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On mixed boundary value problem of impulsive semilinear evolution equations of fractional order

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Abstract

This article studies the existence and uniqueness of solutions for impulsive semilinear evolution equations of fractional order $\alpha \in (1, 2]$ with mixed boundary conditions. Some standard fixed point theorems are applied to prove the main results. An illustrative example is also presented.

Mathematics Subject Classification: 26A33; 34K30; 34K45.

Keywords: evolution equations of fractional order, impulse, mixed boundary conditions, fixed point theorem.

1 Introduction and preliminaries

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, control theory, signal, and image processing, biophysics, electrodynamics of complex medium, polymer rheology, fitting of experimental data, etc. [1-6]. For example, one could mention the problem of anomalous diffusion [7-9], the nonlinear oscillation of earthquake can be modeled with fractional derivative [10], and fluid-dynamic traffic model with fractional derivatives [11] can eliminate the deficiency arising from the assumption to continuum traffic flow and many other [12,13] recent developments in the description of anomalous transport by fractional dynamics. For some recent development on nonlinear fractional differential equations, see [14-24] and the references therein.

Impulsive differential equations, which provide a natural description of observed evolution processes, are regarded as important mathematical tools for the better understanding of several real world problems in applied sciences. The theory of impulsive differential equations of integer order has found its extensive applications in realistic mathematical modelling of a wide variety of practical situations and has emerged as an important area of investigation. The impulsive differential equations of fractional order have also attracted a considerable attention and a variety of results can be found in the articles [25-36].

Motivated by Agarwal and Ahmad's work [33], in this article, we study a mixed boundary value problem for impulsive evolution equations of fractional order given by

$$\begin{cases} {}^cD^{\alpha}u(t) = A(t)u(t) + f(t, u(t)), \quad 1 < \alpha \leq 2, \quad t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = I_k^*(u(t_k)), \quad k = 1, 2, \ldots, p, \\ Tu'(0) = -au(0) - bu(T), \quad Tu'(T) = cu(0) + du(T), \quad a, b, c, d \in \mathbb{R}, \end{cases} \tag{1.1}$$

where ${}^CD^{\alpha}$ is the Caputo fractional derivative, $A(t)$ is a bounded linear operator on J (the function $t \to A(t)$ is continuous in the uniform operator topology), $f \in C(J \times \mathbb{R}, \mathbb{R}), I_k, I_k^* \in C(\mathbb{R}, \mathbb{R}), J = [0, T](T > 0), 0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = T, J' = J \backslash \{t_1, t_2, \ldots, t_p\}$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k (k = 1, 2, ..., p)$, respectively and $\Delta u'(t_k)$ have a similar meaning for $u'(t)$.

It is worthwhile pointing out that the boundary conditions in (1.1) interpolate between Neumann ($a = b = c = d = 0$) and Dirichlet ($a, d \to \infty$ with finite values of b and c) boundary conditions. Note that Zaremba boundary conditions ($u(0) = 0$, $u'(T) = 0$) can be considered as mixed boundary conditions with $a \to \infty$, $c = d = 0$. For more details on Zaremba boundary conditions, see ([37-39]).

Let $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, ..., $J_{p-1} = (t_{p-1}, t_p]$, $J_p = (t_p, T]$, and we introduce the spaces: $PC(J, \mathbb{R}) = \{u : J \to \mathbb{R} | u \in C(J_k)$, $k = 0, 1, ..., p$, and $u(t_k^+)$ exist, $k = 1, 2, ..., p, \}$ with the norm $\|u\| = \sup_{t \in J} |u(t)|$, and $PC^1(J, \mathbb{R}) = \{u : J \to \mathbb{R} | u \in C^1(J_k)$, $k = 0, 1, ..., p$, and $u(t_k^+), u'(t_k^+)$ exist, $k = 1, 2, ..., p, \}$ with the norm $\|u\|_{PC^1} = \max\{\|u\|, \|u'\|\}$. Obviously, $PC(J, \mathbb{R})$ and $PC^1(J, \mathbb{R})$ are Banach spaces.

Definition 1.1 *A function* $u \in PC^1(J, \mathbb{R})$ *with its Caputo derivative of order* α *existing on* J *is a solution of* (1.1) *if it satisfies* (1.1).

For convenience, we give some notations:

$$\lambda_1(t) = \frac{(b+d)T + (ad - bc)t}{\Lambda T}, \qquad \lambda_2(t) = \frac{(b+1)T + (a+b)t}{\Lambda},$$
$$\lambda_3(t) = \frac{(1-d)(T+at) + b(c+1)t}{\Lambda}, \quad \lambda_4 = \frac{|ad| + |bc|}{T|\Lambda|}, \quad \lambda_5 = \frac{|a+b|}{|\Lambda|}, \quad A_1 = \max_{t \in J} |A(t)|,$$

where $\Lambda = (b + 1)(c + d) - (a + b)(d - 1) \neq 0$.

Lemma 1.1 [26]*For a given* $y \in C[0, T]$, *a function* u *is a solution of the impulsive mixed boundary value problem*

$$\begin{cases} {}^CD^{\alpha}u(t) = y(t), \quad 1 < \alpha \leq 2, \quad t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = I_k^*(u(t_k)), \quad k = 1, 2, \ldots, p, \\ Tu'(0) = -au(0) - bu(T), \quad Tu'(T) = cu(0) + du(T), \quad a, b, c, d \in \mathbb{R}, \end{cases} \tag{1.2}$$

if and only if u is a solution of the impulsive fractional integral equation

$$u(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \lambda_1(t) \int_{t_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\ -\lambda_2(t) \int_{t_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + A, \quad t \in J_0; \\ \int_{t_k}^t \frac{(t-s)\alpha - 1}{\Gamma(\alpha)} y(s)ds + \lambda_1(t) \int_{t_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\ -\lambda_2(t) \int_{t_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + \sum_{i=1}^k \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + I_i(u(t_i)) \right] \\ + \sum_{i=1}^{k-1} (t_k - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \right] \\ + \sum_{i=1}^{k} (t - t_k) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \right] + A, \quad t \in J_k, \quad k = 1, 2, \ldots, p, \end{cases} \tag{1.3}$$

where

$$\begin{aligned} A = {} & \lambda_1(t) \sum_{i=1}^p \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + I_i(u(t_i)) \right] \\ & + \lambda_1(t) \sum_{i=1}^{p-1} (t_p - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \right] \\ & - \sum_{i=1}^p [\lambda_3(t) + \lambda_1(t) t_p] \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \right]. \end{aligned}$$

2 Uniqueness and existence results

Define an operator $T : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ as

$$\begin{aligned} Tu(t) = {} & \int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(f(s, u(s)) + A(s)u(s) \right) ds \\ & + \lambda_1(t) \int_{t_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} (f(s, u(s)) + A(s)u(s))ds \\ & - \lambda_2(t) \int_{t_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} (f(s, u(s)) + A(s)u(s))ds \\ & + \sum_{i=1}^k \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} (f(s, u(s)) + A(s)u(s))ds + I_i(u(t_i)) \right] \\ & + \sum_{i=1}^{k-1} (t_k - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} (f(s, u(s)) + A(s)u(s))ds + I_i^*(u(t_i)) \right] \\ & + \sum_{i=1}^{k} (t - t_k) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} (f(s, u(s)) + A(s)u(s))ds + I_i^*(u(t_i)) \right] \\ & + \lambda_1(t) \sum_{i=1}^{p} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} (f(s, u(s)) + A(s)u(s))ds + I_i(u(t_i)) \right] \\ & + \lambda_1(t) \sum_{i=1}^{p-1} (t_p - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} (f(s, u(s)) + A(s)u(s))ds + I_i^*(u(t_i)) \right] \\ & - \sum_{i=1}^{p} [\lambda_3(t) + \lambda_1(t) t_p] \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} (f(s, u(s)) + A(s)u(s))ds + I_i^*(u(t_i)) \right]. \end{aligned} \tag{2.1}$$

Lemma 2.1 *The operator* $T : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ *is completely continuous.*

Proof. Observe that T is continuous in view of the continuity of f, I_k and I_k^*. Let $\Omega \subset PC(J, \mathbb{R})$ be bounded, where $\Omega = \{u \in PC(J, \mathbb{R}): \|u\| \le r\}$. Then, there exist positive constants $L_i > 0 (i = 1, 2, 3)$ such that $|f(t, u(t))| \le L_1$, $|I_k(u)| \le L_2$ and $|I_k^*(u)| \le L_3$ $\forall u \in \Omega$. Thus, $\forall u \in \Omega$, we have

$$
\begin{aligned}
|T_u(t)| \le & \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s)) + A(s)u(s)|\, ds \\
& + |\lambda_1(t)| \int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s)) + A(s)u(s)|\, ds \\
& + |\lambda_2(t)| \int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s)) + A(s)u(s)|\, ds \\
& + \sum_{i=1}^{k} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s)) + A(s)u(s)|\, ds + |I_i(u(t_i))| \right] \\
& + \sum_{i=1}^{k-1} (t_k - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s)) + A(s)u(s)|\, ds + |I_i^*(u(t_i))| \right] \\
& + \sum_{i=1}^{k} (t - t_k) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s)) + A(s)u(s)|\, ds + |I_i^*(u(t_i))| \right] \\
& + |\lambda_1(t)| \sum_{i=1}^{p} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s)) + A(s)u(s)|\, ds + |I_i(u(t_i))| \right] \\
& + |\lambda_1(t)| \sum_{i=1}^{p-1} (t_p - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s)) + A(s)u(s)|\, ds + |I_i^*(u(t_i))| \right] \\
& + \sum_{i=1}^{p} [|\lambda_3(t)| + |\lambda_1(t)|\, t_p] \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s)) + A(s)u(s)|\, ds + |I_i^*(u(t_i))| \right] \\
\le & (L_1 + A_1 r) \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + |\lambda_1(t)| (L_1 + A_1 r) \int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
& + |\lambda_2(t)| (L_1 + A_1 r) \int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + \sum_{i=1}^{p} \left[(L_1 + A_1 r) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} ds + L_2 \right] \\
& + \sum_{i=1}^{p-1} T \left[(L_1 + A_1 r \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + L_3 \right] + \sum_{i=1}^{p} T \left[(L_1 + A_1 r) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + L_3 \right] \\
& + |\lambda_1(t)| \sum_{i=1}^{p} \left[(L_1 + A_1 r) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} ds + L_2 \right] \\
& + |\lambda_1(t)| \sum_{i=1}^{p-1} T \left[(L_1 + A_1 r) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + L_3 \right] \\
& + \sum_{i=1}^{p} [|\lambda_3(t)| + T |\lambda_1(t)|]\, T \left[(L_1 + 7A_1 r) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + L_3 \right] \\
\le & \frac{[(2p-1)(1 + T|\lambda_1(t)|) + p|\lambda_3(t)| + |\lambda_2(t)|] T^{\alpha-1} (L_1 + A_1 r)}{\Gamma(\alpha)} + (1 + |\lambda_1(t)|) p L_2 \\
& + \frac{(1+p)(1 + |\lambda_1(t)| T^{\alpha} (L_1 + A_1 r)}{\Gamma(\alpha+1)} + [(2p-1)(1 + T|\lambda_1(t)|) + p|\lambda_3(t)|] L_3.
\end{aligned}
\tag{2.2}
$$

Since $t \in [0, T]$, therefore there exists a positive constant L, such that $\|Tu\| \le L$, which implies that the operator T is uniformly bounded.

On the other hand, for any $t \in J_k$, $0 \le k \le p$, we have

$$\begin{aligned}
\left|(T_u)'(t)\right| &\le \int_{t_k}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left|f(s,u(s)) + A(s)u(s)\right| ds \\
&+ \lambda_4 \int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f(s,u(s)) + A(s)u(s)\right| ds \\
&+ \lambda_5 \int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left|f(s,u(s)) + A(s)u(s)\right| ds \\
&+ \sum_{i=1}^{p} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left|f(s,u(s)) + A(s)u(s)\right| ds + \left|I_i^*(u(t_i))\right|\right] \\
&+ \lambda_4 \sum_{i=1}^{p} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f(s,u(s)) + A(s)u(s)\right| ds + \left|I_i(u(t_i))\right|\right] \\
&+ \lambda_4 \sum_{i=1}^{p-1} (t_p - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left|f(s,u(s)) + A(s)u(s)\right| ds + \left|I_i^*(u(t_i))\right|\right] \\
&+ \sum_{i=1}^{p} \left|\frac{(a+b)T + (bc-ad)(T-t_p)}{T\Lambda}\right| \\
&\times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left|f(s,u(s)) + A(s)u(s)\right| ds + \left|I_i^*(u(t_i))\right|\right] \\
&\le (L_1 + A_1 r) \int_{t_k}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + \lambda_4 (L_1 + A_1 r) \int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&+ \lambda_5 (L_1 + A_1 r) \int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + \sum_{i=1}^{p} \left[(L_1 + A_1 r) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + L_3\right] \\
&+ \lambda_4 \sum_{i=1}^{p} \left[(L_1 + A_1 r) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} ds + L_2\right] + \lambda_4 \sum_{i=1}^{p-1} \left[(L_1 + A_1 r) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + L_3\right] \\
&+ \sum_{i=1}^{p} (\lambda_5 + T\lambda_4) \left[(L_1 + A_1 r) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + L_3\right] \\
&\le \frac{(1+p)\lambda_4 T^{\alpha}(L_1 + A_1 r)}{\Gamma(\alpha+1)} + [(1+p)(1+\lambda_5) + (1+pT)\lambda_4] \frac{T^{\alpha-1}(L_1 + A_1 r)}{\Gamma(\alpha)} \\
&+ p\lambda_4 L_2 + [p + (1+pT)\lambda_4 + (1+p)\lambda_5] L_3 := \overline{L}.
\end{aligned}$$

Hence, for $t_1, t_2 \in J_k$, $t_1 < t_2$, $0 \le k \le p$, we have

$$\left|(Tu)(t_2) - (Tu)(t_1)\right| \le \int_{t_1}^{t_2} \left|(Tu)'(s)\right| ds \le \overline{L}(t_2 - t_1),$$

which implies that T is equicontinuous on all J_k, $k = 0, 1, 2, \ldots, p$. Thus, by the Arzela-Ascoli Theorem, the operator $T : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous.

We need the following known results to prove the existence of solutions for (1.1).

Theorem 2.1 [40]*Let E be a Banach space. Assume that Ω is an open bounded subset of E with $\theta \in \Omega$ and let $T : \bar{\Omega} \to E$ be a completely continuous operator such that*

$$\|Tu\| \le \|u\|, \quad \forall u \in \partial\Omega.$$

Then T has a fixed point in $\bar{\Omega}$.

Theorem 2.2 *Let* $\lim_{u\to 0} \frac{f(t,u)}{u} = 0, \lim_{u\to 0} \frac{I_k(u)}{u} = 0$ *and* $\lim_{u\to 0} \frac{I_k^*(u)}{u} = 0$, *then the problem* (1.1) *has at least one solution.*

Proof. In view of $\lim_{u\to 0} \frac{f(t,u)}{u} = 0, \lim_{u\to 0} \frac{I_k(u)}{u} = 0$ and $\lim_{u\to 0} \frac{I_k^*(u)}{u} = 0$, then there exists a constant $r > 0$ such that $|f(t, u)| \le \delta_1 |u|$, $|I_k(u)| \le \delta_2 |u|$ and $\left|I_k^*(u)\right| \le \delta_3 |u|$ for $0 < |u| < r$, where $\delta_i > 0(i = 1, 2, 3)$ satisfy the inequality

$$\sup_{t\in J}\left\{\frac{(1+p)(1+|\lambda_1(t)|)T^{\alpha}(\delta_1+A_1)}{\Gamma(\alpha+1)}+\frac{[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|+|\lambda_2(t)|]T^{\alpha-1}(\delta_1+A_1)}{\Gamma(\alpha)}\right. \\ \left.+(1+|\lambda_1(t)|)p\delta_2+[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|]\delta_3\right\}\le 1. \tag{2.3}$$

Let us set $\Omega = \{u \in PC(J, \mathbb{R}) \mid \|u\| < r\}$ and take $u \in PC(J, \mathbb{R})$ such that $\|u\| = r$, that is, $u \in \partial\Omega$. Then, by the process used to obtain (2.2), we have

$$|Tu(t)|\le\sup_{t\in J}\left\{\frac{(1+p)(1+|\lambda_1(t)|)T^{\alpha}(\delta_1+A_1)}{\Gamma(\alpha+1)}+\frac{[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|+|\lambda_2(t)|]T^{\alpha-1}(\delta_1+A_1)}{\Gamma(\alpha)}\right. \\ \left.+(1+|\lambda_1(t)|)p\delta_2+[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|]\delta_3\right\}\|u\|. \tag{2.4}$$

Thus, it follows that $\|Tu\| \le \|u\|$, $u \in \partial\Omega$. Therefore, by Theorem 2.1, the operator T has at least one fixed point, which in turn implies that the problem (1.1) has at least one solution $u \in \bar{\Omega}$.

Theorem 2.3 *Assume that there exist positive constants* $K_i(i = 1, 2, 3)$ *such that*

$$|f(t,u)-f(t,v)|\le K_1|u-v|,\quad |I_k(u)-I_k(v)|\le K_2|u-v|,\quad |I_k^*(u)-I_k^*(v)|\le K_3|u-v|,$$

for $t \in J$, $u, v \in \mathbb{R}$ *and* $k = 1, 2, ..., p$.

Then the problem (1.1) *has a unique solution if* $\mathcal{H} < 1$, *where*

$$\mathcal{H}=\max_{t\in J}\left\{\frac{(1+p)(1+|\lambda_1(t)|)T^{\alpha}(K_1+A_1)}{\Gamma(\alpha+1)}+[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|]K_3\right. \\ \left.+(1+|\lambda_1(t)|)pK_2+\frac{[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|+|\lambda_2(t)|]T^{\alpha-1}(K_1+A_1)}{\Gamma(\alpha)}\right\}. \tag{2.5}$$

Proof. Denote $F(s) = |f(s, u(s)) - f(s,v(s))| + |A(s)u(s) - A(s)v(s)|$.

For $u, v \in PC(J, \mathbb{R})$, we have

$$\begin{aligned}
&|(Tu)(t)-(Tv)(t)| \\
&\le\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds+|\lambda_1(t)|\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds+|\lambda_2(t)|\int_{t_p}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}F(s)ds \\
&\quad+\sum_{i=1}^{k}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds+|I_i(u(t_i))-I_i(v(t_i))|\right]+\sum_{i=1}^{k-1}(t_k-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}F(s)ds\right. \\
&\quad\left.+|I_i^*(u(t_i))-I_i^*(v(t_i))|\right]+\sum_{i=1}^{k}(t-t_k)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}F(s)ds+|I_i^*(u(t_i))-I_i^*(v(t_i))|\right] \\
&\quad+|\lambda_1(t)|\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds+|I_i(u(t_i))-I_i(v(t_i))|\right] \\
&\quad+|\lambda_1(t)|\sum_{i=1}^{p-1}(t_p-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}F(s)ds+|I_i^*(u(t_i))-I_i^*(v(t_i))|\right] \\
&\quad+\sum_{i=1}^{p}[|\lambda_3(t)|+|\lambda_1(t)|t_p]\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}F(s)ds+|I_i^*(u(t_i))-I_i^*(v(t_i))|\right] \\
&\le\left\{\frac{(1+p)(1+|\lambda_1(t)|)T^{\alpha}(K_1+A_1)}{\Gamma(\alpha+1)}+\frac{[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|+|\lambda_2(t)|]T^{\alpha-1}(K_1+A_1)}{\Gamma(\alpha)}\right. \\
&\quad\left.+(1+|\lambda_1(t)|)pK_2+[(2p-1)(1+T|\lambda_1(t)|)+p|\lambda_3(t)|]K_3\right\}\|u-v\|.
\end{aligned}$$

Thus, we obtain $\|Tu-Tv\|\le\mathcal{H}\|u-v\|$, where $\mathcal{H}$ is given by (2.5). As $\mathcal{H} < 1$, therefore, T is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof.

3 Examples

Example 3.1 *Consider the following fractional order impulsive mixed boundary value problem*

$$\begin{cases} {}^{C}D^{\alpha}u(t) = \frac{1}{3}\cos t + e^{u^3(t)} - 1, \quad 0 < t < T, \quad t \neq t_1, \quad 0 < t_1 < T, \\ \Delta u(t_1) = 2\ln(1 + u^2(t_1)), \quad \Delta u'(t_1) = [1 - \cos u(t_1)]^2, \\ Tu'(0) = -\frac{1}{2}u(0) - \frac{1}{3}u(T), \quad Tu'(T) = \frac{1}{4}u(0) + \frac{1}{5}u(T), \end{cases} \tag{3.1}$$

where

$1 < \alpha \leq 2, A(t) = \frac{1}{3}\cos t, f(t,u) = e^{u^3} - 1, I_1(u) = 2\ln(1+u^2), I_1^*(u) = (1 - \cos u)^2, a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{1}{4}, d = \frac{1}{5}$ *and* $p = 1$.

Clearly all the assumptions of Theorem 2.2 hold. Thus, the conclusion of Theorem 2.2 applies and the impulsive fractional mixed boundary value problem (3.1) has at least one solution.

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Authors' contributions
GW and LZ completed the main part of this paper, GS corrected the main theorems and gave an example. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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