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Nodal solutions of second-order two-point boundary value problems

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Abstract

We shall study the existence and multiplicity of nodal solutions of the nonlinear second-order two-point boundary value problems,

$$u'' + f(t, u) = 0, \quad t \in (0, 1), \quad u(0) = u(1) = 0.$$

The proof of our main results is based upon bifurcation techniques.

Mathematics Subject Classifications: 34B07; 34C10; 34C23.

Keywords: nodal solutions, bifurcation

1 Introduction

In [1], Ma and Thompson were considered with determining interval of μ, in which there exist nodal solutions for the boundary value problem (BVP)

$$u''(t) + \mu w(t) f(u) = 0, \quad t \in (0, 1), \quad u(0) = u(1) = 0 \tag{1.1}$$

under the assumptions:

(C1) $w(\cdot) \in C([0, 1], [0, \infty))$ and does not vanish identically on any subinterval of [0, 1];

(C2) $f \in C(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$;

(C3) there exist $f_0, f_\infty \in (0, \infty)$ such that

$$f_0 = \lim_{|s| \to 0} \frac{f(s)}{s}, \quad f_\infty = \lim_{|s| \to \infty} \frac{f(s)}{s}.$$

It is well known that under (C1) assumption, the eigenvalue problem

$$\varphi''(t) + \mu w(t) \varphi(t) = 0, \quad t \in (0, 1), \quad \varphi(0) = \varphi(1) = 0 \tag{1.2}$$

has a countable number of simple eigenvalues μ_k, $k = 1, 2, \ldots$, which satisfy

$$0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots, \quad \text{and} \lim_{k \to \infty} \mu_k = \infty,$$

and let μ_k be the kth eigenvalue of (1.2) and ϕ_k be an eigenfunction corresponding to μ_k, then ϕ_k has exactly $k - 1$ simple zeros in (0,1) (see, e.g., [2]).

Using Rabinowitz bifurcation theorem, they established the following interesting results:

Theorem A (Ma and Thompson [[1], Theorem 1.1]). *Let (C1)-(C3) hold. Assume that for some $k \in \mathbb{N}$, either $\frac{\mu_k}{f_\infty} < \mu < \frac{\mu_k}{f_0}$ or $\frac{\mu_k}{f_0} < \mu < \frac{u_k}{f_\infty}$. Then BVP (1.1) has two* *solutions* u_k^+ *and* u_k^- *such that* u_k^+ *has exactly* $k - 1$ *zeros in (0, 1) and is positive near 0, and* u_k^- *has exactly* $k - 1$ *zeros in (0,1) and is negative near 0.*

In [3], Ma and Thompson studied the existence and multiplicity of nodal solutions for BVP

$$u''(t) + w(t)f(u) = 0, \quad t \in (0, 1), \quad u(0) = u(1) = 0. \tag{1.3}$$

They gave conditions on the ratio $\frac{f(s)}{s}$ at infinity and zero that guarantee the existence of solutions with prescribed nodal properties.

Using Rabinowitz bifurcation theorem also, they established the following two main results:

Theorem B (Ma and Thompson [[1], Theorem 2]). *Let (C1)-(C3) hold. Assume that either (i) or (ii) holds for some* $k \in \mathbb{N}$ *and* $j \in \{0\} \cup \mathbb{N}$;

(i) $f_0 < \mu_k < \ldots < \mu_{k+j} < f_\infty$;

(ii) $f_\infty < \mu_k < \ldots < \mu_{k+j} < f_0$,

where μ_k *denotes the* kth *eigenvalue of (1.2). Then BVP (1.3) has* $2(j + 1)$ *solutions* $u_{k+i}^+, u_{k+i}^-, i = 0, \ldots, j$, *such that* u_{k+i}^+ *has exactly* $k + i - 1$ *zeros in (0, 1) and are positive near 0, and* u_{k+i}^- *has exactly* $k + i - 1$ *zeros in (0,1) and are negative near 0.*

Theorem C (Ma and Thompson [[1], Theorem 3]). *Let (C1)-(C3) hold. Assume that there exists an integer* $k \in \mathbb{N}$ *such that*

$$\mu_{k-1} < \frac{f(s)}{s} < \mu_k,$$

where μ_k *denotes the* kth *eigenvalue of (1.2). Then BVP (1.3) has no nontrivial solution.*

From above literature, we can see that the existence and multiplicity results are largely based on the assumption that t and u are separated in nonlinearity term. It is interesting to know what will happen if t and u are not separated in nonlinearity term? We shall give a confirm answer for this question.

In this article, we consider the existence and multiplicity of nodal solutions for the nonlinear BVP

$$u'' + f(t, u) = 0, \quad t \in (0, 1), \quad u(0) = u(1) = 0 \tag{1.4}$$

under the following assumptions:

(H_1) $\lambda_k \le a(t) \equiv \lim_{|s| \to +\infty} \frac{f(t,s)}{s}$ *uniformly on* $[0, 1]$, *and the inequality is strict on some subset of positive measure in* (0,1), where λ_k denotes the kth eigenvalue of

$$u''(t) + \lambda_u(t) = 0, \quad t \in (0, 1), \quad u(0) = u(1) = 0; \tag{1.5}$$

(H_2) $0 \le \lim_{|s| \to 0} \frac{f(t,s)}{s} \equiv c(t) \le \lambda_k$ *uniformly on* $[0, 1]$, *and all the inequalities are strict on some subset of positive measure in* $(0, 1)$, *where* λ_k *denotes the* kth *eigenvalue of (1.5);*

(H_3) $f(t, s)s > 0$ *for* $t \in (0, 1)$ *and* $s \ne 0$.

Remark 1.1. From (H_1)-(H_3), we can see that there exist a positive constant ϱ and a subinterval $[\alpha, \beta]$ of [0, 1] such that $\alpha < \beta$ and $\frac{f(r,s)}{s} \geq \varrho$ for all $r \in [\alpha, \beta]$ and $s \neq 0$.

In the celebrated study [4], Rabinowitz established Rabinowitz's global bifurcation theory [[4], Theorems 1.27 and 1.40]. However, as pointed out by Dancer [5,6] and López-Gómez [7], the proofs of these theorems contain gaps, the original statement of Theorem 1.40 of [4] is not correct, the original statement of Theorem 1.27 of [4] is stronger than what one can actually prove so far. Although there exist some gaps in the proofs of Rabinowitz's Theorems 1.27, 1.40, and 1.27 has been used several times in the literature to analyze the global behavior of the component of nodal solutions emanating from $u = 0$ in wide classes of boundary value problems for equations and systems [1,2,8,9]. Fortunately, López-Gómez gave a corrected version of unilateral bifurcation theorem in [7].

By applying the bifurcation theorem of López-Gómez [[7], Theorem 6.4.3], we shall establish the following:

Theorem 1.1. *Suppose that $f(t, u)$ satisfies (H_1), (H_2), and (H_3), then problem (1.4) possesses two solutions u_k^+ and u_k^-, such that u_k^+ has exactly $k - 1$ zeros in (0, 1) and is positive near 0, and u_k^- has exactly $k - 1$ zeros in (0,1) and is negative near 0.*

Similarly, we also have the following:

Theorem 1.2. *Suppose that $f(t, u)$ satisfies (H_3) and*

$(H_1')\lambda_k \geq a(x) \equiv \lim_{|s|\to+\infty} \frac{f(t,s)}{s} \geq 0$ *uniformly on* [0, 1], *and all the inequalities are strict on some subset of positive measure in* (0, 1), *where* λ_k *denotes the* kth *eigenvalue of (1.5);*

$(H_2') \lim_{|s|\to 0} \frac{f(t,s)}{s} \equiv c(x) \geq \lambda_k$ *uniformly on* [0, 1], *and the inequality is strict on some subset of positive measure in* (0, 1), *where* λ_k *denotes the* kth *eigenvalue of (1.5), then problem (1.4) possesses two solutions u_k^+ and u_k^-, such that u_k^+ has exactly $k - 1$ zeros in (0,1) and is positive near 0, and u_k^- has exactly $k - 1$ zeros in (0,1) and is negative near 0.*

Remark 1.2. We would like to point out that the assumptions (H_1) and (H_2) are weaker than the corresponding conditions of Theorem A. In fact, if we let $f(t, s) \equiv \mu w(t) f(s)$, then we can get $\lim_{|s|\to+\infty} \frac{f(t,s)}{s} \equiv \mu w(t) f_\infty := a(t)$ and $\lim_{|s|\to 0} \frac{f(t,s)}{s} \equiv \mu w(t) f_0 := c(t)$. By the strict decreasing of $\mu_k(f)$ with respect to weight function f (see [10]), where $\mu_k(f)$ denotes the kth eigenvalue of (1.2) corresponding to weight function f, we can show that our condition $c(t) \leq \lambda_k \leq a(t)$ is equivalent to the condition $\frac{\mu_k}{f_\infty} < \mu < \frac{\mu_k}{f_0}$. Similarly, our condition $c(t) \geq \lambda_k \geq a(t)$ is equivalent to the condition $\frac{\mu_k}{f_0} < \mu < \frac{u_k}{f_\infty}$. Therefore, Theorem A is the corollary of Theorems 1.1 and 1.2.

Using the similar proof with the proof Theorems 1.1 and 1.2, we can obtain the more general results as follows.

Theorem 1.3. *Suppose that (H_3) holds, and either (i) or (ii) holds for some $k \in \mathbb{N}$ and $j \in \{0\} \cup \mathbb{N}$:*

(i) $0 \le c(t) \equiv \lim_{|s|\to 0} \frac{f(t,s)}{s} \le \lambda_k < \cdots < \lambda_{k+j} \le a(t) \equiv \lim_{|s|\to +\infty} \frac{f(t,s)}{s}$ *uniformly on* [0, 1], *and the inequalities are strict on some subset of positive measure in* (0,1), *where* λ_k *denotes the* kth *eigenvalue of (1.5);*

(ii) $0 \le a(t) \equiv \lim_{|s|\to +\infty} \frac{f(t,s)}{s} \le \lambda_k < \cdots < \lambda_{k+j} \le c(t) \equiv \lim_{|s|\to +\infty} \frac{f(t,s)}{s}$ *uniformly on* [0, 1], *and the inequality is strict on some subset of positive measure in* (0, 1), *where* λ_k *denotes the* kth *eigenvalue of (1.5).*

Then BVP (1.4) has $2(j + 1)$ *solutions* $u^+_{k+i}, u^-_{k+i}, i = 0, \ldots, j$, *such that* u^+_{k+i} *has exactly* $k + i - 1$ *zeros in (0,1) and are positive near 0, and* u^-_{k+i} *has exactly* $k + i - 1$ *zeros in (0,1) and are negative near 0.*

Using Sturm Comparison Theorem, we also can get a non-existence result when f satisfies a non-resonance condition.

Theorem 1.4. *Let* (H_3) *hold. Assume that there exists an integer* $k \in \mathbb{N}$ *such that*

$$\lambda_{k-1} < \frac{f(t,u)}{u} < \lambda_k \tag{1.6}$$

for any $t \in [0, 1]$, *where* λ_k *denotes the* kth *eigenvalue of (1.5). Then BVP (1.4) has no nontrivial solution.*

Remark 1.3. Similarly to Remark 1.2, we note that the assumptions (i) and (ii) are weaker than the corresponding conditions of Theorem B. In fact, if we let $f(t, s) \equiv w(t) f(s)$, then we can get $\lim_{|s|\to +\infty} \frac{f(t,s)}{s} \equiv w(t)f_\infty := a(t)$ and $\lim_{|s|\to 0} \frac{f(t,s)}{s} \equiv w(t)f_0 := c(t)$. By the strict decreasing of $\mu_k(f)$ with respect to weight function f (see [11]), where $\mu_k(f)$ denotes the kth eigenvalue of (1.2) corresponding to weight function f, we can show that our condition $c(t) \le \lambda_k < \ldots < \lambda_{k+j} \le a(t)$ is equivalent to the condition $f_0 < \mu_k < \cdots < \mu_{k+j} < f_\infty$. Similarly, our condition $a(t) \le \lambda_k < \cdots < \lambda_{k+j} \le c(t)$ is equivalent to the condition $f_\infty < \mu_k < \ldots < \mu_{k+j} < f_0$. Therefore, Theorem B is the corollary of Theorem 1.3. Similar, we get Theorem C is also the corollary of Theorem 1.4.

2 Preliminary results

To show the nodal solutions of the BVP (1.4), we need only consider an operator equation of the following form

$$u = \lambda A u. \tag{2.1}$$

Equations of the form (2.1) are usually called nonlinear eigenvalue problems. López-Gómez [7] studied a nonlinear eigenvalue problem of the form

$$u = G(r, u), \tag{2.2}$$

where $r \in \mathbb{R}$ is a parameter, $u \in X$, X is a Banach space, θ is the zero element of X, and G: $\mathbb{X} = \mathbb{R} \times X \to X$ is completely continuous. In addition, $G(r, u) = rTu + H(r, u)$, where $H(r, u) = o(||u||)$ as $||u|| \to 0$ uniformly on bounded r interval, and T is a linear completely continuous operator on X. A solution of (2.2) is a pair $(r, u) \in \mathbb{X}$, which satisfies the equation (2.2). The closure of the set nontrivial solutions of (2.2) is denoted by $\mathbb{C}$, let $\Sigma(T)$ denote the set of eigenvalues of linear operator T. López-Gómez [7] established the following results:

Lemma 2.1 [[7], Theorem 6.4.3]. *Assume* $\Sigma(T)$ *is discrete. Let* $\lambda_0 \in \Sigma(T)$ *such that* ind$(0, \lambda_0 T)$ *changes sign as* λ *crosses* λ_0, *then each of the components* $\mathbb{C}^{\nu}_{\lambda_0}, \nu \in \{+, -\}$ *satisfies* $(\lambda_0, \theta) \in \mathbb{C}^{\nu}_{\lambda_0}$, *and either*

(i) meets infinity in $\mathbb{X}$,

(ii) meets (τ, θ), *where* $\tau \neq \lambda_0 \in \Sigma(T)$ *or*

(iii) $\mathbb{C}^{\nu}_{\lambda_0}, \nu \in \{+, -\}$ *contains a point*

$$(\iota, y) \in \mathbb{R} \times (V \backslash \{0\}),$$

where V *is the complement* of $\text{span}\{\varphi_{\lambda_0}\}, \varphi_{\lambda_0}$ *denotes the eigenfunction corresponding to eigenvalue* λ_0.

Lemma 2.2 [[7], Theorem 6.5.1]. *Under the assumptions:*

(A) X *is an order Banach space, whose positive cone, denoted by* P, *is normal and has a nonempty interior;*

(B) The family $\Upsilon(r)$ *has the special form*

$$\Upsilon(r) = I_X - rT,$$

where T *is a compact strongly positive operator, i.e.,* $T(P\backslash\{0\}) \subset int\ P$;

(C) The solutions of $u = rTu + H(r, u)$ *satisfy the strong maximum principle.*

Then the following assertions are true:

(1) Spr (T) is a simple eigenvalue of T, having a positive eigenfunction denoted by $\psi_0 > 0$, *i.e.,* $\psi_0 \in$ int P, *and there is no other eigenvalue of T with a positive eigenfunction;*

(2) For every $y \in$ int P, *the equation*

$$u - rTu = y$$

has exactly one positive solution if $r < \dfrac{1}{\text{Spr}(T)}$, *whereas it does not admit a positive solution if* $r \geq \dfrac{1}{\text{Spr}(T)}$.

Lemma 2.3 [[10], Theorem 2.5]. Assume $T : X \to X$ *is a completely continuous linear operator, and 1 is not an eigenvalue of T, then*

$$ind(I - T, \theta) = (-1)^{\beta},$$

where β *is the sum of the algebraic multiplicities of the eigenvalues of T large than 1, and* $\beta = 0$ *if T has no eigenvalue of this kind.*

Let $Y = C[0, 1]$ with the norm $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$. Let

$$E = \{u \in C^1[0, 1] | u(0) = u(1) = 0\}$$

with the norm

$$\|u\|_E = \max_{t\in[0,1]} |u| + \max_{t\in[0,1]} |u'|.$$

Define L: $D(L) \to Y$ by setting

$$Lu := -u''(t), \quad t \in [0, 1], \quad u \in D(L),$$

where

$$D(L) = \{u \in C^2[0, 1] | u(0) = u(1) = 0\}.$$

Then L^{-1}: $Y \to E$ is compact. Let $\mathbb{E} = \mathbb{R} \times E$ under the product topology. For any C^1 function u, if $u(x_0) = 0$, then x_0 is a simple zero of u, if $u'(x_0) \neq 0$. For any integer $k \in \mathbb{N}$ and $\nu \in \{+, -\}$, define $S_k^\nu \subset C^1[0, 1]$ consisting of functions $u \in C^1[0, 1]$ satisfying the following conditions:

(i) $u(0) = 0$, $\nu u'(0) > 0$;

(ii) u has only simple zeros in [0, 1] and exactly $n - 1$ zeros in (0,1).

Then sets S_k^ν are disjoint and open in E. Finally, let $\phi_k^\nu = \mathbb{R} \times S_k^\nu$.

Furthermore, let $\zeta \in C[0, 1] \times \mathbb{R})$ be such that

$$f(t, u) = c(t)u + \varsigma(t, u)$$

with

$$\lim_{|u|\to 0} \frac{\varsigma(t, u)}{u} = 0 \text{ and } \lim_{|u|\to\infty} \frac{\varsigma(t, u)}{u} = a(t) - c(t) \text{ uniformly on } [0, 1]. \tag{2.3}$$

Let

$$\bar{\varsigma}(t, u) = \max_{0\le|s|\le u} |g(t, u)| \quad \text{for} \quad t \in [0, 1],$$

then $\bar{\varsigma}$ is nondecreasing with respect to u and

$$\lim_{u\to 0^+} \frac{\bar{\varsigma}(t, u)}{|u|} = 0.$$

If $u \in$ E, it follows from (2.3) that

$$\frac{\varsigma(t, u)}{\|u\|_E} \le \frac{\bar{\varsigma}(t, |u|)}{\|u\|_E} \le \frac{\bar{\varsigma}(t, \|u\|_\infty)}{\|u\|_E} \le \frac{\bar{\varsigma}(t, \|u\|_E)}{\|u\|_E} \to 0, \quad \text{as } \|u\|_E \to 0$$

uniformly for $t \in [0, 1]$.

Let us study

$$Lu - \mu c(t)u = \mu\varsigma(t, u) \tag{2.4}$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

Equation (2.4) can be converted to the equivalent equation

$$u(t) = \mu L^{-1}[c(t)u(t)] + \mu L^{-1}[\varsigma(t, u(t))].$$

Further we note that $||L^{-1}[\zeta(t, u(t))]||_E = o(||u||_E)$ for u near 0 in E.

Lemma 2.4. *For each $k \in \mathbb{N}$ and $\nu \in \{+. -\}$, there exists a continuum $C_k^\nu \subset \phi_k^\nu$ of solutions of (2.4) with the properties:*

(i) $(\lambda_k, \theta) \in C_k^\nu$;

(ii) $C_k^\nu \setminus \{(\lambda_k, \theta)\} \subset \phi_k^\nu$;

(iii) C_k^ν *is unbounded in* $\mathbb{E}$, *where* λ_k *denotes the* kth *eigenvalue of (1.5).*

Proof. It is easy to see that the problem (2.4) is of the form considered in [7], and satisfies the general hypotheses imposed in that article.

Combining Lemma 2.1 with Lemma 2.3, we know that there exists a continuum $C_k^\nu \subset \mathbb{E}$ of solutions of (2.4) such that:

(a) C_k^ν is unbounded and $(\lambda_k, \theta) \in C_k^\nu, C_k^\nu \backslash \{(\lambda_k, \theta)\} \subset \phi_k^\nu$;

(b) or $(\lambda_j, \theta) \in C_k^\nu$, where $j \in \mathbb{N}$, λ_j is another eigenvalue of (1.5) and different from λ_k;

(c) or C_k^ν contains a point

$$(\iota, y) \in \mathbb{R} \times (V \backslash \{0\}),$$

where V is the complement of span$\{\phi_k\}$, ϕ_k denotes the eigenfunction corresponding to eigenvalue λ_k.

We finally prove that the first choice of the (a) is the only possibility.

In fact, all functions belong to the continuum sets C_k^ν have exactly $k - 1$ simple zeros, this implies that it is impossible to exist $(\lambda_j, \theta) \in C_k^\nu, j \in \mathbb{N}$.

Next, we shall prove (c) is impossible, suppose (c) occurs, then C_k^ν is bounded and without loss of generality, suppose there exists a point $(\iota, y) \in \mathbb{R} \times (V \backslash \{\theta\}) \cap C_k^+$. Moreover, it follows from Lemma 2.1 that

$$C_k^+ \cap \{(\lambda, \theta) : \lambda \in \mathbb{R}\} = \{(\lambda_k, \theta)\}.$$

Note that as the complement V of span$\{\phi_k\}$ in E, we can take

$$V := R[I_E - \lambda_k L].$$

Thus, for this choice of V, the component C_k^+ cannot contain a point

$$(\iota, y) \in \mathbb{R} \times (V \backslash \{\theta\}) \cap C_k^+.$$

Indeed, if

$$(\iota, y) \in \mathbb{R} \times (V \backslash \{\theta\}) \cap C_k^+.$$

then $y > 0$ in $(0, a_0)$, where a_0 denotes the first zero point of y, and there exists $u \in E$ for which

$$u - \lambda_k L u = y > 0, \quad \text{in} \quad (0, a_0).$$

Thus, for each sufficiently large $\alpha > 0$, we have that $u + \alpha\phi_k >> 0$ in $(0, a_0)$ and

$$u + \alpha\varphi_k - \lambda_k L(u + \alpha\varphi_k) = y > 0 \quad \text{in} \quad (0, a_0).$$

Define

$$P = \{u \in E | u(t) \geq 0, \quad t \in [0, a_0]\}.$$

Hence, according to Lemma 2.2

$$\text{Spr}(\lambda_k L) < 1,$$

which is impossible since $\text{Spr}(L) = \dfrac{1}{\lambda_k f_0}$.

Lemma 2.5. *If $(\mu, u) \in \mathbb{E}$ is a non-trivial solution of (2.4), then $u \in S_k^\nu$ for ν and some $k \in \mathbb{N}$.*

Proof. Taking into account Lemma 2.4, we only need to prove that $C_k^\nu \subset \Phi_k^\nu \cup \{(\lambda_k, \theta)\}$.

Suppose $C_k^\nu \not\subset \Phi_k^\nu \cup \{(\lambda_k, \theta)\}$. Then there exists $(\mu^*, u) \in C_k^\nu \cap (\mathbb{R} \times \partial S_k^\nu)$ such that $(\mu^*, u) \neq (\lambda_k, \theta), u \notin S_k^\nu$, and $(\mu_j, u_j) \to (\mu^*, u)$ with $(\mu_j, u_j) \in C_k^\nu \cap (\mathbb{R} \times S_k^\nu)$. Since $u \in \partial S_k^\nu$, so $u \equiv 0$. Let $c_j := \dfrac{u_j}{\|u_j\|_E}$, then c_j should be a solution of problem,

$$c_j = \mu_j L^{-1} \left[c(t)c_j(t) + \frac{\varsigma(t, u_j(t))}{\|u_j\|_E} \right]. \tag{2.5}$$

By (2.3), (2.5) and the compactness of L^{-1}, we obtain that for some convenient subsequence $c_j \to c_0 \neq 0$ as $j \to +\infty$. Now c_0 verifies the equation

$$-c_0''(t) = \mu^* c(t) c_o(t), \quad t \in (0, 1)$$

and $||c_0||_E = 1$. Hence $\mu^* = \lambda_i$, for some $i \neq k$, $i \in \mathbb{N}$. Therefore, $(\mu_j, u_j) \to (\lambda_i, \theta)$ with $(\mu_j, u_j) \in C_k \cap (\mathbb{R} \times S_k^\nu)$. This contradicts to Lemma 2.3.

3 Proof of main results

Proof of Theorems 1.1 and 1.2. We only prove Theorem 1.1 since the proof of Theorem 1.2 is similar. It is clear that any solution of (2.4) of the form (1, u) yields a solution u of (1.4). We shall show C_k^ν crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

By the strict decreasing of $\mu_k(c(t))$ with respect to $c(t)$ (see [11]), where $\mu_k(c(t))$ is the kth eigenvalue of (1.2) corresponding to the weight function $c(t)$, we have $\mu_k(c(t)) > \mu_k(\lambda_k) = 1$.

Let $(\mu_j, u_j) \in \mathcal{C}_k^\nu$ with $u_j \not\equiv 0$ satisfies

$$\mu_j + \|u_j\|_E \to +\infty.$$

We note that $\mu_j > 0$ for all $j \in \mathbb{N}$, since (0,0) is the only solution of (2.4) for $\mu = 0$ and $\mathcal{C}_k^\nu \cap (\{0\} \times E) = \emptyset$.

Step 1: We show that if there exists a constant $M > 0$, such that

$$\mu_j \subset (0, M]$$

for $j \in \mathbb{N}$ large enough, then C_k^ν crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

In this case it follows that

$$\|u_j\|_E \to \infty.$$

Let $\xi \in C([0, 1] \times \mathbb{R})$ be such that

$$f(t, u) = a(t)u + \xi(t, u)$$

with

$$\lim_{|u| \to +\infty} \frac{\xi(t, u)}{u} = 0 \quad \text{and} \quad \lim_{|u| \to 0} \frac{\xi(t, u)}{u} = c(t) - a(t), \text{ uniformly on } [0, 1]. \tag{3.1}$$

We divide the equation

$$Lu_j - \mu_j a(t)u_j = \mu_j \xi(t, u_j), \tag{3.2}$$

set $\bar{u}_j = \frac{u_j}{\|\bar{u}_j\|_E}$. Since $\bar{u}_j$ is bounded in C^2 [0, 1], after taking a subsequence if necessary, we have that $\bar{u}_j \to \bar{u}$ for some $\bar{u} \in E$ with $||u||_E = 1$. By (3.1), using the similar proof of (2.3), we have that

$$\lim_{j\to+\infty} \frac{\xi(t, u_j(t))}{\|u_j\|_E} = 0 \quad \text{in } Y.$$

By the compactness of L we obtain

$$-\bar{u}'' - \bar{\mu}a(t)\bar{u} = 0,$$

where $\bar{\mu} = \lim_{j\to+\infty} \mu_j$, again choosing a subsequence and relabeling if necessary.

It is clear that $\bar{u} \in \overline{\mathcal{C}_k^\nu} \subseteq \mathcal{C}_k^\nu$ since $\mathcal{C}_k^\nu$ is closed in $\mathbb{R} \times E$. Therefore, $\bar{\mu}(a(t))$ is the kth eigenvalue of

$$u''(t) + \mu a(t)u(t) = 0,\ t \in (0, 1), \quad u(0) = u(1) = 0.$$

By the strict decreasing of $\bar{\mu}(a(t))$ with respect to $a(t)$ (see [11]), where $\bar{\mu}(a(t))$ is the kth eigenvalue of (1.2) corresponding to the weight function $a(t)$, we have $\bar{\mu}(a(t)) < \bar{\mu}(\lambda_k) = 1$. Therefore, $\mathcal{C}_k^\nu$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

Step 2: We show that there exists a constant M such that $\mu_j \in (0, M]$ for $j \in \mathbb{N}$ large enough.

On the contrary, we suppose that

$$\lim_{j\to+\infty} \mu_j = +\infty.$$

On the other hand, we note that

$$-u_j'' = \mu_j \frac{f(t, u_j)}{u_j} u_j.$$

In view of Remark 1.1, we have $\mu_j \frac{f(t, u_j)}{u_j} > \lambda_k$ on $[\alpha, \beta]$ and for j large enough and all $t \in [0, 1]$. By Lemma 3.2 of [12], we get u_j must change its sign more than k times on $[\alpha, \beta]$ for j large enough, which contradicts the act that $u_j \in S_k^\mu$.

Therefore,

$$\mu_j \le M$$

for some constant number $M > 0$ and $j \in \mathbb{N}$ sufficiently large.

Proof of Theorem 1.3. Repeating the arguments used in the proof of Theorems 1.1 and 1.2, we see that for $\nu \in \{+, -\}$ and each $i \in \{k, k + 1,..., k + j\}$

$$C_i^\nu \cap (\{1\} \times E) \neq \emptyset.$$

The results follows.

Proof of Theorem 1.4. Assume to the contrary that BVP (1.4) has a solution $u \in E$, we see that u satisfies

$$u''(t) + b(t)u(t) = 0, \quad t \in (0, 1),$$

where $b(t) = \dfrac{f(t, u)}{u}$.

Note that $c(t) \equiv \lim_{|s| \to 0} \dfrac{f(t, s)}{s} \le \lambda_{k+1} < \infty$ and hence $\dfrac{f(t, u)}{u}$ can be regarded as a continuous function on $\mathbb{R}$. Thus we get $b(\cdot) \in C[0, 1]$. Also, notice that a nontrivial solution of (1.4) has a finite number of zeros. From (2.8) and the above fact $\lambda_k < b(t) < \lambda_{k+1}$ for all $t \in [0, 1]$.

We know that the eigenfunction ϕ_k corresponding to λ_k has exactly $k - 1$ zeros in [0, 1]. Applying Lemma 2.4 of [13] to ϕ_k and u, we see that u has at least k zeros in I. By Lemma 2.4 of [13] again to u and ϕ_{k+1}, we get that ϕ_{k+1} has at least $k + 1$ zeros in [0, 1]. This is a contradiction.

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Authors' contributions
GD conceived of the study, and participated in its design and coordination and helped to draft the manuscript. BY drafted the manuscript. RM participated in the design of the study. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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