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Nonlinear fractional differential equations with nonlocal fractional integro-differential boundary conditions

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Abstract
We study the existence of solutions for a class of nonlinear Caputo-type fractional boundary value problems with nonlocal fractional integro-differential boundary conditions. We apply some fixed point principles and Leray-Schauder degree theory to obtain the main results. Some examples are discussed for the illustration of the main work.

MSC: 34A08; 34A12; 34B15

Keywords: fractional differential equations; fractional boundary conditions; separated boundary conditions; fixed point theorems

1 Introduction

Nonlocal boundary value problems of fractional differential equations have been extensively studied in the recent years. In fact, the subject of fractional calculus has been quite attractive and exciting due to its applications in the modeling of many physical and engineering problems. For theoretical and practical development of the subject, we refer to the books [1–5]. Some recent results on fractional boundary value problems can be found in [6–14] and references therein. In [11], the authors studied a boundary value problem of fractional differential equations with fractional separated boundary conditions.

In this article, motivated by [11], we consider a fractional boundary value problem with fractional integro-differential boundary conditions given by

$$\begin{cases} {}^c\mathcal{D}^\alpha x(t) = f(t, x(t)), \quad 1 < \alpha \le 2, t \in [0,1], \\ \alpha_1 x(0) + \beta_1({}^c\mathcal{D}^p x(0)) = \gamma_1 \int_0^\eta \frac{(\eta - s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)\, ds, \quad 0 < p < 1, \\ \alpha_2 x(1) + \beta_2({}^c\mathcal{D}^p x(1)) = \gamma_2 \int_0^\sigma \frac{(\sigma - s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)\, ds, \quad 0 < \eta, \sigma < 1, \end{cases} \tag{1.1}$$

where ${}^c\mathcal{D}^\alpha$ denotes the Caputo fractional derivative of order α, f is a given continuous function, and α_i, β_i, γ_i $(i = 1, 2)$ are suitably chosen real constants.

The main aim of the present study is to obtain some existence results for the problem (1.1). As a first step, we transform the given problem to a fixed point problem and show the existence of fixed points for the transformed problem which in turn correspond to the solutions of the actual problem. The methods used to prove the existence results are standard; however, their exposition in the framework of the problem (1.1) is new.

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2 Preliminaries

Let us recall some basic definitions of fractional calculus [1, 2].

Definition 2.1 For $(n-1)$-times absolutely continuous function $g:[0,\infty)\to\mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$ {}^{c}D^{q}g(t)=\frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}g^{(n)}(s)\,ds, \quad n-1<q<n, n=[q]+1, $$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 The Riemann-Liouville fractional integral of order q is defined as

$$ I^{q}g(t)=\frac{1}{\Gamma(q)}\int_0^t \frac{g(s)}{(t-s)^{1-q}}\,ds, \quad q>0, $$

provided the integral exists.

To define the solution of the boundary value problem (1.1), we need the following lemma, which deals with a linear variant of the problem (1.1).

Lemma 2.3 *For a given $y\in C([0,1],\mathbb{R})$, the unique solution of the linear fractional boundary value problem*

$$ \begin{cases} {}^{c}D^{\alpha}x(t)=y(t), \quad 1<\alpha\leq 2, \\ \alpha_1x(0)+\beta_1({}^{c}D^{p}x(0))=\gamma_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}x(s)\,ds, \quad 0<p<1, \\ \alpha_2x(1)+\beta_2({}^{c}D^{p}x(1))=\gamma_2\int_0^{\sigma}\frac{(\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)}x(s)\,ds, \quad 0<\eta,\sigma<1 \end{cases} \tag{2.1} $$

is given by

$$ \begin{aligned} x(t)={}&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+\mu_1(t)\int_0^{\eta}\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}y(s)\,ds \\ &+\mu_2(t)\biggl\{\gamma_2\int_0^{\sigma}\frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}y(s)\,ds-\beta_2\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds \\ &-\alpha_2\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\biggr\}, \end{aligned} \tag{2.2} $$

where

$$ \begin{aligned} &\mu_1(t)=\gamma_1(\Delta_1-\Delta_2t), \qquad \mu_2(t)=\gamma_1\Delta_3+\Delta_4t, \\ &\Delta_1=\frac{1}{\Delta}\biggl(\alpha_2-\frac{\gamma_2\sigma^{\alpha}}{\Gamma(\alpha+1)}+\frac{\beta_2}{\Gamma(2-p)}\biggr), \qquad \Delta_2=\frac{1}{\Delta}\biggl(\alpha_2-\frac{\gamma_2\sigma^{\alpha-1}}{\Gamma(\alpha)}\biggr), \\ &\Delta_3=\frac{\eta^{\alpha}}{\Delta\Gamma(\alpha+1)}, \qquad \Delta_4=\frac{1}{\Delta}\biggl(\alpha_1-\frac{\gamma_1\eta^{\alpha-1}}{\Gamma(\alpha)}\biggr), \\ &\Delta=\biggl(\alpha_1-\frac{\gamma_1\eta^{\alpha-1}}{\Gamma(\alpha)}\biggr)\biggl(\alpha_2-\frac{\gamma_2\sigma^{\alpha}}{\Gamma(\alpha+1)}+\frac{\beta_2}{\Gamma(2-p)}\biggr)+\frac{\gamma_1\eta^{\alpha}}{\Gamma(\alpha+1)}\biggl(\alpha_2-\frac{\gamma_2\sigma^{\alpha-1}}{\Gamma(\alpha)}\biggr)\neq 0. \end{aligned} \tag{2.3} $$

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Proof It is well known [2] that the solution of the fractional differential equation in (2.1) can be written as

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + c_0 + c_1 t. \tag{2.4}$$

Using ${}^cD^p b = 0$ (b is a constant), ${}^cD^p t = \frac{t^{1-p}}{\Gamma(2-p)}$, ${}^cD^p I^q y(t) = I^{q-p} y(t)$, (2.4) gives

$${}^cD^p x(t) = \int_0^t \frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds - c_1 \frac{t^{1-p}}{\Gamma(2-p)}. \tag{2.5}$$

Using the integral boundary conditions of the problem (2.1) together with (2.3), (2.4), and (2.5) yields

$$\begin{aligned} c_0 &= \frac{1}{\Delta}\biggl\{\gamma_1\biggl(\alpha_2 - \frac{\gamma_2\sigma^\alpha}{\Gamma(\alpha+1)} + \frac{\beta_2}{\Gamma(2-p)}\biggr)\int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} y(s)\,ds \\ &\quad + \frac{\gamma_1\eta^\alpha}{\Gamma(\alpha+1)}\biggl(\gamma_2\int_0^\sigma \frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} y(s)\,ds \\ &\quad - \beta_2\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds - \alpha_2\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds\biggr)\biggr\}, \\ c_1 &= \frac{1}{\Delta}\biggl\{-\gamma_1\biggl(\alpha_2 - \frac{\gamma_2\sigma^{\alpha-1}}{\Gamma(\alpha)}\biggr)\int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} y(s)\,ds \\ &\quad + \biggl(\alpha_1 - \frac{\gamma_1\eta^{\alpha-1}}{\Gamma(\alpha)}\biggr)\biggl(\gamma_2\int_0^\sigma \frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} y(s)\,ds \\ &\quad - \beta_2\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds - \alpha_2\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds\biggr)\biggr\}. \end{aligned}$$

Substituting the values of c_0, c_1 in (2.4), we get (2.2). This completes the proof. □

Remark 2.4 Notice that the solution (2.2) is independent of the parameter β_1, which distinguishes the present work from the one containing the fractional differential equation of (2.1) with the boundary conditions of the form:

$$\begin{aligned} \alpha_1 x(0) + \beta_1 x'(0) &= \gamma_1\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)\,ds, \\ \alpha_2 x(1) + \beta_2 x'(1) &= \gamma_2\int_0^\sigma \frac{(\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)\,ds. \end{aligned} \tag{2.6}$$

In case $\beta_1 = 0 = \beta_2$, the boundary conditions in (2.1) coincide with (2.6) and consequently the corresponding solutions become identical.

3 Main results

Let $\mathcal{C} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ endowed with the usual supremum norm.

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In view of Lemma 2.3, we define an operator $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned}(\mathcal{F}x)(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s)\big)\,ds + \mu_1(t)\int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f\big(s,x(s)\big)\,ds \\ & + \mu_2(t)\bigg\{\gamma_2\int_0^\sigma \frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f\big(s,x(s)\big)\,ds - \beta_2\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\big(s,x(s)\big)\,ds \\ & - \alpha_2\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s)\big)\,ds\bigg\}. \qquad (3.1)\end{aligned}$$

Observe that the problem (1.1) has solutions if and only if the operator equation $\mathcal{F}x = x$ has fixed points.

In the sequel, we use the following notation:

$$\begin{aligned}\omega &= \max_{t\in[0,1]}\bigg\{\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{|\mu_1(t)|\eta^{2\alpha-1}}{\Gamma(2\alpha)} + \big|\mu_2(t)\big|\bigg(\frac{|\gamma_2|\sigma^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{|\beta_2|}{\Gamma(\alpha-p+1)} + \frac{|\alpha_2|}{\Gamma(\alpha+1)}\bigg)\bigg\} \\ &= \frac{1+|\alpha_2|\tilde{\mu}_2}{\Gamma(\alpha+1)} + \frac{\tilde{\mu}_1\eta^{2\alpha-1} + \tilde{\mu}_2|\gamma_2|\sigma^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{|\beta_2|\tilde{\mu}_2}{\Gamma(\alpha-p+1)}, \qquad (3.2)\end{aligned}$$

where $\tilde{\mu}_1 = |\gamma_1|(|\Delta_1| + |\Delta_2|)$, $\tilde{\mu}_2 = |\gamma_1\Delta_3| + |\Delta_4|$ with Δ_i $(i = 1,2,3,4)$ given by (2.3).

Our first result is based on the Leray-Schauder nonlinear alternative [15].

Lemma 3.1 (Nonlinear alternative for single valued maps) *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $F:\overline{U}\to C$ is a continuous, compact* (*that is, $F(\overline{U})$ is a relatively compact subset of C*) *map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$* (*the boundary of U in C*) *and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Theorem 3.2 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a jointly continuous function. Assume that*:

(A_1) *there exist a function $p \in C([0,1],\mathbb{R}^+)$ and a nondecreasing function $\psi:\mathbb{R}^+\to\mathbb{R}^+$ such that $|f(t,x)| \le p(t)\psi(\|x\|)$, $\forall(t,x)\in[0,1]\times\mathbb{R}$;*
(A_2) *there exists a constant $M > 0$ such that*

$$\frac{M}{\psi(M)\{\frac{1+|\alpha_2|\tilde{\mu}_2}{\Gamma(\alpha+1)} + \frac{\tilde{\mu}_1\eta^{2\alpha-1}+\tilde{\mu}_2|\gamma_2|\sigma^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{|\beta_2|\tilde{\mu}_2}{\Gamma(\alpha-p+1)}\}\|p\|} > 1.$$

Then the boundary value problem (1.1) *has at least one solution on* $[0,1]$.

Proof Consider the operator $F:\mathcal{C}\to\mathcal{C}$ defined by (3.1). We show that *F maps bounded sets into bounded sets in* $C([0,1],\mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}\|\mathcal{F}x\| \le \sup_{t\in[0,1]}\bigg\{ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(s,x(s)\big)\big|\,ds + \big|\mu_1(t)\big|\int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\big|f\big(s,x(s)\big)\big|\,ds \\ & + \big|\mu_2(t)\big|\bigg(|\gamma_2|\int_0^\sigma \frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\big|f\big(s,x(s)\big)\big|\,ds \\ & + |\beta_2|\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\big|f\big(s,x(s)\big)\big|\,ds\end{aligned}$$

$$
\begin{aligned}
&+ |\alpha_2| \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \big|f\big(s,x(s)\big)\big|\,ds\bigg)\bigg\} \\
&\le \psi(r)\bigg\{\frac{1+|\alpha_2|\tilde{\mu}_2}{\Gamma(\alpha+1)} + \frac{\tilde{\mu}_1\eta^{2\alpha-1}+\tilde{\mu}_2|\gamma_2|\sigma^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{|\beta_2|\tilde{\mu}_2}{\Gamma(\alpha-p+1)}\bigg\}\|p\|.
\end{aligned}
$$

Next, we show that F *maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. Let $t', t'' \in [0,1]$ with $t' < t''$ and $x \in B_r$, where B_r is a bounded set of $C([0,1],\mathbb{R})$. Then we obtain

$$
\begin{aligned}
&\big|(\mathcal{F}x)(t'') - (\mathcal{F}x)(t')\big| \\
&= \bigg|\frac{1}{\Gamma(q)}\int_0^{t''} \big(t''-s\big)^{q-1} f\big(s,x(s)\big)\,ds - \frac{1}{\Gamma(q)}\int_0^{t'} \big(t'-s\big)^{q-1} f\big(s,x(s)\big)\,ds \\
&\quad - \gamma_1\Delta_2\big(t''-t'\big)\int_0^{\eta} \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f\big(s,x(s)\big)\,ds \\
&\quad + \Delta_4\big(t''-t'\big)\bigg\{\gamma_2\int_0^{\sigma} \frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f\big(s,x(s)\big)\,ds - \beta_2\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\big(s,x(s)\big)\,ds \\
&\quad - \alpha_2\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s)\big)\,ds\bigg\}\bigg| \\
&\le \frac{1}{\Gamma(q)}\int_0^{t'} \big|\big(t''-s\big)^{q-1} - \big(t'-s\big)^{q-1}\big|\psi(r)p(s)\,ds \\
&\quad + \frac{1}{\Gamma(q)}\int_{t'}^{t''} \big|t''-s\big|^{q-1}\psi(r)p(s)\,ds \\
&\quad + \big|\gamma_1\Delta_2\big(t''-t'\big)\big|\int_0^{\eta} \frac{|\eta-s|^{2\alpha-2}}{\Gamma(2\alpha-1)}\psi(r)p(s)\,ds \\
&\quad + \big|\Delta_4\big(t''-t'\big)\big|\bigg\{|\gamma_2|\int_0^{\sigma} \frac{|\sigma-s|^{2\alpha-2}}{\Gamma(2\alpha-1)}\psi(r)p(s)\,ds - |\beta_2|\int_0^1 \frac{|1-s|^{\alpha-p-1}}{\Gamma(\alpha-p)}\psi(r)p(s)\,ds \\
&\quad - |\alpha_2|\int_0^1 \frac{|1-s|^{\alpha-1}}{\Gamma(\alpha)}\psi(r)p(s)\,ds\bigg\}.
\end{aligned}
$$

Obviously, the right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $t'' - t' \to 0$. As $\mathcal{F}$ satisfies the above assumptions, therefore, it follows by the Arzelá-Ascoli theorem that $\mathcal{F} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

Let x be a solution. Then for $t \in [0,1]$, using the computations in proving that $\mathcal{F}$ is bounded, we have

$$
\begin{aligned}
\big|x(t)\big| &= \big|\lambda(\mathcal{F}x)(t)\big| \\
&\le \psi\big(\|x\|\big)\bigg\{\frac{1+|\alpha_2|\tilde{\mu}_2}{\Gamma(\alpha+1)} + \frac{\tilde{\mu}_1\eta^{2\alpha-1}+\tilde{\mu}_2|\gamma_2|\sigma^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{|\beta_2|\tilde{\mu}_2}{\Gamma(\alpha-p+1)}\bigg\}\|p\|.
\end{aligned}
$$

Consequently, we have

$$
\frac{\|x\|}{\psi(\|x\|)\{\frac{1+|\alpha_2|\tilde{\mu}_2}{\Gamma(\alpha+1)} + \frac{\tilde{\mu}_1\eta^{2\alpha-1}+\tilde{\mu}_2|\gamma_2|\sigma^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{|\beta_2|\tilde{\mu}_2}{\Gamma(\alpha-p+1)}\}\|p\|} \le 1.
$$

In view of (A_2), there exists M such that $\|x\| \neq M$. Let us set

$$
U = \big\{x \in C\big([0,1],\mathbb{R}\big) : \|x\| < M+1\big\}.
$$

Note that the operator $\mathcal{F}:\overline{U}\to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x=\lambda\mathcal{F}(x)$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.1), we deduce that $\mathcal{F}$ has a fixed point $x\in\overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

In the special case when $p(t)=1$ and $\psi(|x|)=\kappa|x|+N$ (κ and N are suitable constants) in the statement of Theorem 3.2, we have the following corollary.

Corollary 3.3 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that there exist constants $0\le\kappa<1/\omega$, where ω is given by* (3.2) *and $N>0$ such that $|f(t,x)|\le\kappa|x|+N$ for all $t\in[0,1]$, $x\in C[0,1]$. Then the boundary value problem* (1.1) *has at least one solution.*

Next, we prove an existence and uniqueness result by means of Banach's contraction mapping principle.

Theorem 3.4 *Suppose that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and satisfies the following assumption*:

(A_3) $|f(t,x)-f(t,y)|\le L|x-y|$, $\forall t\in[0,1]$, $L>0$, $x,y\in\mathbb{R}$.

Then the boundary value problem (1.1) *has a unique solution provided*

$$\omega<1/L, \tag{3.3}$$

where ω is given by (3.2).

Proof With $r\ge M\omega/(1-L\omega)$, we define $B_r=\{x\in\mathcal{F}:\|x\|\le r\}$, where $M=\sup_{t\in[0,1]}|f(t,0)|<\infty$ and ω is given by (3.2). Then we show that $\mathcal{F}B_r\subset B_r$. For $x\in B_r$, we have

$$\begin{aligned}\|\mathcal{F}x\| &= \sup_{t\in[0,1]}\bigl|(\mathcal{F}x)(t)\bigr| \\ &\le \sup_{t\in[0,1]}\biggl\{\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds+\bigl|\mu_1(t)\bigr|\int_0^\eta\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\ &\quad+\bigl|\mu_2(t)\bigr|\biggl(|\gamma_2|\int_0^\sigma\frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds+|\beta_2|\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\ &\quad+|\alpha_2|\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds\biggr)\biggr\}.\end{aligned}$$

Using $|f(s,x(s))|\le|f(s,x(s))-f(s,0)|+|f(s,0)|\le L\|x\|+M\le Lr+M$, the above expression yields

$$\begin{aligned}\|\mathcal{F}x\| &\le (Lr+M)\sup_{t\in[0,1]}\biggl\{\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds+\bigl|\mu_1(t)\bigr|\int_0^\eta\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\,ds \\ &\quad+\bigl|\mu_2(t)\bigr|\biggl(|\gamma_2|\int_0^\sigma\frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\,ds+|\beta_2|\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\,ds \\ &\quad+|\alpha_2|\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds\biggr)\biggr\}\end{aligned}$$

$$\begin{aligned}&\leq (Lr+M)\left\{\frac{1+|\alpha_2|\tilde{\mu}_2}{\Gamma(\alpha+1)}+\frac{\tilde{\mu}_1\eta^{2\alpha-1}+\tilde{\mu}_2|\gamma_2|\sigma^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{|\beta_2|\tilde{\mu}_2}{\Gamma(\alpha-p+1)}\right\}\\&=(Lr+M)\omega\leq r,\end{aligned}$$

where we used (3.2). Now, for $x,y\in\mathcal{C}$ and for each $t\in[0,1]$, we obtain

$$\begin{aligned}&\big\|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\big\|\\&\leq\sup_{t\in[0,1]}\left\{\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(s,x(s)\big)-f\big(s,y(s)\big)\big|\,ds\right.\\&\quad+\big|\mu_1(t)\big|\int_0^\eta\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\big|f\big(s,x(s)\big)-f\big(s,y(s)\big)\big|\,ds\\&\quad+\big|\mu_2(t)\big|\left(|\gamma_2|\int_0^\sigma\frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\big|f\big(s,x(s)\big)-f\big(s,y(s)\big)\big|\,ds\right.\\&\quad+|\beta_2|\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\big|f\big(s,x(s)\big)-f\big(s,y(s)\big)\big|\,ds\\&\quad\left.\left.+|\alpha_2|\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(s,x(s)\big)-f\big(s,y(s)\big)\big|\,ds\right)\right\}\\&\leq L\left\{\frac{1+|\alpha_2|\tilde{\mu}_2}{\Gamma(\alpha+1)}+\frac{\tilde{\mu}_1\eta^{2\alpha-1}+\tilde{\mu}_2|\gamma_2|\sigma^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{|\beta_2|\tilde{\mu}_2}{\Gamma(\alpha-p+1)}\right\}\|x-y\|\\&=L\omega\|x-y\|.\end{aligned}$$

Note that ω depends only on the parameters involved in the problem. As $L\omega<1$, therefore, $\mathcal{F}$ is a contraction. Hence, by Banach's contraction mapping principle, the problem (1.1) has a unique solution on $[0,1]$. □

Now, we prove the existence of solutions of (1.1) by applying Krasnoselskii's fixed point theorem [16].

Theorem 3.5 (Krasnoselskii's fixed point theorem) *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that* (i) $Ax+By\in M$ *whenever $x,y\in M$*; (ii) *A is compact and continuous*; (iii) *B is a contraction mapping. Then there exists $z\in M$ such that $z=Az+Bz$.*

Theorem 3.6 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a jointly continuous function satisfying the assumption* (A_3). *In addition we assume that*:

(A_4) $|f(t,x)|\leq\mu(t)$, $\forall(t,x)\in[0,1]\times\mathbb{R}$, *and* $\mu\in C([0,1],\mathbb{R}^+)$.

Then the problem (1.1) *has at least one solution on* $[0,1]$ *if*

$$\frac{|\alpha_2|\tilde{\mu}_2}{\Gamma(\alpha+1)}+\frac{\tilde{\mu}_1\eta^{2\alpha-1}+\tilde{\mu}_2|\gamma_2|\sigma^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{|\beta_2|\tilde{\mu}_2}{\Gamma(\alpha-p+1)}<1. \tag{3.4}$$

Proof Letting $\sup_{t\in[0,1]}|\mu(t)|=\|\mu\|$, we choose a real number $\bar{r}$ satisfying the inequality

$$\bar{r}\geq\|\mu\|\left[\frac{1+|\alpha_2|\tilde{\mu}_2}{\Gamma(\alpha+1)}+\frac{\tilde{\mu}_1\eta^{2\alpha-1}+\tilde{\mu}_2|\gamma_2|\sigma^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{|\beta_2|\tilde{\mu}_2}{\Gamma(\alpha-p+1)}\right],$$

and consider $B_{\bar{r}} = \{x \in \mathcal{C} : \|x\| \leq \bar{r}\}$. We define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\bar{r}}$ as

$$
\begin{aligned}
(\mathcal{P}x)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,x(s)\bigr)\,ds, \\
(\mathcal{Q}x)(t) &= \mu_1(t)\int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f\bigl(s,x(s)\bigr)\,ds + \mu_2(t)\biggl(\gamma_2\int_0^\sigma \frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f\bigl(s,x(s)\bigr)\,ds \\
&\quad - \beta_2\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\bigl(s,x(s)\bigr)\,ds - \alpha_2\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,x(s)\bigr)\,ds\biggr).
\end{aligned}
$$

For $x, y \in B_{\bar{r}}$, we find that

$$
\begin{aligned}
&\|\mathcal{P}x + \mathcal{Q}y\| \\
&\quad \leq \|\mu\|\biggl[\frac{1+|\alpha_2|\tilde{\mu}_2}{\Gamma(\alpha+1)} + \frac{\tilde{\mu}_1\eta^{2\alpha-1} + \tilde{\mu}_2|\gamma_2|\sigma^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{|\beta_2|\tilde{\mu}_2}{\Gamma(\alpha-p+1)}\biggr] \leq \bar{r}.
\end{aligned}
$$

Thus, $\mathcal{P}x + \mathcal{Q}y \in B_{\bar{r}}$. It follows from the assumption (A_3) together with (3.4) that $\mathcal{Q}$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\bar{r}}$ as

$$
\|\mathcal{P}x\| \leq \frac{\|\mu\|}{\Gamma(q+1)}.
$$

Now, we prove the compactness of the operator $\mathcal{P}$.

In view of (A_1), we define $\sup_{(t,x)\in[0,1]\times B_{\bar{r}}} |f(t,x)| = \bar{f}$, and consequently, for $t_1, t_2 \in [0,1]$, we have

$$
\begin{aligned}
\bigl|(\mathcal{P}x)(t_1) - (\mathcal{P}x)(t_2)\bigr| &= \biggl|\frac{1}{\Gamma(q)}\int_0^{t_1}\bigl[(t_2-s)^{q-1} - (t_1-s)^{q-1}\bigr]f\bigl(s,x(s)\bigr)\,ds \\
&\quad + \int_{t_1}^{t_2}(t_2-s)^{q-1}f\bigl(s,x(s)\bigr)\,ds\biggr| \\
&\leq \frac{\bar{f}}{\Gamma(q+1)}\bigl[2\bigl(|t_2-t_1|\bigr)^q + \bigl|t_1^q - t_2^q\bigr|\bigr],
\end{aligned}
$$

which is independent of x. Thus, $\mathcal{P}$ is equicontinuous. Hence, by the Arzelá-Ascoli theorem, $\mathcal{P}$ is compact on $B_{\bar{r}}$. Thus, all the assumptions of Theorem 3.5 are satisfied. So, the conclusion of Theorem 3.5 implies that the boundary value problem (1.1) has at least one solution on $[0,1]$. □

4 Examples

Example 4.1 Consider the following boundary value problem:

$$
\begin{cases}
{}^cD^{3/2}x(t) = \frac{1}{(12\pi)}\sin(2\pi x) + \frac{|x|}{1+|x|} + \frac{1}{30}|x| + t^2 + 2, \quad t \in [0,1], \\
x(0) + 3({}^cD^{1/3}x(0)) = \int_0^{1/4} \frac{(1/4-s)^{-1/2}}{\Gamma(1/2)}x(s)\,ds, \\
x(1) + 2({}^cD^{1/3}x(1)) = 3\int_0^{3/4} \frac{(3/4-s)^{-1/2}}{\Gamma(1/2)}x(s)\,ds.
\end{cases}
\tag{4.1}
$$

Here, $\alpha = 3/2$, $p = 1/3$, $\alpha_1 = 1$, $\beta_1 = 3$, $\gamma_1 = 1$, $\eta = 1/4$, $\alpha_2 = 1$, $\beta_2 = 2$, $\gamma_2 = 3$, $\sigma = 3/4$, and

$$
\omega = \frac{1+|\alpha_2|\tilde{\mu}_2}{\Gamma(\alpha+1)} + \frac{\tilde{\mu}_1\eta^{2\alpha-1} + \tilde{\mu}_2|\gamma_2|\sigma^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{|\beta_2|\tilde{\mu}_2}{\Gamma(\alpha-p+1)} \simeq 3.578386.
$$

Clearly,

$$\left|f(t,x)\right| = \left|\frac{1}{(12\pi)}\sin(2\pi x) + \frac{|x|}{1+|x|} + \frac{1}{30}|x| + t^2 + 2\right| \le \frac{1}{5}|x| + 4.$$

Clearly, $N = 4$ and

$$\kappa = \frac{1}{5} < \frac{1}{\omega} = \frac{1}{3.578386}.$$

Thus, all the conditions of Corollary 3.3 are satisfied and consequently the problem (4.1) has at least one solution.

Example 4.2 Consider the following fractional boundary value problem:

$$\begin{cases} {}^c\mathcal{D}^{3/2}x(t) = \frac{L}{2}(x + \tan^{-1}x) + \frac{\sqrt{t(t+3)}}{t^4+2}, \quad t \in [0,1], \\ x(0) + 3({}^c\mathcal{D}^{1/3}x(0)) = \int_0^{1/4} \frac{(1/4-s)^{-1/2}}{\Gamma(1/2)} x(s)\,ds, \\ x(1) + 2({}^c\mathcal{D}^{1/3}x(1)) = 3\int_0^{3/4} \frac{(3/4-s)^{-1/2}}{\Gamma(1/2)} x(s)\,ds, \end{cases} \tag{4.2}$$

where $\alpha, p, \alpha_i, \beta_i, \gamma_i, (i = 1,2)\ \eta, \sigma$ are the same as given in (4.1) and $f(t,x) = \frac{L}{2}(x + \tan^{-1}x) + \frac{\sqrt{t(t+3)}}{t^4+2}$. Clearly, $|f(t,x) - f(t,y)| \le L|x - y|$ and thus, for $L < 1/\omega = 1/3.578386$, all the conditions of Theorem 3.4 are satisfied. Hence, the boundary value problem (4.2) has a unique solution on $[0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, BA and AA contributed to each part of this work equally and read and approved the final version of the manuscript.

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