightarrow +\infty} \frac{g(t, u)}{u} = +\infty$ uniformly for $t \in [a, b]$;

(H₃) There exists $r > 0$ such that

$$\int_0^1 (1-s)^{\alpha-1} b(s) \, ds < \frac{m_1}{M_1^2} r,$$

$$M_1 \int_0^1 (1-s)^{\alpha-1} (a(s) + b(s)) \, ds < \frac{r}{\max_{0 \leq u \leq r} \{h(u), 1\}}.$$

Let

$$P = \{u \mid u(t) \geq 0, u \in E\},$$

$$Q = \left\{ u \mid u \in P, u(t) \geq \frac{m_1}{M_1} \|u\| e(t) \right\}. \tag{4}$$

Obviously, Q is a cone in a Banach space E and (E, Q) is an ordering Banach space.

Let

$$x_0(t) = \int_0^1 G(t,s) b(s) \, ds, \quad t \in I, \tag{5}$$

where $b(t)$ is defined as that in (H₁). It follows from Lemma 2.4 and (H₃) that

$$0 \leq x_0(t) \leq M_1 e(t) \int_0^1 (1-s)^{\alpha-1} b(s) \, ds < +\infty, \quad t \in I. \tag{6}$$

So, $x_0 \in P$ and it satisfies

$$\begin{cases} D_{0+}^\alpha x_0(t) + x_0(t) = 0, & 0 < t < 1, \\ x_0(0) = x_0'(0) = x_0''(0) = 0, & x_0(1) = \lambda \int_0^\eta x_0(s) \, ds. \end{cases} \tag{7}$$

For any $u \in Q \setminus \{\theta\}$, $u(t) \geq \frac{m_1}{M_1} \|u\| e(t)$, $t \in I$. Consequently, by (6) and Lemma 2.4, we have

$$\begin{aligned} x_0(t) &\leq M_1 e(t) \int_0^1 (1-s)^{\alpha-1} b(s) \, ds \\ &\leq M_1 \|u\| \frac{m_1}{M_1} e(t) \cdot \frac{1}{\|u\| \frac{m_1}{M_1}} \int_0^1 (1-s)^{\alpha-1} b(s) \, ds \\ &\leq \frac{u(t)}{\|u\|} \frac{M_1^2}{m_1} \int_0^1 (1-s)^{\alpha-1} b(s) \, ds. \end{aligned} \tag{8}$$

For any $k \in E$, denote

$$[k(t)]^+ = \begin{cases} k(t), & k(t) \geq 0, \\ 0, & k(t) < 0. \end{cases}$$

We define an operator A as follows:

$$(Au)(t) = \int_0^1 G(t,s) (f(s, [u(s) - x_0(s)]^+) + b(s)) \, ds, \quad \forall u \in P. \tag{9}$$

Lemma 2.5 *Suppose that (H₁)-(H₃) hold. Then $A : Q \rightarrow Q$ is completely continuous.*

Proof For any $u \in P$, it is clear that $[u(s) - x_0(s)]^+ \leq u(s) \leq \|u\|$. By (H₁), we get

$$\begin{aligned} f(s, [u(s) - x_0(s)]^+) + b(s) &\leq a(s)h([u(s) - x_0(s)]^+) + b(s) \\ &\leq \max_{0 \leq r \leq \|u\|} \{h(r), 1\}(a(s) + b(s)), \quad \forall s \in J. \end{aligned} \tag{10}$$

By (10) and Lemma 2.4, we have

$$\begin{aligned} &\int_0^1 G(t, s)(f(s, [u(s) - x_0(s)]^+) + b(s)) \, ds \\ &\leq M_1 e(t) \int_0^1 (1-s)^{\alpha-1}(a(s) + b(s)) \, ds \cdot \max_{0 \leq r \leq \|u\|} \{h(r), 1\} < +\infty, \quad t \in J, \end{aligned} \tag{11}$$

which together with (H₃) means that operator A defined by (9) is well defined.

Now, we show that $A : Q \rightarrow Q$.

For any $u \in Q$, by (H₁) we have by (9) and Lemma 2.4 that

$$(Au)(t) \leq M_1 \int_0^1 s(1-s)^{\alpha-1}[f(s, [u(s) - x_0(s)]^+) + b(s)] \, ds, \quad \forall t \in I,$$

which means that

$$\|Au\| \leq M_1 \int_0^1 s(1-s)^{\alpha-1}[f(s, [u(s) - x_0(s)]^+) + b(s)] \, ds. \tag{12}$$

It follows from (12) and Lemma 2.4 that

$$\begin{aligned} (Au)(t) &= \int_0^1 G(t, s)(f(s, [u(s) - x_0(s)]^+) + b(s)) \, ds \\ &\geq m_1 e(t) \int_0^1 s(1-s)^{\alpha-1}[f(s, [u(s) - x_0(s)]^+) + b(s)] \, ds \\ &= \frac{m_1}{M_1} e(t) \int_0^1 M_1 s(1-s)^{\alpha-1}[f(s, [u(s) - x_0(s)]^+) + b(s)] \, ds \\ &\geq \frac{m_1}{M_1} e(t) \|Au\|. \end{aligned}$$

Thus, A maps Q into Q .

Finally, we prove that A maps Q into Q is completely continuous.

Let $D \subset Q$ be any bounded set. Then there exists a constant $L_1 > 0$ such that $\|u\| \leq L_1$ for any $u \in D$. Notice that $[u(s) - x_0(s)]^+ \leq u(s) \leq L_1$, for any $u \in D, s \in I$, by (H₃) and (11), we have

$$|(Au)(t)| \leq M_1 \int_0^1 (1-s)^{\alpha-1}(a(s) + b(s)) \, ds \cdot \max_{0 \leq r \leq L_1} \{h(r), 1\} < +\infty.$$

Therefore, $A(D)$ is uniformly bounded.

On the other hand, since $G(t, s)$ is continuous on $[0, 1] \times [0, 1]$, it is uniformly continuous on $[0, 1] \times [0, 1]$ as well. Thus, for fixed $s \in I$ and for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for any $t_1, t_2 \in [0, 1]$ and $|t_1 - t_2| < \delta$,

$$|G(t_1, s) - G(t_2, s)| < \frac{\varepsilon}{\max_{0 \leq r \leq L_1} \{h(r), 1\} \int_0^1 (a(s) + b(s)) \, ds}. \tag{13}$$

Therefore, for any $x \in D$, we get by (10) and (13)

$$\begin{aligned} & |(Au)(t_1) - (Au)(t_2)| \\ & \leq \int_0^1 |G(t_1, s) - G(t_2, s)| (f(s, [u(s) - x_0(s)]^+) + b(s)) \, ds \\ & \leq \frac{\varepsilon}{\max_{0 \leq r \leq L_1} \{h(r), 1\} \int_0^1 (a(s) + b(s)) \, ds} \int_0^1 (a(s) + b(s)) \, ds \cdot \max_{0 \leq r \leq L_1} \{h(r), 1\} = \varepsilon, \end{aligned}$$

which implies that the operator A is equicontinuous. Thus, the Ascoli-Arzelà theorem guarantees that $A(D)$ is a relatively compact set.

Let $u_n, u_0 \in Q$, $u_n \rightarrow u_0$ ($n \rightarrow \infty$). Then $\{u_n\}$ is bounded. Let $L_2 = \sup\{\|u_n\|, n = 0, 1, 2, \dots\}$, by (10), we get

$$f(s, [u_n(s) - x_0(s)]^+) + b(s) \leq \left(\max_{0 \leq r \leq L_2} \{h(r), 1\} \right) (a(s) + b(s)), \quad n = 0, 1, 2, \dots \tag{14}$$

By (9), we have

$$\begin{aligned} & |(Au_n)(t) - (Au_0)(t)| \\ & \leq \int_0^1 G(t, s) |f(s, [u_n(s) - x_0(s)]^+) - f(s, [u_0(s) - x_0(s)]^+)| \, ds \\ & \leq M_1 \int_0^1 s(1-s)^{\alpha-1} |f(s, [u_n(s) - x_0(s)]^+) - f(s, [u_0(s) - x_0(s)]^+)| \, ds. \end{aligned} \tag{15}$$

It follows from (14), (15), (H_1) , (H_3) , and the Lebesgue dominated convergence theorem that A is continuous. Thus, we have proved the continuity of the operator A . This completes the complete continuity of A . \square

To prove the main result, we need the following well-known fixed point theorem.

Lemma 2.6 (Fixed point theorem of cone expansion and compression of norm type [22])
 Let Ω_1 and Ω_2 be two bounded open sets in a Banach space E such that $\theta \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$, $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$ be a completely continuous operator, where θ denotes the zero element of E and P a cone of E . Suppose that one of the two conditions holds:

- (i) $\|Au\| \leq \|u\|, \forall u \in P \cap \partial\Omega_1; \|Au\| \geq \|u\|, \forall u \in P \cap \partial\Omega_2;$
- (ii) $\|Au\| \geq \|u\|, \forall u \in P \cap \partial\Omega_1; \|Au\| \leq \|u\|, \forall u \in P \cap \partial\Omega_2.$

Then A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3 Main result

Theorem 3.1 Assume that conditions (H_1) - (H_3) are satisfied. Then the singular semipositone BVP (1) has at least one positive solution $\omega(t)$. Furthermore, there exist two constants

$M > m > 0$ such that

$$me(t) \leq \omega(t) \leq Me(t), \quad t \in I. \tag{16}$$

Proof Firstly, we show that the operator A has a fixed point in Q . Let

$$\Omega_r = \{u \in E \mid \|u\| < r\},$$

where r is the same as that defined in (H_3) . For any $u \in \partial\Omega_r \cap Q$, by (10) and (12), we have that

$$\begin{aligned} (Au)(t) &= \int_0^1 G(t,s) (f(s, [u(s) - x_0(s)]^+) + b(s)) \, ds \\ &\leq M_1 \int_0^1 s(1-s)^{\alpha-1} (a(s) + b(s)) \, ds \cdot \max_{0 \leq u \leq r} \{h(u), 1\} < +\infty, \quad t \in J. \end{aligned}$$

Therefore,

$$\|Au\| \leq M_1 \int_0^1 s(1-s)^{\alpha-1} (a(s) + b(s)) \, ds \cdot \max_{0 \leq u \leq r} \{h(u), 1\} < r,$$

which together with (H_3) implies that

$$\|Au\| \leq r = \|u\|, \quad \forall u \in \partial\Omega_r \cap Q. \tag{17}$$

For $[a, b]$ in (H_2) , it is clear that

$$e(t) \geq a^{\alpha-1}, \quad t \in [a, b]. \tag{18}$$

By (H_3) , we know that there exists a natural number m_0 big enough such that

$$\int_0^1 (1-s)^{\alpha-1} b(s) \, ds < \frac{m_0}{m_0 + 1} \frac{m_1}{M_1^2} r. \tag{19}$$

Choose

$$M > \frac{M_1(m_0 + 1)}{m_1 a^{\alpha-1} \min_{t \in [a,b]} \int_0^1 G(t,s) \, ds}. \tag{20}$$

By (H_2) , we know there exists $R_1 > r$ such that

$$g(t, u) \geq Mu, \quad \forall t \in [a, b], u \geq R_1. \tag{21}$$

Take

$$R > \max \left\{ r, 1, \frac{R_1(m_0 + 1)M_1}{a^{\alpha-1}m_1} \right\}. \tag{22}$$

In the following, we are in a position to show that

$$\|Au\| \geq \|u\|, \quad \forall u \in \partial\Omega_R \cap Q. \tag{23}$$

For any $u \in \partial\Omega_R \cap Q$, by (8) we get

$$x_0(t) \leq \frac{u(t)}{\|u\|} \frac{M_1^2}{m_1} \int_0^1 (1-s)^{\alpha-1} b(s) \, ds,$$

which together with (18), (19), (22), and (H₃) implies that

$$\begin{aligned} & u(t) - x_0(t) \\ & \geq \left(1 - \frac{u(t)}{R} \frac{M_1^2}{m_1} \int_0^1 (1-s)^{\alpha-1} b(s) \, ds\right) u(t) \\ & \geq \left(1 - \frac{1}{R} \frac{m_0}{m_0+1} r\right) u(t) \geq \frac{1}{m_0+1} \|u\| \frac{m_1}{M_1} e(t) \\ & \geq \frac{R}{m_0+1} \frac{m_1}{M_1} a^{\alpha-1} \geq R_1, \quad t \in [a, b]. \end{aligned} \tag{24}$$

For $u \in \partial\Omega_R \cap Q$, $t \in [a, b]$, it follows from (H₁), (20), (21), (22), and (24) that

$$\begin{aligned} (Au)(t) &= \int_0^1 G(t,s) (f(s, [u(s) - x_0(s)]^+) + b(s)) \, ds \\ &\geq \int_0^1 G(t,s) g(s, [u(s) - x_0(s)]^+) \, ds \\ &\geq \int_0^1 G(t,s) M(u(s) - x_0(s)) \, ds \\ &\geq M \frac{R}{m_0+1} \frac{m_1}{M_1} a^{\alpha-1} \min_{t \in [a,b]} \int_0^1 G(t,s) \, ds > R, \quad t \in [a, b]. \end{aligned} \tag{25}$$

By (25), we know that (23) holds. So, (17), (23), and Lemma 2.6 guarantee that A has at least one fixed point $z_0(t)$ in $(\overline{\Omega}_R \setminus \Omega_r) \cap Q$ and $r \leq \|z_0\| \leq R$. Furthermore,

$$z_0(t) = \int_0^1 G(t,s) (f(s, [z_0(s) - x_0(s)]^+) + b(s)) \, ds, \quad t \in I. \tag{26}$$

By simple computation, we have that

$$\begin{cases} D_{0+}^\alpha z_0(t) + f(t, [z_0(t) - x_0(t)]^+) + b(t) = 0, & 0 < t < 1, \\ z_0(0) = z_0'(0) = z_0''(0) = 0, & z_0(1) = \lambda \int_0^\eta z_0(s) \, ds. \end{cases} \tag{27}$$

Secondly, we show BVP (1) has a positive solution. It follows from (8) and the fact $\|z_0\| \geq r$ that

$$x_0(t) \leq \frac{z_0(t)}{\|z_0\|} \frac{M_1^2}{m_1} \int_0^1 (1-s)^{\alpha-1} b(s) \, ds \leq \frac{z_0(t)}{r} \frac{M_1^2}{m_1} \int_0^1 (1-s)^{\alpha-1} b(s) \, ds,$$

which combined with (19) implies that

$$z_0(t) - x_0(t) \geq \left(1 - \frac{1}{r} \frac{M_1^2}{m_1} \int_0^1 (1-s)^{\alpha-1} b(s) \, ds\right) z_0(t) \geq \frac{1}{m_0+1} z_0(t) \geq 0. \tag{28}$$

By (27) and (28), we have

$$\begin{cases} D_{0+}^\alpha z_0(t) + f(t, [z_0(t) - x_0(t)]) + b(t) = 0, & 0 < t < 1, \\ z_0(0) = z_0'(0) = z_0''(0) = 0, & z_0(1) = \lambda \int_0^\eta z_0(s) \, ds. \end{cases} \tag{29}$$

Let $\omega(t) = z_0(t) - x_0(t)$, $t \in I$. It follows from (28) and $z_0 \in Q$ that

$$\omega(t) \geq \frac{1}{m_0 + 1} z_0(t) \geq \frac{m_1 r}{(m_0 + 1) M_1} e(t) > 0, \quad t \in J. \tag{30}$$

By (7), (29), and (30), we obtain

$$\begin{cases} D_{0+}^\alpha \omega(t) + f(t, \omega(t)) = 0, & 0 < t < 1, \\ \omega(0) = \omega'(0) = \omega''(0) = 0, & \omega(1) = \lambda \int_0^\eta \omega(s) \, ds. \end{cases}$$

Thus, we have proved that $\omega(t)$ is a positive solution for BVP (1).

Finally, we show that (16) holds. From (26) and Lemma 2.4, we know that

$$z_0(t) \leq M_1 e(t) \int_0^1 (1-s)^{\alpha-1} (f(s, [z_0(s) - x_0(s)]^+) + b(s)) \, ds, \quad t \in I. \tag{31}$$

Since $\omega(t) \leq z_0(t)$, (30), and (31) mean that (16) holds for $m = \frac{m_1 r}{(m_0 + 1) M_1}$ and $M = M_1 \int_0^1 (1-s)^{\alpha-1} (f(s, [z_0(s) - x_0(s)]^+) + b(s)) \, ds$ holds. This completes the proof of Theorem 3.1. \square

4 Example

Consider the following singular semipositone fractional differential equations:

$$\begin{cases} D_{0+}^{\frac{7}{2}} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = 16\sqrt{2} \int_0^{\frac{1}{2}} u(s) \, ds, \end{cases} \tag{32}$$

where $f(t, u) = \frac{t^2}{8\sqrt{1-t}} u^{\frac{5}{2}} + \frac{t^{\frac{1}{4}}}{20(1-t)} (u - u^{\frac{1}{2}} - \frac{3}{4})$. It is clear (32) has the form of (1), where $\alpha = \frac{7}{2}$, $\lambda = 16\sqrt{2}$, $\eta = \frac{1}{2}$. By simple computation, we know that $0 < \frac{\lambda \eta^\alpha}{\alpha} \approx 0.5714 < 1$, $p(0) \approx 0.4286$. Let

$$\begin{aligned} a(t) &= \frac{t^2}{8\sqrt{1-t}} + \frac{t^{\frac{1}{4}}}{20(1-t)}, & b(t) &= \frac{t^{\frac{1}{4}}}{20(1-t)}, \\ g(t, u) &= \frac{t^2}{8\sqrt{1-t}} u^{\frac{5}{2}}, & h(u) &= u^{\frac{5}{2}} + u - u^{\frac{1}{2}} + \frac{1}{4}. \end{aligned}$$

Notice that

$$-1 \leq u - u^{\frac{1}{2}} - \frac{3}{4} \leq u - u^{\frac{1}{2}} + \frac{1}{4},$$

we have

$$-\frac{t^{\frac{1}{4}}}{20(1-t)} \leq \frac{t^{\frac{1}{4}}}{20(1-t)} \left(u - u^{\frac{1}{2}} - \frac{3}{4} \right) \leq \frac{t^{\frac{1}{4}}}{20(1-t)} \left(u - u^{\frac{1}{2}} + \frac{1}{4} \right). \tag{33}$$

It follows from the left side of (33) that

$$-b(t) \leq f(t, u) - g(t, u). \quad (34)$$

Considering $u - u^{\frac{1}{2}} + \frac{1}{4} \geq 0$, we get

$$f(t, u) \leq \frac{t^2}{8\sqrt{1-t}} u^{\frac{5}{2}} + \frac{t^{\frac{1}{4}}}{20(1-t)} \left(u - u^{\frac{1}{2}} + \frac{1}{4} \right) \leq a(t)h(u). \quad (35)$$

By (34) and (35) we know (H_1) holds. Obviously, (H_2) holds for $[a, b] = [\frac{1}{4}, \frac{3}{4}]$.

Now, we check (H_3) . By simple computation, we have $m_1 = 0.4012$, $M_1 = 3.9619$, $\frac{m_1}{M_1} = 0.0256$, $\int_0^1 (1-s)^{\alpha-1} b(s) ds \approx 0.0136$, $M_1 \int_0^1 (1-s)^{\alpha-1} (a(s) + b(s)) ds \approx 0.1245$. Take $r = 1$, then $\max_{0 \leq u \leq r} \{h(u), 1\} \approx 1.2500$, $\frac{r}{\max_{0 \leq u \leq r} \{h(u), 1\}} \approx 0.8000$. Thus, (H_3) is valid. It follows from Theorem 3.1 that BVP (32) has at least one positive solution.

Competing interests

The author declares that he has no competing interests.

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