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Existence of anti-periodic solutions for second-order ordinary differential equations involving the Fučík spectrum

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Abstract

In this paper, we study the existence of anti-periodic solutions for a second-order ordinary differential equation. Using the interaction of the nonlinearity with the Fučík spectrum related to the anti-periodic boundary conditions, we apply the Leray-Schauder degree theory and the Borsuk theorem to establish new results on the existence of anti-periodic solutions of second-order ordinary differential equations. Our nonlinearity may cross multiple consecutive branches of the Fučík spectrum curves, and recent results in the literature are complemented and generalized.

Keywords: anti-periodic solutions; Fučík spectrum; Leray-Schauder degree theory; Borsuk theorem

1 Introduction and main results

In this paper, we study the existence of anti-periodic solutions for the following second-order ordinary differential equation:

$$-x'' = f(t, x), \tag{1.1}$$

where $f \in C(\mathbb{R}^2, \mathbb{R}), f(t + \frac{T}{2}, -s) = -f(t, s), \forall t, s \in \mathbb{R}$ and T is a positive constant. A function $x(t)$ is called an anti-periodic solution of (1.1) if $x(t)$ satisfies (1.1) and $x(t + \frac{T}{2}) = -x(t)$ for all $t \in \mathbb{R}$. Note that to obtain anti-periodic solutions of (1.1), it suffices to find solutions of the following anti-periodic boundary value problem:

$$\begin{cases} x'' = -f(t, x), \\ x^{(i)}(0) = -x^{(i)}(\frac{T}{2}), \quad i = 0, 1. \end{cases} \tag{1.2}$$

In what follows, we will consider problem (1.2) directly.

The problem of the existence of solutions of (1.1) under various boundary conditions has been widely investigated in the literature and many results have been obtained (see [1–13]). Usually, the asymptotic interaction of the ratio $\frac{f(t,s)}{s}$ with the Fučík spectrum of $-x''$ under various boundary conditions was required as a nonresonance condition to obtain the solvability of equation (1.1). Recall that the Fučík spectrum of $-x''$ with an anti-periodic

boundary condition is the set of real number pairs $(\lambda_+, \lambda_-) \in \mathbb{R}^2$ such that the problem

$$\begin{cases} -x'' = \lambda_+ x^+ - \lambda_- x^-, \\ x^{(i)}(0) = -x^{(i)}(\frac{T}{2}), \quad i = 0, 1 \end{cases} \tag{1.3}$$

has nontrivial solutions, where $x^+ = \max\{0, x\}$, $x^- = \max\{0, -x\}$; while the concept of Fučík spectrum was firstly introduced in the 1970s by Fučík [14] and Dancer [15] independently under the periodic boundary condition. Since the work of Fonda [6], some investigation has been devoted to the nonresonance condition of (1.1) by studying the asymptotic interaction of the ratio $\frac{2F(t,s)}{s^2}$, where $F(t,s) = \int_0^s f(t, \tau) \, d\tau$, with the spectrum of $-x''$ under different boundary conditions; for instance, see [10] for the periodic boundary condition, [16] for the two-point boundary condition. Note that

$$\liminf_{s \to \pm\infty} \frac{f(t,s)}{s} \leq \liminf_{s \to \pm\infty} \frac{2F(t,s)}{s^2} \leq \limsup_{s \to \pm\infty} \frac{2F(t,s)}{s^2} \leq \limsup_{s \to \pm\infty} \frac{f(t,s)}{s},$$

we can see that the conditions on the ratio $\frac{2F(t,s)}{s^2}$ are more general than those on the ratio $\frac{f(t,s)}{s}$. In fact, by using the asymptotic interaction of the ratio $\frac{2F(t,s)}{s^2}$ with the spectrum of $-x''$, the ratio $\frac{f(t,s)}{s}$ can cross multiple spectrum curves of $-x''$. In this paper, we are interested in the nonresonance condition on the ratio $\frac{2F(t,s)}{s^2}$ for the solvability of (1.1) involving the Fučík spectrum of $-x''$ under the anti-periodic boundary condition.

Note that the study of anti-periodic solutions for nonlinear differential equations is closely related to the study of periodic solutions. In fact, since $f(t,s) = -f(t + \frac{T}{2}, -s) = f(t + T, s)$, $x(t)$ is a T-periodic solution of (1.1) if $x(t)$ is a $\frac{T}{2}$-anti-periodic solution of (1.1). Many results on the periodic solutions of (1.1) have been worked out. For some recent work, one can see [2–5, 8–10, 17]. As special periodic solutions, the existence of anti-periodic solutions plays a key role in characterizing the behavior of nonlinear differential equations coming from some models in applied sciences. During the last thirty years, anti-periodic problems of nonlinear differential equations have been extensively studied since the pioneering work by Okochi [18]. For example, in [19], anti-periodic trigonometric polynomials are used to investigate the interpolation problems, and anti-periodic wavelets are studied in [20]. Also, some existence results of ordinary differential equations are presented in [17, 21–24]. Anti-periodic boundary conditions for partial differential equations and abstract differential equations are considered in [25–32]. For recent developments involving the existence of anti-periodic solutions, one can also see [33–35] and the references therein.

Denote by Σ the Fučík spectrum of the operator $-x''$ under the anti-periodic boundary condition. Simple computation implies that $\Sigma = \bigcup_{m=1}^{+\infty} \Sigma_m$, where

$$\Sigma_m = \left\{ (\lambda_+, \lambda_-) \in \mathbb{R}^2 : \frac{(m+1)\pi}{\sqrt{\lambda_+}} + \frac{m\pi}{\sqrt{\lambda_-}} = \frac{T}{2} \text{ or } \frac{m\pi}{\sqrt{\lambda_+}} + \frac{(m+1)\pi}{\sqrt{\lambda_-}} = \frac{T}{2}, m \in \mathbb{N} \right\}.$$

It is easily seen that the set Σ can be seen as a subset of the Fučík spectrum of $-x''$ under the corresponding Dirichlet boundary condition; one can see the definition of the set Σ_{2i+1}, $i \in \mathbb{N}$, or Figure 1 in [12]. Without loss of generality, we assume that φ_m is an eigenfunction of (1.3) corresponding to $(\lambda_+, \lambda_-) \in \Sigma_m$ such that $\varphi_m(0) = 0$ and $\varphi'_m(0) = a \in \mathbb{R} \setminus \{0\}$. Denote $\Sigma_{m,1} = \{(\lambda_+, \lambda_-) \in \mathbb{R}^2 : \frac{(m+1)\pi}{\sqrt{\lambda_+}} + \frac{m\pi}{\sqrt{\lambda_-}} = \frac{T}{2}, m \in \mathbb{Z}^+\}$ and $\Sigma_{m,2} = \{(\lambda_+, \lambda_-) \in \mathbb{R}^2 : \frac{m\pi}{\sqrt{\lambda_+}} + \frac{(m+1)\pi}{\sqrt{\lambda_-}} =$

$\frac{T}{2}, m \in \mathbb{Z}^+\}$. Then if $a > 0$, we obtain only a one-dimensional function φ_m, denoted by $\varphi_{m,1}$, corresponding to the point $(\lambda_+, \lambda_-) \in \Sigma_{m,1}$, and if $a < 0$, we obtain only a one-dimensional function φ_m, denoted by $\varphi_{m,2}$, corresponding to the point $(\lambda_+, \lambda_-) \in \Sigma_{m,2}$.

In this paper, together with the Leray-Schauder degree theory and the Borsuk theorem, we obtain new existence results of anti-periodic solutions of (1.1) when the nonlinearity $f(t,s)$ is asymptotically linear in s at infinity and the ratio $\frac{F(t,s)}{s^2}$ stays asymptotically at infinity in some rectangular domain between Fučík spectrum curves Σ_m and Σ_{m+1}.

Our main result is as follows.

Theorem 1.1 *Assume that $f \in C(\mathbb{R}^2, \mathbb{R}), f(t + \frac{T}{2}, -s) = -f(t,s)$. If the following conditions*:
  (i) *There exist positive constants ρ, C_1, M such that*

$$\rho \leq \frac{f(t,s)}{s} \leq C_1, \quad \forall t \in \mathbb{R}, \forall |s| \geq M; \tag{1.4}$$

  (ii) *There exist connect subset $\Gamma \subset \mathbb{R}^2 \setminus \Sigma$, constants $p_1, q_1, p_2, q_2 > 0$ and a point of the type $(\lambda, \lambda) \in \mathbb{R}^2$ such that*

$$(\lambda, \lambda) \in [p_1, q_1] \times [p_2, q_2] \subset \Gamma \tag{1.5}$$

  and

$$p_1 \leq \liminf_{s \to +\infty} \frac{2F(t,s)}{s^2} \leq \limsup_{s \to +\infty} \frac{2F(t,s)}{s^2} \leq p_2,$$

$$q_1 \leq \liminf_{s \to -\infty} \frac{2F(t,s)}{s^2} \leq \limsup_{s \to -\infty} \frac{2F(t,s)}{s^2} \leq q_2,$$

  hold uniformly for all $t \in \mathbb{R}$,
then (1.1) admits a $\frac{T}{2}$-anti-periodic solution.

In particular, if $\lambda_+ = \lambda_-$, then problem (1.3) becomes the following linear eigenvalue problem:

$$\begin{cases} -x'' = \lambda x, \\ x^{(i)}(0) = -x^{(i)}(\frac{T}{2}), \quad i = 0, 1. \end{cases} \tag{1.6}$$

Simple computation implies that the operator $-x''$ with the anti-periodic boundary condition has a sequence of eigenvalues $\lambda_m = \frac{4(2m-1)^2\pi^2}{T^2}$, $m \in \mathbb{Z}^+$, and the corresponding eigenspace is two-dimensional.

Corollary 1.2 *Assume that $f \in C(\mathbb{R}^2, \mathbb{R}), f(t,s) = -f(t + \frac{T}{2}, -s)$. If (1.4) holds and there exist constants p, q and $m \in \mathbb{Z}^+$ such that*

$$\frac{4(2m-1)^2\pi^2}{T^2} < p \leq \liminf_{|s| \to +\infty} \frac{2F(t,s)}{s^2} \leq \limsup_{|s| \to +\infty} \frac{2F(t,s)}{s^2} \leq q < \frac{4(2m+1)^2\pi^2}{T^2}$$

holds uniformly for all $t \in \mathbb{R}$, then (1.1) admits a $\frac{T}{2}$-anti-periodic solution.

Remark It is well known that (1.1) has a $\frac{T}{2}$-anti-periodic solution if

$$\limsup_{|s|\to+\infty}\frac{f(t,s)}{s}\le\sigma_1<\frac{4\pi^2}{T^2}=\lambda_1,\quad\forall t\in\mathbb{R},$$

for some $\sigma_1>0$ (see Theorem 3.1 in [22]), which implies that the ratio $\frac{f(t,s)}{s}$ stays at infinity asymptotically below the first eigenvalue λ_1 of (1.6). In this paper, this requirement on the ratio $\frac{f(t,s)}{s}$ can be relaxed to (1.4), with some additional restrictions imposed on the ratio $\frac{2F(t,s)}{s^2}$. In fact, the conditions relative to the ratios $\frac{f(t,s)}{s}$ and $\frac{2F(t,s)}{s^2}$ as in Theorem 1.1 and Corollary 1.2 may lead to that the ratio $\frac{f(t,s)}{s}$ oscillates and crosses multiple consecutive eigenvalues or branches of the Fučík spectrum curves of the operator $-x''$. In what follows, we give an example to show this.

Denote $\lambda_m=\frac{4(2m-1)^2\pi^2}{T^2}$ for some positive integer $m\ge1$. Define

$$f(t,s)=\cos\left(\frac{2\pi}{T}t\right)+\frac{\lambda_m+\lambda_{m+1}}{2}s+\left(\frac{\lambda_m+\lambda_{m+1}}{2}-\delta\right)s\cos s,\quad\forall t\in\mathbb{R},s\in\mathbb{R},$$

where $\delta\in(0,\frac{\lambda_1}{100})$. Clearly,

$$f\left(t+\frac{T}{2},-s\right)=\cos\left[\frac{2\pi}{T}\left(t+\frac{T}{2}\right)\right]-\left[\frac{\lambda_m+\lambda_{m+1}}{2}s+\left(\frac{\lambda_m+\lambda_{m+1}}{2}-\delta\right)s\cos s\right]$$
$$=-f(t,s).$$

In addition,

$$\frac{f(t,s)}{s}=\frac{\cos(\frac{2\pi}{T}t)}{s}+\frac{\lambda_m+\lambda_{m+1}}{2}+\left(\frac{\lambda_m+\lambda_{m+1}}{2}-\delta\right)\cos s,$$

$$\frac{2F(t,s)}{s^2}=\frac{2\cos(\frac{2\pi}{T}t)}{s}+\frac{\lambda_m+\lambda_{m+1}}{2}+\left(\frac{\lambda_m+\lambda_{m+1}}{2}-\delta\right)\frac{s\sin s+\cos s}{s^2}$$

for all $t\in\mathbb{R}$, $s\in\mathbb{R}$, which imply that

$$\delta=\liminf_{|s|\to+\infty}\frac{f(t,s)}{s}\le\limsup_{|s|\to+\infty}\frac{f(t,s)}{s}=\lambda_m+\lambda_{m+1}-\delta,\tag{1.7}$$

$$\lim_{|s|\to+\infty}\frac{2F(t,s)}{s^2}=\frac{\lambda_m+\lambda_{m+1}}{2}\tag{1.8}$$

for all $t\in\mathbb{R}$. It is obvious that (1.7) implies that the assumption (i) of Theorem 1.1 holds. Take $p_1=\frac{\lambda_m+\lambda_{m+1}}{2}-\sigma_1$, $p_2=\frac{\lambda_m+\lambda_{m+1}}{2}+\sigma_1$, $q_1=\frac{\lambda_m+\lambda_{m+1}}{2}-\sigma_2$, $q_2=\frac{\lambda_m+\lambda_{m+1}}{2}-\sigma_2$ such that $[p_1,p_2]\times[q_1,q_2]\subset\mathbb{R}^2\setminus\Sigma$. Then (1.8) implies that the assumption (ii) of Theorem 1.1 holds. Thus, by Theorem 1.1 we can obtain a $\frac{T}{2}$-anti-periodic solution of equation (1.1). Here the ratio $\frac{2F(t,s)}{s^2}$ stays at infinity in the rectangular domain $[p_1,p_2]\times[q_1,q_2]$ between Fučík spectrum curves Σ_m and Σ_{m+1}, while the ratio $\frac{f(t,s)}{s}$ can cross at infinity multiple Fučík spectrum curves $\Sigma_1,\Sigma_2,\ldots,\Sigma_{m+1}$.

This paper is organized as follows. In Section 2, some necessary preliminaries are presented. In Section 3, we give the proof of Theorem 1.1.

2 Preliminaries

Assume that $T > 0$. Define

$$C_{\frac{T}{2}}^k = \left\{ x \in C^k(\mathbb{R}, \mathbb{R}) : x\left(t + \frac{T}{2}\right) = -x(t), \forall t \in \mathbb{R} \right\},$$

$$\|x\|_{C^k} = \max_{t \in [0,T]} |x(t)| + \cdots + \max_{t \in [0,T]} |x^{(k)}(t)|, \quad \forall x \in C^k([0, T], \mathbb{R}), k = 0, 1, 2, \ldots.$$

For $x \in C_{\frac{T}{2}}^k$, we can write the Fourier series expansion as follows:

$$x(t) = \sum_{i=0}^{\infty} \left[a_{2i+1} \cos \frac{2\pi(2i+1)t}{T} + b_{2i+1} \sin \frac{2\pi(2i+1)t}{T} \right].$$

Define an operator $J : C_{\frac{T}{2}}^k \to C_{\frac{T}{2}}^{k+1}$ by

$$(Jx)(t) = \int_0^t x(s)\, ds - \frac{T}{2\pi} \sum_{i=0}^{\infty} \frac{b_{2i+1}}{2i+1}$$

$$= \frac{T}{2\pi} \sum_{i=0}^{\infty} \left[\frac{a_{2i+1}}{2i+1} \sin \frac{2\pi(2i+1)t}{T} - \frac{b_{2i+1}}{2i+1} \cos \frac{2\pi(2i+1)t}{T} \right].$$

Clearly,

$$\frac{dJx(t)}{dt} = x(t), \qquad (Jx)(0) = -\frac{T}{2\pi} \sum_{i=0}^{\infty} \frac{b_{2i+1}}{2i+1},$$

which implies that

$$\frac{d^2(J^2x(t))}{dt^2} = x(t). \tag{2.1}$$

Furthermore, we obtain

$$|Jx(t)| \leq \int_0^T |x(s)|\, ds + \frac{T}{2\pi} \sum_{i=0}^{\infty} \frac{|b_{2i+1}|}{2i+1} \leq T\|x\|_{C^k} + \frac{T}{2\pi} \left(\sum_{i=0}^{\infty} b_{2i+1}^2 \right)^{\frac{1}{2}} \left(\sum_{i=0}^{\infty} \frac{1}{(2i+1)^2} \right)^{\frac{1}{2}}.$$

Note that

$$\left(\sum_{i=0}^{\infty} \frac{1}{(2i+1)^2} \right)^{\frac{1}{2}} = \frac{\pi}{2\sqrt{2}},$$

using the Parseval equality $\int_0^T |x(s)|^2\, ds = \frac{T}{2} \sum_{i=0}^{\infty} [a_{2i+1}^2 + b_{2i+1}^2]$, we get

$$|Jx(t)| \leq T\|x\|_{C^k} + \frac{T}{4\sqrt{2}} \sum_{i=0}^{\infty} [a_{2i+1}^2 + b_{2i+1}^2]$$

$$\leq T\|x\|_{C^k} + \frac{T}{4\sqrt{2}} \frac{2}{T} \int_0^T |x(s)|^2\, ds$$

$$\leq \frac{5T}{4} \|x\|_{C^k}, \quad \forall t \in [0, T],$$

which implies that the operator J is continuous. In view of the Arzela-Ascoli theorem, it is easy to see that J is completely continuous.

Denote by deg the Leray-Schauder degree. We need the following results.

Lemma 2.1 ([36, p.58]) *Let Ω be a bounded open region in a real Banach space X. Assume that $K : \overline{\Omega} \to \mathbb{R}$ is completely continuous and $p \notin (I - K)(\partial\Omega)$. Then the equation $(I - K)(x) = p$ has a solution in Ω if $\deg(I - K, \Omega, p) \neq 0$.*

Lemma 2.2 ([36, Borsuk theorem, p.58]) *Assume that X is a real Banach space. Let Ω be a symmetric bounded open region with $\theta \in \Omega$. Assume that $K : \overline{\Omega} \to \mathbb{R}$ is completely continuous and odd with $\theta \notin (I - K)(\partial\Omega)$. Then $\deg(I - K, \Omega, \theta)$ is odd.*

3 Proof of Theorem 1.1

Proof of Theorem 1.1 Consider the following homotopy problem:

$$x'' = -\mu f(t, x) - (1 - \mu)\lambda x \equiv \varphi(\mu, t, x(t)), \tag{3.1}$$

$$x^{(i)}(0) = -x^{(i)}\left(\frac{T}{2}\right), \quad i = 0, 1, \tag{3.2}$$

where $(\lambda, \lambda) \in [p_1, p_2] \times [q_1, q_2]$, $\mu \in [0, 1]$.

We first prove that the set of all possible solutions of problem (3.1)-(3.2) is bounded. Assume by contradiction that there exist a sequence of number $\{\mu_n\} \subset [0, 1]$ and corresponding solutions $\{x_n\}$ of (3.1)-(3.2) such that

$$\|x_n\|_{C^1} \to +\infty. \tag{3.3}$$

Set $z_n = \frac{x_n}{\|x_n\|_{C^1}}$. Obviously, $\|z_n\|_{C^1} = 1$ and z_n satisfies

$$-z_n'' = \frac{\mu_n f(t, x_n)}{\|x_n\|_{C^1}} + (1 - \mu_n)\lambda z_n, \tag{3.4}$$

$$z_n^{(i)}(0) = -z_n^{(i)}\left(\frac{T}{2}\right), \quad i = 0, 1. \tag{3.5}$$

By (1.4), (3.3) and the fact that f is continuous, there exist $n_0 \in \mathbb{Z}^+$, $C_1 > 0$ such that

$$\frac{|f(t, x_n)|}{\|x_n\|_{C^1}} = \left|\frac{f(t, x_n)}{x_n}\right| \frac{|x_n|}{\|x_n\|_{C^1}} \leq C_1 \quad \text{for } n \geq n_0.$$

In view of $\mu_n \in [0, 1]$, together with the choice of (λ, λ), it follows that there exists $M_1 > 0$ such that, for all $n \geq n_0$,

$$\left|z_n''(t)\right| \leq M_1, \quad \forall t \in [0, T].$$

It is easily seen that $\{z_n(t)\}$ and $\{z_n'(t)\}$ are uniformly bounded and equicontinuous on $[0, T]$. Then, using the Arzela-Ascoli theorem, there exist uniformly convergent subsequences on $[0, T]$ for $\{z_n(t)\}$ and $\{z_n'(t)\}$ respectively, which are still denoted as $\{z_n(t)\}$ and

$\{z'_n(t)\}$, such that

$$\lim_{n \to \infty} z_n(t) = z(t), \qquad \lim_{n \to \infty} z'_n(t) = z'(t). \tag{3.6}$$

Clearly, $\|z\|_{C^1} = 1$. Since $x_n(t)$ is a solution of (3.1)-(3.2), for each n, we get

$$\int_0^T x_n(t)\,dt = \int_0^{\frac{T}{2}} x_n(t)\,dt + \int_0^{\frac{T}{2}} x_n\left(t + \frac{T}{2}\right)dt = 0,$$

which implies that there exists $t_n \in [0, T]$ such that $x_n(t_n) = 0$. Then

$$\lim_{n \to \infty} z_n(t_n) = \lim_{n \to \infty} \frac{x_n(t_n)}{\|x_n\|_{C^1}} = 0. \tag{3.7}$$

Owing to that the sequences $\{t_n\}$ and $\{\mu_n\}$ are uniformly bounded, there exist $t_0 \in [0, T]$ and $\mu_0 \in [0, 1]$ such that, passing to subsequences if possible,

$$\lim_{n \to \infty} t_n = t_0, \qquad \lim_{n \to \infty} \mu_n = \mu_0. \tag{3.8}$$

Multiplying both sides of (3.4) by $z'_n(t)$ and integrating from t_n to t, we get

$$[z'_n(t_n)]^2 - [z'_n(t)]^2$$
$$= \mu_n \frac{2F(t, x_n(t))}{x_n^2(t)} \cdot \frac{x_n^2(t)}{\|x_n\|_{C^1}^2} - \mu_n \frac{2F(t, x_n(t_n))}{\|x_n\|_{C^1}^2}$$
$$+ (1 - \mu_n)\lambda\left[(z_n(t))^2 - (z_n(t_n))^2\right].$$

Taking a superior limit as $n \to \infty$, by (3.3) and (3.6)-(3.8), we obtain

$$[z'(t_0)]^2 - [z'(t)]^2 = \mu_0 \limsup_{n \to \infty} \frac{2F(t, x_n(t))}{x_n^2(t)} \cdot z^2(t) + (1 - \mu_0)\lambda z^2(t).$$

By the assumption (ii) and the choice of λ, if $z(t) > 0$, we have

$$[z'(t_0)]^2 - [z'(t)]^2 \le p_2 z^2(t).$$

Similarly, we obtain

$$[z'(t_0)]^2 - [z'(t)]^2 \ge p_1 z^2(t) \quad \text{for } z(t) > 0,$$
$$[z'(t_0)]^2 - [z'(t)]^2 \le q_2 z^2(t) \quad \text{for } z(t) < 0,$$
$$[z'(t_0)]^2 - [z'(t)]^2 \ge q_1 z^2(t) \quad \text{for } z(t) < 0.$$

Note that $z(t) \in C^1[0, T]$, the above inequalities can be rewritten as the following equivalent forms:

$$-p_2[z(t)]^2 \le [z'(t)]^2 - [z'(t_0)]^2 \le -p_1[z(t)]^2, \quad z(t) \ge 0, \tag{3.9}$$
$$-q_2[z(t)]^2 \le [z'(t)]^2 - [z'(t_0)]^2 \le -q_1[z(t)]^2, \quad z(t) \le 0. \tag{3.10}$$

It is easy to see that $z'(t_0) \neq 0$. In fact, if not, in view of (3.7)-(3.10), we get $z(t) = 0$, $z'(t) = 0$, $\forall t \in [0, T]$, which is contrary to $\|z\|_{C^1} = 1$.

We claim that $z'(t)$ has only finite zero points on $[0, T]$. In fact, if not, we may assume that there are infinitely many zero points $\{\zeta_i\} \subset [0, T]$ of $z'(t)$. Without loss of generality, we assume that there exists $\zeta_0 \in [0, T]$ such that $\lim_{i \to \infty} \zeta_i = \zeta_0$. Letting $t = \zeta_i$ in (3.9)-(3.10) and taking $i \to \infty$, we can obtain that $z(\zeta_0) \neq 0$. Without loss of generality, we assume that $z(\zeta_0) > 0$. Since $z(t)$ is continuous, there exist $\eta, \delta > 0$ such that $z(t) \geq \eta > 0$, $\forall t \in [t_0 - \delta, t_0 + \delta]$. Then there exists $n_1 > 0$ such that, if $n > n_1$, we have

$$z_n(t) \geq \eta, \quad \forall t \in [t_0 - \delta, t_0 + \delta]. \tag{3.11}$$

Clearly, $z_n^-(t) = 0$, $\forall t \in [t_0 - \delta, t_0 + \delta]$. Take $\zeta_*, \zeta^* \in [t_0 - \delta, t_0 + \delta]$ with $\zeta_* < \zeta^*$ such that $z'(\zeta^*) = z'(\zeta_*) = 0$. Integrating (3.4) from ζ_* to ζ^*,

$$z_n'(\zeta_*) - z_n'(\zeta^*) = \mu_n \frac{1}{\|x_n\|_{C^1}} \int_{\zeta_*}^{\zeta^*} f\big(t, x_n(s)\big)\, ds + (1 - \mu_n) \int_{\zeta_*}^{\zeta^*} \lambda z_n(s)\, ds. \tag{3.12}$$

By (3.3), (3.11), we obtain

$$x_n(t) = z_n(t)\|x_n\|_{C^1} \geq \eta\|x_n\|_{C^1} \to +\infty \quad \text{as } n \to +\infty$$

holds uniformly for $t \in [\zeta_*, \zeta^*]$. Thus, using (1.4), we get

$$\frac{f(t, x_n(t))}{x_n(t)} \geq \rho, \quad \forall t \in [\zeta_*, \zeta^*],$$

which implies that

$$\frac{f(t, x_n(t))}{\|x_n\|_{C^1}} = \frac{f(t, x_n(t))}{x_n(t)} \cdot z_n(t) \geq \rho \cdot \eta > 0, \quad \forall t \in [\zeta_*, \zeta^*].$$

Then, together with (3.6), (3.8) and (3.12), we obtain

$$0 \geq \mu_0 \cdot \rho \cdot \eta\big(\zeta^* - \zeta_*\big) + (1 - \mu_0) \cdot \lambda \cdot \eta\big(\zeta^* - \zeta_*\big) > 0,$$

a contradiction.

Now, we show that (3.9)-(3.10) has only a trivial anti-periodic solution. In fact, if not, we assume that (3.9)-(3.10) has a nontrivial anti-periodic solution $\bar{z}(t)$. Without loss of generality, we assume $t_0 = 0$. Firstly, we consider the case that $\bar{z}'(0) > 0$. Assume that z_1, z_2 satisfy the following equations respectively:

$$\big[z_1'(t)\big]^2 - \big[z_1'(0)\big]^2 = -p_2\big[z_1(t)\big]^2, \quad z_1(t) \geq 0, \tag{3.13}$$

$$\big[z_2'(t)\big]^2 - \big[z_2'(0)\big]^2 = -p_1\big[z_2(t)\big]^2, \quad z_2(t) \geq 0 \tag{3.14}$$

with

$$\bar{z}(0) = z_1(0) = z_2(0), \tag{3.15}$$

$$z_1'(0) \leq \bar{z}'(0) \leq z_2'(0). \tag{3.16}$$

Take t_1 as the first zero point of $\overline{z}(t)$ on $(0, T]$. Then by (3.13)-(3.16) it follows that

$$z_1(t) \le \overline{z}(t) \le z_2(t), \quad \forall t \in [0, t_1]. \tag{3.17}$$

In fact, by (3.15)-(3.16) and the fact that $\overline{z}$, z_1, z_2 are continuous differential, it is easy to see that there exists sufficiently small $\epsilon \in (0, t_1)$ such that

$$z_1(t) \le \overline{z}(t) \le z_2(t), \quad t \in (0, \epsilon),$$

$$z_1'(t) \le \overline{z}'(t) \le z_2'(t), \quad t \in (0, \epsilon).$$

If there is $\overline{t} \in (\epsilon, t_1)$ such that $\overline{z}(\overline{t}) = z_1(t)$, then comparing (3.9) with (3.13), we can obtain that $\overline{z}'(\overline{t}) \ge z_1'(\overline{t})$, which implies that if $t > \overline{t}$, we have $\overline{z}(\overline{t}) \ge z_1(\overline{t})$. Then $\overline{z}(t) \ge z_1(t)$ for $t \in (0, t_1]$. Similarly, we have $\overline{z}(t) \le z_2(t)$, $\forall t \in [0, t_1]$. Hence, (3.17) holds.

Similarly, if z_1, z_2 satisfy

$$\left[z_1'(t)\right]^2 - \left[z_1'(0)\right]^2 = -q_2\left[z_1(t)\right]^2, \quad z_1(t) \le 0, \tag{3.18}$$

$$\left[z_2'(t)\right]^2 - \left[z_2'(0)\right]^2 = -q_1\left[z_2(t)\right]^2, \quad z_2(t) \le 0 \tag{3.19}$$

and

$$\overline{z}(t_1) = z_1(t_1) = z_2(t_1),$$

$$z_1'(t_1) \le \overline{z}'(t_1) \le z_2'(t_1),$$

then we obtain

$$z_1(t) \le \overline{z}(t) \le z_2(t), \quad \forall t \in [t_1, t_2],$$

where t_2 is the first zero point on (t_1, T).

Since $z'(t)$ has finite zero points, (3.13), (3.14), (3.18), (3.19) can be transformed into the following equations respectively:

$$z''(t) = -p_2 z(t), \qquad z''(t) = -p_1 z(t), \quad z(t) \ge 0, \tag{3.20}$$

$$z''(t) = -q_2 z(t), \qquad z''(t) = -q_1 z(t), \quad z(t) \le 0. \tag{3.21}$$

Then there exist $A, B, C, D > 0$ such that

$$A \sin \sqrt{p_2} t \le \overline{z}(t) \le B \sin \sqrt{p_1} t, \quad \forall t \in [0, t_1],$$

$$-C \sin \sqrt{q_2}(t - t_1) \le \overline{z}(t) \le -D \sin \sqrt{q_1}(t - t_1), \quad \forall t \in [t_1, t_2].$$

It is easy to get

$$\frac{\pi}{\sqrt{p_2}} \le t_1 \le \frac{\pi}{\sqrt{p_1}},$$

$$\frac{\pi}{\sqrt{q_2}} + \frac{\pi}{\sqrt{p_2}} \le t_2 \le \frac{\pi}{\sqrt{q_1}} + \frac{\pi}{\sqrt{p_1}}.$$

Since $\bar{z}$ is anti-periodic and $\bar{z}'(0) > 0$, there exists $m \in \mathbb{Z}^+$ such that

$$\frac{(m+1)\pi}{\sqrt{q_2}} + \frac{m\pi}{\sqrt{p_2}} \le t_m = \frac{T}{2} \le \frac{(m+1)\pi}{\sqrt{q_1}} + \frac{m\pi}{\sqrt{p_1}},$$

which implies that there exists a real number pair $(p^*, q^*) \in [p_1, p_2] \times [q_1, q_2]$ such that

$$\frac{(m+1)\pi}{\sqrt{q^*}} + \frac{m\pi}{\sqrt{p^*}} = \frac{T}{2}. \tag{3.22}$$

On the other hand, in view of the assumption (ii), by the definition of Σ and $(p^*, q^*) \in [p_1, p_2] \times [q_1, q_2]$, it follows that

$$\frac{(m+1)\pi}{\sqrt{p^*}} + \frac{m\pi}{\sqrt{q^*}} \ne \frac{T}{2}, \quad \forall m \in \mathbb{Z}^+,$$

which is contrary to (3.22).

If $\bar{z}'(0) < 0$, then by the assumption (ii), we can obtain a contradiction using similar arguments.

In a word, we can see that there exists $C > 0$ independent of μ such that

$$\|x\|_{C^1} \le C. \tag{3.23}$$

Set

$$\Omega = \left\{ x \in C^1_{\frac{T}{2}} : \|x\|_{C^1} < C + 1 \right\}.$$

Clearly, Ω is a bounded open set in $C^1_{\frac{T}{2}}$. Note that, for $x \in C^1_{\frac{T}{2}}$, using the assumption on f, we obtain

$$\begin{aligned}
\varphi\left(\mu, t + \frac{T}{2}, x\left(t + \frac{T}{2}\right)\right) \\
= -\mu f\left(t + \frac{T}{2}, x\left(t + \frac{T}{2}\right)\right) - (1 - \mu)\lambda x\left(t + \frac{T}{2}\right) \\
= \mu f(t, x) + (1 - \mu)\lambda x(t) \\
= -\varphi\left(\mu, t, x(t)\right),
\end{aligned}$$

which implies that $\varphi \in C^0_{\frac{T}{2}}$.

Define $G_\mu : \overline{\Omega} \to C^2_{\frac{T}{2}}$ by

$$G_\mu\left(x(t)\right) = J^2 \varphi\left(\mu, t, x(t)\right).$$

Clearly, G_μ is completely continuous, and by (2.1) and (3.1) it follows that the fixed point of G_1 in $\overline{\Omega}$ is the anti-periodic solution of problem (1.1). Define the homotopy $H : \overline{\Omega} \times [0, 1] \to C^1_{\frac{T}{2}}$ as follows:

$$H(x, \mu) = x - G_\mu(x).$$

In view of (3.23), it follows that

$$H(x, \mu) \neq 0, \quad \forall (x, \mu) \in \partial\Omega \times [0, 1].$$

Hence,

$$\deg(I - G_1, \Omega, 0) = \deg(I - G_0, \Omega, 0).$$

Note that the operator G_0 is odd. By Lemma 2.2 it follows that $\deg(I - G_0, \Omega, 0) \neq 0$. Thus,

$$\deg(I - G_1, \Omega, 0) \neq 0.$$

Now, using Lemma 2.1, we can see that (1.2) has a solution and hence (1.1) has a $\frac{T}{2}$-anti-periodic solution. The proof is complete. □

Authors' contributions
All authors read and approved the final manuscript.

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