Chen et al. Boundary Value Problems 2012, **2012**:19
http://www.boundaryvalueproblems.com/content/2012/1/19

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

L^∞ estimates of solutions for the quasilinear parabolic equation with nonlinear gradient term and L^1 data

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Abstract

In this article, we study the quasilinear parabolic problem

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^m \nabla u) + u|u|^{\beta-2}|\nabla u|^q = u|u|^{\alpha-2}|\nabla u|^p + g(u), & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega; \quad u(x,t) = 0, & x \in \partial\Omega, \ t \ge 0, \end{cases} \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $m > 0$ and $g(u)$ satisfies $|g(u)| \le K_1|u|^{1+\nu}$ with $0 \le \nu < m$. By the Moser's technique, we prove that if $\alpha, \beta > 1$, $0 \le p < q$, $1 \le q < m + 2$, $p + \alpha < q + \beta$, there exists a weak solution $u(t) \in L^\infty([0,\infty), L^1) \cap L^\infty_{\rm loc}((0,\infty) W_0^{1,m+2})$ for all $u_0 \in L^1(\Omega)$. Furthermore, if $2q \le m + 2$, we derive the L^∞ estimate for $\nabla u(t)$. The asymptotic behavior of global weak solution $u(t)$ for small initial data $u_0 \in L^2(\Omega)$ also be established if $p + \alpha > \max\{m + 2, q + \beta\}$.

2000 Mathematics Subject Classification: 35K20; 35K59; 35K65.

Keywords: quasilinear parabolic equation, L^∞ estimates, asymptotic behavior of solution

1 Introduction

In this article, we are concerned with the initial boundary value problem of the quasilinear parabolic equation with nonlinear gradient term

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^m \nabla u) + u|u|^{\beta-2}|\nabla u|^q = u|u|^{\alpha-2}|\nabla u|^p + g(u), & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \quad u(x,t) = 0, & x \in \partial\Omega, \ t \ge 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $m > 0$, $\alpha, \beta > 1$, $0 \le p < q$, $1 \le q < m + 2$.

Recently, Andreu et al. in [1] considered the following quasilinear parabolic problem

$$\begin{cases} u_t - \Delta u + u|u|^{\beta-2}|\nabla u|^q = u|u|^{\alpha-2}|\nabla u|^p, & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \quad u(x,t) = 0, & x \in \partial\Omega, \ t \ge 0, \end{cases} \tag{1.2}$$

where $\alpha, \beta > 1$, $0 \le p < q \le 2$, $p + \alpha < q + \beta$ and $u_0 \in L^1(\Omega)$. By the so-called stability theorem with the initial data, they proved that there exists a generalized solution $u(t) \in C([0, T], L^1)$ for (1.2), in which $u(t)$ satisfies $A_k(u) \in L^2([0,T], W_0^{1,2})$ and

$$\int_{\Omega} J_k(u(t)-\phi(t))\mathrm{d}x + \int_0^t \int_{\Omega} (\nabla u \cdot \nabla A_k(u-\phi) + u|u|^{\beta-2}|\nabla u|^q A_k(u-\phi))\mathrm{d}x\mathrm{d}s$$
$$= \int_0^t \int_{\Omega} (u|u|^{\alpha-2}|\nabla u|^p A_k(u-\phi) - A_k(u-\phi)\phi_s)\mathrm{d}x\mathrm{d}s + \int_{\Omega} J_k(u_0-\phi(0))\mathrm{d}x \tag{1.3}$$

for $\forall t \in [0, T]$ and $\forall \phi \in L^2([0,T], W_0^{1,2}) \cap L^\infty(Q_T)$, where $Q_T = \Omega \times (0, T]$, and for any $k > 0$,

$$A_k(u) = \begin{cases} -k & u \le -k, \\ u & -k \le u \le k, \\ k & u \ge k. \end{cases} \tag{1.4}$$

$J_k(u)$ is the primitive of $A_k(u)$ such that $J_k(0) = 0$. The problem similar to (1.2) has also been extensively considered, see [2-6] and the references therein. It is an interesting problem to prove the existence of global solution $u(t)$ of (1.2) or (1.1) and to derive the L^∞ estimate for $u(t)$ and $\nabla u(t)$.

Porzio in [7] also investigated the solution of Leray-Lions type problem

$$\begin{cases} u_t = \mathrm{div}(a(x,t,u,\nabla u)), & (x,t) \in \Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \end{cases} \tag{1.5}$$

where $a(x, t, s, \xi)$ is a Carathéodory function satisfying the following structure condition

$$a(x,t,s,\xi)\xi \ge \theta|\xi|^m, \quad for \ \forall (x,t,s,\xi) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^1 \times \mathbb{R}^N \tag{1.6}$$

with $\theta > 0$ and $u_0 \in L^q(\Omega)$, $q \ge 1$. By the integral inequalities method, Porzio derived the L^∞ decay estimate of the form

$$\|u(t)\|_{L^\infty(\Omega)} \le C\|u_0\|^{\alpha}_{L^q(\Omega)} t^{-\lambda}, \quad t > 0 \tag{1.7}$$

with $C = C(N, q, m, \theta)$, $\alpha = mq(N(m-2)+mq)^{-1}$, $\lambda = N(N(m-2)+mq)^{-1}$.

In this article, we will consider the global existence of solution $u(t)$ of (1.1) with $u_0 \in L^1(\Omega)$ and give the L^∞ estimates for $u(t)$ under the similar condition in [1]. More specially, we will study the behavior of solution $u(t)$ as $t \to 0^+$. Obviously, if $m = 0$ and $g \equiv 0$, problem (1.1) is reduced to (1.2). We remark that the methods used in our article are different from that of [1]. In L^∞ estimates, we use an improved Morser's technique as in [8-10]. Since the equation in (1.1) contains the nonlinear gradient term $u|u|^{\alpha-2}|\nabla u|^p$ and $u|u|^{\beta-2}|\nabla u|^q$, it is difficult to derive L^∞ estimates for $u(t)$ and $\nabla u(t)$.

This article is organized as follows. In Section 2, we state the main results and present some Lemmas which will be used later. In Section 3, we use these Lemmas to derive L^∞ estimates of $u(t)$. Also the proof of the main results will be given in Section 3. The L^∞ estimates of $\nabla u(t)$ are considered in Section 4. The asymptotic behavior of solution for the small initial data $u_0(x)$ is investigated in Section 5.

2 Preliminaries and main results

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $\|\cdot\|_r$, $\|\cdot\|_{1,r}$ denote the Sobolev space $L^r(\Omega)$ and $W^{1,r}(\Omega)$ norms, respectively, $1 \le r \le \infty$. We often drop the letter Ω in these notations.

Let us state our precise assumptions on the parameters p, q, α, β and the function $g(u)$.

(H_1) the parameters $\alpha, \beta > 1$, $0 \le p < q < m + 2 < N$, $p + \alpha < q + \beta$ and $q(\alpha - 1) \ge p(\beta - 1)$,

(H_2) the function $g(u) \in C^1$ and $\exists K_1 \ge 0$ and $0 \le \nu < \max\{q + \beta - 2, m]$, such that

$$|g(u)| \le K_1 |u|^{1+\nu}, \quad \forall u \in \mathbb{R}^1,$$

(H_3) the initial data $u_0 \in L^1(\Omega)$,

(H_4) $2q \le 2 + m$, $\alpha, \beta < 2 + m(1 + 1/N)/2$,

(H_5) the mean curvature of $H(x)$ of $\partial\Omega$ at x is non-positive with respect to the outward normal.

Remark 2.1 The assumptions (H_1) and (H_3) are similar to as in [1].

Definition 2.2 A measurable function $u(t) = u(x, t)$ on $\Omega \times [0, \infty)$ is said to be a global weak solution of the problem (1.1) if $u(t)$ is in the class

$$C([0,\infty), L^1) \cap L^{\infty}_{\mathrm{loc}}((0,\infty), W_0^{1,m+2})$$

and $u|u|^{\beta-2}|\nabla u|^q,\ u|u|^{\alpha-2}|\nabla u|^p \in L^1_{\mathrm{loc}}([0,\infty)\times\Omega)$, and for any $\phi = \phi(x,t) \in C^1([0,\infty), C_0^1(\Omega))$ the equality

$$\int_0^T \int_\Omega \left\{-u\phi_t + |\nabla u|^m \nabla u \nabla\phi + u|u|^{\beta-2}|\nabla u|^q \phi\right\} \mathrm{d}x\mathrm{d}t = \int_\Omega (u_0(x)\phi(x,0) - u(x,T)\phi(x,T))\mathrm{d}x + \int_0^T \int_\Omega (u|u|^{\alpha-2}|\nabla u|^p + g(u))\phi \mathrm{d}x\mathrm{d}t \tag{2.1}$$

is valid for any $T > 0$.

Remark 2.3 In [1], the concept of generalized solution for (1.2) was introduced. A similar concept can be found in [7,11]. By the definition, we know that weak solution is the generalized solution. Conversely, a generalized solution is not necessarily weak solution.

Our main results read as follows.

Theorem 2.4 Assume (H_1)-(H_3). Then the problem (1.1) admits a global weak solution $u(t)$ which satisfies

$$u(t) \in L^{\infty}([0,\infty), L^1) \cap C([0,\infty), L^1) \cap L^{\infty}_{\mathrm{loc}}((0,\infty), W_0^{1,m+2}),\quad u_t \in L^2_{\mathrm{loc}}((0,\infty), L^2) \tag{2.2}$$

and the estimates

$$\|u(t)\|_\infty \le C_0 t^{-\lambda}, \qquad 0 < t \le T. \tag{2.3}$$

Furthermore, if (H_4) is satisfied, the solution $u(t)$ has the following estimates

$$\int_0^T s^{1+r} \left\| u_t(s) \right\|_2^2 \mathrm{d}s \le C_0, \tag{2.4}$$

$$\left\| \nabla u(t) \right\|_{m+2} \le C_0 t^{-(1+\lambda)/(m+2)}, \qquad 0 < t \le T, \tag{2.5}$$

with $r > \lambda = N(mN + m + 2)^{-1}$ and $C_0 = C_0(T, \|u_0\|_1)$.

Theorem 2.5 Assume (H_1)-(H_5). Then the solution $u(t)$ of (1.1) has the following L^∞ gradient estimate

$$\left\| \nabla u(t) \right\|_\infty \le C_0 t^{-\sigma}, \qquad 0 < t \le T, \tag{2.6}$$

with $\sigma = (2 + 2\lambda + N)(mN + 2m + 4)^{-1}$ and $C_0 = C_0(T, \|u_0\|_1)$.

Remark 2.6 The estimates (2.3) and (2.6) give the behavior of $\|u(t)\|_\infty$ and $\|\nabla u(t)\|_\infty$ as

Theorem 2.7 Assume the parameters $\alpha, \beta > 1$, $\gamma \ge 0$, $0 \le q < m + 2 < N$ and $p < m + 2 < p + \alpha$, $\alpha \le (m + 2 - p)(1 + 2N^{-1})$.

Then, $\exists d_0 > 0$, such that $u_0 \in L^2(\Omega)$ with $\|u_0\|_2 < d_0$, the initial boundary value problem

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^m \nabla u) + \gamma u|u|^{\beta-2}|\nabla u|^q = |u|^{\alpha-2}u|\nabla u|^p, \ x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \quad u(x,t) = 0, \quad x \in \partial\Omega, \ t \ge 0, \end{cases} \tag{2.7}$$

admits a solution $u(t) \in L^\infty([0,\infty), L^2) \cap W_0^{1,m+2}$, which satisfies

$$\left\| u(t) \right\|_2 \le C(1+t)^{-1/m}, \qquad t \ge 0. \tag{2.8}$$

where $C = C(\|u_0\|_2)$.

Theorem 2.8 Assume the parameters $\gamma > 0$, $\alpha, \beta > 1$, $1 \le p < q < m + 2 < N$ and $\tau = N(\mu - q)(q + \beta) \le 2(q^2 + N\beta)$ with $\mu = (q\alpha - p\beta)/(q - p) > q + \beta$.

Then, $\exists d_0 > 0$, such that $u_0 \in L^2$ with $\|u_0\|_2 < d_0$, the initial boundary value problem

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^m \nabla u) + \gamma u|u|^{\beta-2}|\nabla u|^q = |u|^{\alpha-2}u|\nabla u|^p, \ x \in \Omega, t > 0 \\ u(x,0) = u_0(x), \quad x \in \Omega; \quad u(x,t) = 0, \quad x \in \partial\Omega, \ t \ge 0, \end{cases} \tag{2.9}$$

admits a solution $u(t) \in L^\infty([0,\infty), L^2) \cap W_0^{1,m+2}$ which satisfies

$$\left\| u(t) \right\|_2 \le C(1+t)^{-1/(q+\beta-2)}, \quad t \ge 0. \tag{2.10}$$

where $C = C(\|u_0\|_2)$.

To obtain the above results, we will need the following Lemmas.

Lemma 2.9 (Gagliardo-Nirenberg type inequality) Let $\beta \ge 0$, $N > p \ge 1$, $q \ge 1 + \beta$ and $1 \le r \le q \le pN(1 + \beta)/(N - p)$. Then for $|u|^\beta u \in W^{1,p}(\Omega)$, we have

$$\|u\|_q \le C_0^{1/(\beta+1)} \|u\|_r^{1-\theta} \left\| |u|^\beta u \right\|_{1,p}^{\theta/(\beta+1)}$$

with $\theta = (1 + \beta)(r^{-1} - q^{-1})/(N^{-1} - p^{-1} + (1 + \beta)r^{-1})$, where the constant C_0 depends only on p, N.

The Proof of Lemma 2.9 can be obtained from the well-known Gagliardo-Nirenberg-Sobolev inequality and the interpolation inequality and is omitted here.

Lemma 2.10 [10] Let $y(t)$ be a nonnegative differentiable function on (0, T] satisfying

$$y'(t) + At^{\lambda\theta-1}y^{1+\theta}(t) \le Bt^{-k}y(t) + Ct^{-\delta}, \qquad 0 < t \le T$$

with A, $\theta > 0$, $\lambda\theta \ge 1$, B, $C \ge 0$, $k \le 1$. Then, we have

$$y(t) \le A^{-1/\theta}(2\lambda + 2BT^{1-k})^{1/\theta}t^{-\lambda} + 2C(\lambda + BT^{1-k})^{-1}t^{1-\delta}, \qquad 0 < t \le T.$$

3 L^∞ estimate for $u(t)$

In this section, we derive a priori estimates of the assumed solutions $u(t)$ and give a proof of Theorem 2.4. The solutions are in fact given as limits of smooth solutions of appropriate approximate equations and we may assume for our estimates that the solutions under consideration are sufficiently smooth.

Let $u_{0,i} \in C_0^2(\Omega)$ and $u_{0,i} \to u_0$ in $L^1(\Omega)$ as $i \to \infty$. For i = 1, 2, ..., we consider the approximate problem of (1.1)

$$\begin{cases} u_t - \operatorname{div}\left((|\nabla u|^2 + i^{-1})^{\frac{m}{2}}\nabla u\right) + u|u|^{\beta-2}|\nabla u|^q = u|u|^{\alpha-2}|\nabla u|^p + g(u), \ x \in \Omega, t > 0, \\ u(x,0) = u_{0,i}(x), \quad x \in \Omega, \quad u(x,t) = 0, \quad x \in \partial\Omega, \ t \ge 0. \end{cases} \tag{3.1}$$

The problem (3.1) is a standard quasilinear parabolic equation and admits a unique smooth solution $u_i(t)$(see Chapter 6 in [12]). We will derive estimates for $u_i(t)$. For the simplicity of notation, we write u instead of u_i and u^k for $|u|^{k-1}u$ where $k > 0$. Also, let C, C_j be generic constants independent of k, i, n changeable from line to line.

Lemma 3.1 Let (H_1)-(H_3) hold. Suppose that $u(t)$ is the solution of (3.1), then $u(t) \in L^\infty([0, \infty), L^1)$.

Proof Let n = 1, 2, ..., and

$$f_n(s) = \begin{cases} 1, & \frac{1}{n} \le s \\ ns(2 - ns), & 0 \le s \le \frac{1}{n} \\ -ns(2 + ns), & -\frac{1}{n} \le s \le 0 \\ -1, & s < -\frac{1}{n}. \end{cases}$$

It is obvious that $f_n(s)$ is odd and continuously differentiable in $\mathbb{R}^1$. Furthermore, $|f_n(s)| \le 1, f_n'(s) \ge 0$ and $f_n(s) \to \operatorname{sign}(s)$ uniformly in $\mathbb{R}^1$.

Multiplying the equation in (3.1) by $f_n(u)$ and integrating on Ω, we get

$$\begin{aligned} &\int_\Omega f_n(u)u_t \mathrm{d}x + \int_\Omega |\nabla u|^{m+2}f'_n(u)\mathrm{d}x + \int_\Omega u|u|^{\beta-2}f_n(u)|\nabla u|^q \mathrm{d}x \\ &\le \int_\Omega u|u|^{\alpha-2}f_n(u)|\nabla u|^p \mathrm{d}x + \int_\Omega u|u|^{\beta-2}f_n(u)|\nabla u|^q \mathrm{d}x \end{aligned} \tag{3.2}$$

and the application of the Young inequality gives

$$\int_\Omega u|u|^{\alpha-2}f_n(u)|\nabla u|^p \mathrm{d}x \le \frac{1}{4}\int_\Omega u|u|^{\beta-2}f_n(u)|\nabla u|^q \mathrm{d}x + C_1\int_\Omega |u|^{\mu-1}\mathrm{d}x, \tag{3.3}$$

where $\mu = (q\alpha - p\beta)(q - p)^{-1} \ge 1$, i.e $q(\alpha - 1) \ge p(\beta - 1)$.

In order to get the estimate for the third term of left-hand side in (3.2), we denote

$$F_n(u) = \int_0^u \left(s|s|^{\beta-2} f_n(s)\right)^{1/q} \mathrm{d}s, \quad u \in \mathbb{R}^1.$$

It is easy to verify that $F_n(u)$ is odd in $\mathbb{R}^1$. Then, we obtain from the Sobolev inequality that

$$\begin{aligned} \frac{1}{4}\int_\Omega u|u|^{\beta-2} f_n(u)|\nabla u|^q \mathrm{d}x &= \frac{1}{4}\int_\Omega \left|\nabla F_n(u)\right|^q \mathrm{d}x \\ &\ge \lambda_0 \int_\Omega \left|F_n(u)\right|^q \mathrm{d}x = \lambda_0 \int_{\Omega_n} \left|F_n(u)\right|^q \mathrm{d}x + \lambda_0 \int_{\Omega_n^c} \left|F_n(u)\right|^q \mathrm{d}x \end{aligned} \tag{3.4}$$

with some $\lambda_0 > 0$ and

$$\Omega_n = \{x \in \Omega |\, \left|u(x,t)\right| \ge n^{-1}\}, \quad \Omega_n^c = \Omega \backslash \Omega_n, \quad n = 1, 2, \ldots.$$

We note that $|F_n(u)|^q \le n^{-(q+\beta-1)}$ in Ω_n^c and

$$\int_{\Omega_n^c} \left|F_n(u)\right|^q \mathrm{d}x \le n^{-(q+\beta-1)}\, |\Omega|.$$

On the other hand, we have $|u(x, t)| \ge n^{-1}$ in Ω_n and

$$\left|F_n(u)\right| \ge \int_{n^{-1}}^{|u|} \left(s|s|^{\beta-2} f_n(s)\right)^{1/q} \mathrm{d}s \ge \frac{q}{q+\beta-1}\left(|u|^{\frac{q+\beta-1}{q}} - n^{-\frac{q+\beta-1}{q}}\right) \quad \text{in } \Omega_n.$$

This implies that there exists $\lambda_1 > 0$, such that

$$\lambda_0 \int_{\Omega_n} \left|F_n(u)\right|^q \mathrm{d}x \ge \lambda_1 \int_{\Omega_n} |u|^{q+\beta-1} \mathrm{d}x - \lambda_1 \,|\Omega|\, n^{-(q+\beta-1)} \tag{3.5}$$

Then it follows from (3.4)-(3.5) that

$$\frac{1}{4}\int_\Omega u|u|^{\beta-2} f_n(u)|\nabla u|^q \mathrm{d}x \ge \lambda_1 \int_\Omega |u|^{q+\beta-1} \mathrm{d}x - C_2 n^{-(q+\beta-1)} \tag{3.6}$$

with some $C_2 > 0$.

Similarly, we have from the assumption (H_2) and the Young inequality that

$$\begin{aligned} \int_\Omega \left|g(u) f_n(u)\right| \mathrm{d}x &\le K_1 \int_\Omega |u|^{1+\nu} \left|f_n(u)\right| \mathrm{d}x \\ &\le K_1 \int_\Omega |u|^{1+\nu} \mathrm{d}x \le \frac{\lambda_1}{2}\int_{\Omega_n} |u|^{q+\beta-1} \mathrm{d}x + C_2\left(1 + n^{-1-\nu}\right). \end{aligned} \tag{3.7}$$

Furthermore, the assumption $\mu < q + \beta$ implies that

$$C_1 \int_{\Omega_n} |u|^{\mu-1} \mathrm{d}x \le \frac{\lambda_1}{2}\int_{\Omega_n} |u|^{q+\beta-1} \mathrm{d}x + C_2. \tag{3.8}$$

Then (3.2)-(3.3) and (3.6)-(3.8) give that

$$\int_\Omega f_n(u)u_t \mathrm{d}x + \frac{1}{2}\int_\Omega u|u|^{\beta-2}f_n(u)|\nabla u|^q \mathrm{d}x \le C_3\left(1 + n^{-1-\nu} + n^{-(q+\beta-1)}\right). \tag{3.9}$$

Letting $n \to \infty$ in (3.9) yields

$$\frac{d}{\mathrm{d}t}\|u(t)\|_1 + \frac{1}{2}\int_\Omega |u|^{\beta-1}|\nabla u|^q \mathrm{d}x \le C_3. \tag{3.10}$$

Note that

$$\int_\Omega |u|^{\beta-1}|\nabla u|^q \mathrm{d}x = \left(\frac{q}{q+\beta-1}\right)^q \int_\Omega \left|\nabla u^{1+\frac{\beta-1}{q}}\right|^q \mathrm{d}x \ge 2\lambda_2 \ge \|u\|_1^{q+\beta-1}$$

with some $\lambda_2 > 0$. Then (3.10) becomes

$$\frac{d}{\mathrm{d}t}\|u(t)\|_1 + \lambda_2\|u(t)\|_1^{q+\beta-1} \le C_3. \tag{3.11}$$

This gives that $u(t) \in L^\infty([0, \infty), L^1)$ if $u_0 \in L^1$.

Remark 3.2 The differential inequality (3.10) implies that the solution $u_i(t)$ of (3.1) satisfies

$$\int_0^T\int_\Omega |u_i|^{\beta-1}|\nabla u_i|^q \mathrm{d}x\mathrm{d}t \le C_0 \quad \text{for } i = 1, 2, \ldots. \tag{3.12}$$

with$C_0 = C_0(T, \|u_0\|_1)$.

Lemma 3.3 Assume (H_1)-(H_4). Then, for any $T > 0$, the solution $u(t)$ of (3.1) also satisfies the following estimates:

$$\|u(t)\|_\infty \le C_0 t^{-\lambda}, \qquad 0 < t \le T, \tag{3.13}$$

where $\lambda = N(mN + m + 2)^{-1}$, $C_0 = C_0(T, \|u_0\|_1)$.

Proof Multiplying the equation in (3.1) by u^{k-1}, $k \ge 2$, we have

$$\begin{aligned}&\frac{1}{k}\frac{d}{\mathrm{d}t}\|u(t)\|_k^k + (k-1)\left(\frac{m+2}{k+2}\right)^{m+2}\left\|\nabla u^{\frac{k+m}{m+2}}\right\|_{m+2}^{m+2} + \int_\Omega |u|^{\beta+k-2}|\nabla u|^q \mathrm{d}x \\ &\quad \le \int_\Omega |u|^{\alpha+k-2}|\nabla u|^p \mathrm{d}x + K_1\int_\Omega |u|^{\nu+k}\mathrm{d}x.\end{aligned} \tag{3.14}$$

It follows from the Hӧlder and Sobolev inequalities that

$$\begin{aligned}K_1\int_\Omega |u|^{\nu+k}\mathrm{d}x &\le C\|u\|_k^{\theta_1}\|u\|_1^{\theta_2}\|u\|_s^{\theta_3} \le C\|u\|_k^{\theta_1}\left\|\nabla u^{\frac{k+m}{m+2}}\right\|_{m+2}^{\frac{(m+2)\theta_3}{k+m}} \\ &\le \frac{k-1}{2}\left(\frac{m+2}{k+2}\right)^{m+2}\left\|\nabla u^{\frac{k+m}{m+2}}\right\|_{m+2}^{m+2} + Ck^\sigma\|u\|_k^k,\end{aligned}$$

in which $\theta_1 = k\lambda(m - \nu + (m + 2)N^{-1})$, $\theta_2 = \nu\lambda(m + 2)N^{-1}$, $\theta_3 = \nu\lambda(k + m)$, $\sigma = \nu\lambda$, $s = N(k + m)(N - m - 2)^{-1}$.

Note that

$$\int_\Omega |u|^{\alpha+k-2}|\nabla u|^p \mathrm{d}x \le \frac{1}{4}\int_\Omega |u|^{\beta+k-2}|\nabla u|^q \mathrm{d}x + C\int_\Omega |u|^{\mu+k-2}\mathrm{d}x$$

and

$$\frac{1}{2}\int_\Omega |u|^{\beta+k-2}|\nabla u|^q \mathrm{d}x \quad \ge C_1 k^{-q}\int_\Omega \left|\nabla u^{\frac{q+\beta+k-2}{q}}\right|^q \mathrm{d}x$$

with some C_1 independent of k and $\mu = (q\alpha - p\beta)(q - p)^{-1} < q + \beta$.

Without loss of generality, we assume $k > 3 - \mu$. Similarly, we derive

$$C\int_\Omega |u|^{\mu+k-2}\mathrm{d}x \le C\,\|u\|_{k-2}^{\mu_1}\,\|u\|_1^{\mu_2}\,\|u\|_{k^*}^{\mu_3} \le C\xi_1^{\mu_2}\,\|u\|_k^{\mu_1}\,\|u\|_{k^*}^{\mu_3}$$

$$\le C\,\|u\|_k^{\mu_1}\left\|\nabla u^{q_k/q}\right\|_q^{q\mu_3/q_k} \equiv A_k$$

with $\xi_1 = \sup_{t\ge 0}\|u(t)\|_1$ and

$$\mu_1 = \lambda_0(k-2)(q+\beta-\mu+qN^{-1}),\ \ \mu_2 = \lambda_0\mu qN^{-1},\ \ \mu_3 = \lambda_0\mu q_k,$$
$$\lambda_0 = (q+\beta+q/N)^{-1},\ \ k^* = q_kN(N-q)^{-1},\ \ q_k = q+\beta+k-2.$$

Then, for any $\eta > 0$,

$$A_k \le C\eta\left\|\nabla u^{q_k/q}\right\|_q^q + C\eta^{-\theta'/\theta}\,\|u\|_k^{\mu_1\theta'} \tag{3.15}$$

with $\mu\lambda_0\theta = 1$, $(1 - \mu\lambda_0)\theta' = 1$.

Note that $\mu_1\theta' < k$. Let $\eta = \frac{C_1}{2C}k^{-q}$. Then it follows from (3.15) that

$$A_k \le \frac{C_1}{2}k^{-q}\left\|\nabla u^{q_k/q}\right\|_q^q + Ck^\gamma(\|u\|_k^k + 1) \tag{3.16}$$

with $\gamma = q\theta'\theta^{-1} = q\mu\lambda_0/(1 - \mu\lambda_0)$. Then, (3.14) becomes

$$\frac{1}{k}\frac{d}{\mathrm{d}t}\|u\|_k^k + \frac{k-1}{2}\left(\frac{m+2}{k+2}\right)^{m+2}\left\|\nabla u^{\frac{k+m}{m+2}}\right\|_{m+2}^{m+2} + \frac{C_1}{2}k^{-q}\left\|\nabla u^{q_k/q}\right\|_q^q \le Ck^{\sigma_0}(\|u\|_k^k + 1)$$

or

$$\frac{d}{\mathrm{d}t}\|u\|_k^k + C_1k^{-m}\left\|\nabla u^{\frac{k+m}{m+2}}\right\|_{m+2}^{m+2} \le Ck^{1+\sigma_0}(\|u\|_k^k + 1) \tag{3.17}$$

with $\sigma_0 = \max\{\sigma, \gamma\} = \max\{\nu\lambda, \gamma\}$.

Now we employ an improved Moser's technique as in [8,9]. Let $\{k_n\}$ be a sequence defined by $k_1 = 1$, $k_n = R^{n-2}(R - m - 1) + m(R - 1)^{-1}(n = 2, 3, \ldots)$ with $R > \max\{m + 1, m + 4 - \mu\}$ such that $k_n \ge 3 - \mu(n \ge 2)$. Obviously, $k_n \to \infty$ as $n \to \infty$.

By Lemma 2.9, we have

$$\|u(t)\|_{k_n} \le C_0^{\frac{m+2}{m+k_n}}\,\|u(t)\|_{k_{n-1}}^{1-\theta_n}\left\|\nabla u^{\frac{m+k_n}{m+2}}\right\|_{m+2}^{\frac{\theta_n(m+2)}{m+k_n}} \tag{3.18}$$

with $\theta_n = RN(1 - k_{n-1}k_n^{-1})(m + 2 + N(R-1))^{-1}$.

Then, inserting (3.18) into (3.17) ($k = k_n$), we find that

$$\frac{d}{dt}\left\|u(t)\right\|_{k_n}^{k_n} + C_1 C_0^{-\frac{m+2}{\theta_n}} k_n^{-m}\left\|u(t)\right\|_{k_{n-1}}^{(1-1/\theta_n)(m+k_n)}\left\|u(t)\right\|_{k_n}^{(m+k_n)/\theta_n} \le C k_n^{1+\sigma_0}\left(\left\|u(t)\right\|_{k_n}^{k_n}+1\right), \quad 0 < t \le T, \tag{3.19}$$

or

$$\frac{d}{dt}\left\|u(t)\right\|_{k_n}^{k_n} + C_1 C_0^{-\frac{m+2}{\theta_n}} k_n^{-m}\left\|u(t)\right\|_{k_{n-1}}^{m-\beta_n}\left\|u(t)\right\|_{k_n}^{k_n+\beta_n} \le C k_n^{1+\sigma_0}\left(\left\|u(t)\right\|_{k_n}^{k_n}+1\right), \tag{3.20}$$

where $\beta_n = (m + k_n)\theta_n^{-1} - k_n,\ n = 2, 3, \ldots$. It is easy to see that

$$\theta_n \to \theta_0 = \frac{N(R-1)}{m+2+N(R-1)}, \quad \beta_n k_n^{-1} \to \frac{m+2}{N(R-1)}, \quad \text{as } n \to \infty.$$

Denote

$$y_n(t) = \left\|u(t)\right\|_{k_n}^{k_n}, \quad 0 < t \le T.$$

Then (3.20) can be rewritten as follows

$$y_n'(t) + C_1 C^{-\frac{m+2}{\theta_n}} k_n^{-m}(y_n(t))^{1+\beta_n/k_n}\left\|u(t)\right\|_{k_{n-1}}^{m-\beta_n} \le C k_n^{1+\sigma_0}(y_n(t)+1). \tag{3.21}$$

We claim that there exist a bounded sequence $\{\xi_n\}$ and a convergent sequence $\{\lambda_n\}$, such that

$$\left\|u(t)\right\|_{k_n} \le \xi_n t^{-\lambda_n}, \quad 0 < t \le T. \tag{3.22}$$

Indeed, by Lemma 3.1, the estimate (3.22) holds for $n = 1$ if we take $\lambda_1 = 0$, $\xi_1 = \sup_{t\ge 0}\|u(t)\|_1$. If (3.22) is true for $n - 1$, then we have from (3.21) and (3.22) that

$$y_n'(t) + C_1 C^{-\frac{m+2}{\theta_n}} k_n^{-m}(\xi_{n-1})^{m-\beta_n} t^{\Lambda_n\tau_n - 1} y_n^{1+\tau_n}(t) \le C k_n^{1+\sigma_0}(y_n(t)+1), \quad 0 \le t \le T, \tag{3.23}$$

where

$$\tau_n = \frac{\beta_n}{k_n}, \quad \Lambda_n = k_n\lambda_n, \quad \lambda_n = \frac{1+\lambda_{n-1}(\beta_n - m)}{\beta_n}.$$

Applying Lemma 2.10 to (3.23), we have

$$y_n(t) \le \left(C_1 C^{-\frac{m+2}{\theta_n}} k_n^{-m}\xi_{n-1}^{m-\beta_n}\right)^{-1/\tau_n}\left(2k_n\lambda_n + 2CTk_n^{1+\sigma_0}\right)^{1/\tau_n} t^{-k_n\lambda_n}. \tag{3.24}$$

This implies that for $t \in (0, T)$,

$$\left\|u(t)\right\|_{k_n} \le \left(C_1 C^{-\frac{m+2}{\theta_n}} k_n^{-m}\xi_{n-1}^{m-\beta_n}\right)^{-1/\beta_n}\left(2k_n\lambda_n + 2CTk_n^{1+\sigma_0}\right)^{1/\beta_n} t^{-\lambda_n} \le \xi_n t^{-\lambda_n}, \tag{3.25}$$

where

$$\xi_n = \xi_{n-1}\left(C_1 C^{-\frac{m+2}{\theta_n}} k_n^{-m}\right)^{-1/\beta_n}\left(2k_n\lambda_n + 4CTk_n^{1+\sigma_0}\right)^{1/\beta_n}, \tag{3.26}$$

in which the fact $k_n \sim \beta_n$ as $n \to \infty$ has been used.

It is not difficult to show that $\{\xi_n\}$ is bounded. Furthermore, by Lemma 4 in [9], we have

$$\frac{1+\lambda_{n-1}(\beta_n - m)}{\beta_n} \to \lambda = \frac{N}{m+2+mN}, \quad \text{as } n \to \infty.$$

Letting $n \to \infty$ in (3.22) implies that (3.13) and we finish the Proof of Lemma 3.3.

Lemma 3.4. Let (H_1)-(H_4) hold. Then, the solution $u(t)$ of (3.1) has the following estimates

$$\int_0^T s^{1+r} \|u_t(s)\|_2^2 \, ds \le C_0 \tag{3.27}$$

and

$$\|\nabla u(t)\|_{m+2} \le C_0 t^{-(1+\lambda)/(m+2)}, \qquad 0 < t \le T, \tag{3.28}$$

with $r > \lambda = N(mN + m + 2)^{-1}$, $C_0 = C_0(T, \|u_0\|_1)$.

Proof We first choose $r > \lambda$ and $\eta(t) \in C[0, \infty) \cap C^1(0, \infty)$ such that $\eta(t) = t^r$ when $t \in [0, 1]$; $\eta(t) = 2$, when $t \ge 2$ and $\eta(t)$, $\eta'(t) \ge 0$ in $[0, \infty)$. Multiplying the equation in (3.1) by $\eta(t)u$, we have

$$\begin{aligned} &\frac{1}{2}\eta(t)\|u(t)\|_2^2 + \int_0^t \eta(s)\|\nabla u(s)\|_{m+2}^{m+2} ds + \int_0^t \int_\Omega |u|^\beta |\nabla u|^q \eta(s) dx ds \\ &\le \frac{1}{2}\int_0^t \eta'(s)\|u(s)\|_2^2 ds + \int_0^t \int_\Omega |u|^\alpha |\nabla u|^p \eta(s) dx ds + K_1 \int_0^t \int_\Omega |u|^{2+\nu} \eta(s) dx ds. \end{aligned} \tag{3.29}$$

Note that

$$\int_0^t \int_\Omega |u|^\alpha |\nabla u|^p \eta(s) dx ds \le \frac{1}{2}\int_0^t \int_\Omega |u|^\beta |\nabla u|^q \eta(s) dx ds + C \int_0^t \int_\Omega |u|^\mu \eta(s) dx ds.$$

Hence, we have

$$\begin{aligned} &\frac{1}{2}\eta(t)\|u(t)\|_2^2 + \int_0^t \eta(s)\|\nabla u(s)\|_{m+2}^{m+2} ds + \frac{1}{2}\int_0^t \int_\Omega |u|^\beta |\nabla u|^q \eta(s) dx ds \\ &\le \frac{1}{2}\int_0^t \eta'(s)\|u(s)\|_2^2 ds + C\int_0^t \int_\Omega |u|^\mu \eta(s) dx ds + K_1 \int_0^t \int_\Omega |u|^{2+\nu} \eta(s) dx ds. \end{aligned} \tag{3.30}$$

By Lemma 3.1 and the estimate (3.13), we get

$$\int_0^t \eta'(s)\|u(s)\|_2^2 ds \le C \int_0^t s^{r-1}\|u(t)\|_1 \|u(t)\|_\infty ds \le C t^{r-\lambda}, \quad 0 \le t < T. \tag{3.31}$$

Since $\mu < q + \beta$, we have from Sobolev inequality that

$$C\int_0^t\int_\Omega |u|^\mu \eta(s)\mathrm{d}x\mathrm{d}s \le \frac{1}{4}\int_0^t\int_\Omega |u|^\beta |\nabla u|^q \eta(s)\mathrm{d}x\mathrm{d}s + C\int_0^t \eta(s)\mathrm{d}s. \tag{3.32}$$

Similarly, we have from $2 + \nu < q + \beta$ that

$$K_1\int_0^t\int_\Omega |u|^{2+\nu} \eta(s)\mathrm{d}x\mathrm{d}s \le \frac{1}{4}\int_0^t\int_\Omega |u|^\beta |\nabla u|^q \eta(s)\mathrm{d}x\mathrm{d}s + C\int_0^t \eta(s)\mathrm{d}s. \tag{3.33}$$

Therefore, it follows from (3.30)-(3.33) that

$$\int_0^t\int_\Omega |\nabla u|^{m+2} \eta(s)\mathrm{d}x\mathrm{d}s \le Ct^{r-\lambda}, \quad 0 \le t \le T. \tag{3.34}$$

Next, let $G(u) = \int_0^u g(s)\mathrm{d}s, u \in \mathbb{R}^1, \rho(t) = \int_0^t \eta(s)\mathrm{d}s, t \in (0, \infty)$. Furthermore, multiplying the equation in (3.1) by $\rho(t)u_t$ yields

$$\begin{aligned}
&\rho(t)\left\|u_t(t)\right\|_2^2 + \frac{1}{m+2}\frac{d}{\mathrm{d}t}\int_\Omega \rho(t)(|\nabla u|^2 + i^{-1})^{\frac{m+2}{2}}\mathrm{d}x + \rho'(t)\int_\Omega \mathrm{G}(u)\mathrm{d}x \\
&\le \frac{\rho'(t)}{m+2}\int_\Omega (|\nabla u|^2 + i^{-1})^{\frac{m+2}{2}}\mathrm{d}x + \frac{d}{\mathrm{d}t}\int_\Omega \rho(t)G(u)\mathrm{d}x \\
&+ \int_\Omega \rho(t)|u|^{\beta-1}|u_t||\nabla u|^q\mathrm{d}x + \int_\Omega \rho(t)|u|^{\alpha-1}|u_t||\nabla u|^p\mathrm{d}x.
\end{aligned} \tag{3.35}$$

By the assumption $p < q$ and the Cauchy inequality, we deduce

$$\int_\Omega |u|^{\beta-1}|u_t||\nabla u|^q\mathrm{d}x \le \frac{1}{4}\left\|u_t(t)\right\|_2^2 + C\int_\Omega |u|^{2(\beta-1)}|\nabla u|^{2q}\mathrm{d}x \tag{3.36}$$

and

$$\begin{aligned}
\int_\Omega |u|^{\alpha-1}|u_t||\nabla u|^p\mathrm{d}x &\le \frac{1}{4}\left\|u_t(t)\right\|_2^2 + C\int_\Omega |u|^{2(\alpha-1)}|\nabla u|^{2p}\mathrm{d}x \\
&\le \frac{1}{4}\left\|u_t(t)\right\|_2^2 + C\int_\Omega |u|^{2(\beta-1)}|\nabla u|^{2q}\mathrm{d}x + C\int_\Omega |u|^{2(\mu-1)}\mathrm{d}x
\end{aligned} \tag{3.37}$$

and

$$\int_\Omega \left|G(u)\right|\mathrm{d}x \le C_1\int_\Omega |u|^{2+\nu}\mathrm{d}x \le Ch^{2+\nu}(t) \tag{3.38}$$

with $h(t) = \|u(t)\|_\infty$.

Now, it follows from (H_4) and (3.35)-(3.38) that

$$\begin{aligned}\frac{1}{2}\int_0^t \rho(s)\,\|u_t(s)\|_2^2\,ds + \frac{1}{m+2}\rho(t)\,\|\nabla u(t)\|_{m+2}^{m+2} &\le \frac{1}{2}\int_0^t \eta(s)\,\|\nabla u(s)\|_{m+2}^{m+2}\,\mathrm{d}s + C\rho(t)h^{2+\nu}(t)\\ &+ C\int_0^t \rho(s)h^{2(\beta-1)}(s)(1+\|\nabla u(s)\|_{m+2}^{m+2})\mathrm{d}s + C\int_0^t (\rho(s)h^{2(\mu-1)}(s)+\eta(s)h^{2+\nu}(s))\mathrm{d}s\\ &\le C(t^{r-\lambda}+t^{r+2-2(\beta-1)\lambda}+t^{r+2-2(\mu-1)\lambda}+t^{r+1-(2+\nu)\lambda})\\ &+ C\int_0^t \rho(s)h^{2(\beta-1)}(s)\,\|\nabla u(s)\|_{m+2}^{m+2}\,\mathrm{d}s,\end{aligned} \tag{3.39}$$

or

$$\begin{aligned}&\frac{1}{2}\int_0^t \rho(s)\,\|u_t(s)\|_2^2\,\mathrm{d}s + \frac{\rho(t)}{m+2}\,\|\nabla u(t)\|_{m+2}^{m+2}\\ &\le C_0 t^{r-\lambda} + C_0\int_0^t \rho(s)h^{2(\beta-1)}(s)\,\|\nabla u(s)\|_{m+2}^{m+2}\,\mathrm{d}s\end{aligned} \tag{3.40}$$

where $C_0 = C_0(T, \|u_0\|_1)$ and the fact $2 + \lambda \ge 2(\mu - 1)\lambda$ has been used.

Since the function $h^{2(\beta-1)}(t) \in L^1([0, T])$, the application of the Gronwall inequality to (3.40) gives

$$\int_0^t \rho(s)\,\|u_t(s)\|_2^2\,\mathrm{d}s + \rho(t)\,\|\nabla u\|_{m+2}^{m+2} \le C_0 t^{r-\lambda}, \quad 0 < t \le T. \tag{3.41}$$

Hence,

$$\|\nabla u\|_{\mathrm{m+2}} \le C_0 t^{-(1+\lambda)/(m+2)}, \quad 0 < t \le T. \tag{3.42}$$

and the Proof of Lemma 3.4 is completed.

Proof of Theorem 2.4 Noticing that the estimate constant C_0 in (3.12)-(3.13) and (3.27)-(3.28) is independent of i, we have from the standard compact argument as in [1,13,14] that there exists a subsequence (still denoted by u_i) and a function $u \in L^s([0,T], W_0^{1,s}(\Omega)), (1 \le s \le m+2)$ satisfying

$$\begin{aligned}&u_i \rightharpoonup u \quad \text{weakly in } L^s([0,T], W_0^{1,s}(\Omega)),\\ &u_i \to u \quad \text{in } L^s(Q_T) \text{ and a.e. in } Q_T,\\ &|u_i|^{\beta-1}|\nabla u_i|^q \to |u|^{\beta-1}|\nabla u|^q \quad \text{in } L^1(Q_T),\\ &|u_i|^{\alpha-1}|\nabla u_i|^p \to |u|^{\alpha-1}|\nabla u|^p \quad \text{in } L^1(Q_T),\\ &u_i \to u \quad \text{in } C([0,T];\ L^1(\Omega)),\\ &\frac{\partial u_i}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{weakly in } L^2_{\mathrm{loc}}(0,T;L^2).\end{aligned} \tag{3.43}$$

Since $A_i(u_i) = -\mathrm{div}((|\nabla u_i|^2 + i^{-1})^{\frac{m}{2}}\nabla u_i)$ is bounded in $(W_0^{1,m+2})^* = W_0^{-1,\frac{m+2}{m+1}}$, we see further that

$$A_i(u_i) \to \chi \quad \text{weakly} * \text{ in } L^\infty_{loc}(0,T;(W_0^{1,m+2})*) \tag{3.44}$$

for some $\chi \in L^{\infty}_{\text{loc}}(0, T], (W_0^{1,m+2})*)$. As the Proof of Theorem 1 in [9], we have $\chi = A(u) = -\text{div}((\|\nabla u\|^m \nabla u)$.

Then, the function u is a global weak solution of (1.1). Furthermore, it follows from Lemma 3.4 that $u(t)$ satisfies the estimate (2.4)-(2.5). The Proof of Theorem 2.4 is now completed.

4 L^{∞} estimate for $\nabla u(t)$

In this section, we use an argument similar to that in [9,10,15] and give the Proof of Theorem 2.5. Hence, we only consider the estimate of $\|\nabla u\|_{\infty}$ for the smooth solution $u(t)$ of (3.1). As above, let C, C_j be the generic constants independent of k and i. Denote

$$\left|D^2 u\right|^2 = \sum_{i,j=1}^{N} u_{ij}^2, \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Multiplying (3.1) by $-\text{div}(|\nabla u|^{k-2}\nabla u)$, $k \geq m + 2$ and integrating by parts, we have

$$\begin{aligned} &\frac{1}{k}\frac{d}{\mathrm{d}t}\left(\|\nabla u(t)\|_k^k\right) + \int_{\Omega} |\nabla u|^{k+m-2}\left|D^2 u\right|^2 \mathrm{d}x + \frac{k-2}{4}\int_{\Omega} |\nabla u|^{k+m-4}\left|\nabla(|\nabla u|^2)\right|^2 \mathrm{d}x \\ &-(N-1)\int_{\partial\Omega} H(x)|\nabla u|^{k+m}\mathrm{d}S \\ &= \int_{\Omega} u|u|^{\beta-2}|\nabla u|^q \text{div}(|\nabla u|^{k-2}\nabla u)\mathrm{d}x - \int_{\Omega} u\,|\nabla u|^p |u|^{\alpha-2}\text{div}(|\nabla u|^{k-2}\nabla u)\mathrm{d}x \\ &+ \int_{\Omega} g(u)\text{div}(|\nabla u|^{k-2}\nabla u)\mathrm{d}x \equiv I + II + III. \end{aligned} \tag{4.1}$$

Since

$$\text{div}(|\nabla u|^{k-2}\nabla u) = |\nabla u|^{k-2}\Delta u + \frac{k-2}{2}|\nabla u|^{k-4}\nabla u \nabla(|\nabla u|^2), \tag{4.2}$$

we have

$$\left|\text{div}(|\nabla u|^{k-2}\nabla u)\right| \leq (k-1)|\nabla u|^{k-2}\left|D^2 u\right| \tag{4.3}$$

and

$$\begin{aligned} |I| &\leq (k-1)\int_{\Omega} |u|^{\beta-1}|\nabla u|^{q+k-2}\left|D^2 u\right| \mathrm{d}x \\ &= (k-1)\int_{\Omega} |\nabla u|^{\frac{k+m-2}{2}}\left|D^2 u\right| |\nabla u|^{\frac{k+2q-m-2}{2}}|u|^{\beta-1}\mathrm{d}x \\ &\leq \frac{1}{4}\int_{\Omega} |\nabla u|^{k+m-2}\left|D^2 u\right|^2 \mathrm{d}x + C_0 k^2 \int_{\Omega} |\nabla u|^{k+2q-m-2}|u|^{2(\beta-1)}\mathrm{d}x. \end{aligned} \tag{4.4}$$

Similarly, we obtain the following estimates

$$|II| \leq \frac{1}{4}\int_{\Omega} |\nabla u|^{m+k-2}\left|D^2 u\right|^2 \mathrm{d}x + C_0 k^2 \int_{\Omega} |\nabla u|^{k+2p-m-2}|u|^{2(\alpha-1)}\mathrm{d}x \tag{4.5}$$

and

$$III = \int_\Omega g(u)\mathrm{div}(|\nabla u|^{k-2}\nabla u)\mathrm{d}x = -\int_\Omega g'(u)|\nabla u|^k \mathrm{d}x \\ \leq K_1 \int_\Omega |y|^\nu |\nabla u|^k \mathrm{d}x \leq Ch^\nu(t)\,\|\nabla u(t)\|_k^k, \tag{4.6}$$

where $h(t) = \|u(t)\|_\infty \leq Ct^{-\lambda}$.

Moreover, we assume that $2q \leq m + 2$, $2p \leq m + 2$, then (4.1) becomes

$$\frac{1}{k}\frac{d}{\mathrm{d}t}\left(\|\nabla u\|_k^k\right) + \frac{1}{2}\int_\Omega |\nabla u|^{k+m-2}|D^2u|^2\mathrm{d}x + \frac{k-2}{4}\int_\Omega |\nabla u|^{k+m-4}|\nabla(|\nabla u|^2)|^2\mathrm{d}x \\ -(N-1)\int_{\partial\Omega} H(x)|\nabla u|^{k+m}\mathrm{d}S \\ \leq C_0k^2\int_\Omega \left(|\nabla u|^{k+2q-m-2}|u|^{2(\beta-1)} + |\nabla u|^{k+2p-m-2}|u|^{2(\alpha-1)}\right)\mathrm{d}x + Ch^\nu(t)\,\|\nabla u(t)\|_k^k \\ \leq C_0k^2h_1(t)\left(1 + \|\nabla u(t)\|_k^k\right), \tag{4.7}$$

where $h_1(t) = \max\{h^{2(\alpha-1)}(t), h^{2(\beta-1)}(t), h^\nu(t)\}$. Since $\alpha, \beta < 2 + \frac{m}{2}\left(1 + \frac{1}{N}\right)$, $\nu < m + 2 + \frac{m}{N}$, we get $h_1(t) \in L^1([0.T])$ for any $T > 0$.

If $H(x) \leq 0$ on $\partial\Omega$ and $N > 1$, then by an argument of elliptic eigenvalue problem in [15], there exists $\lambda_1 > 0$, such that

$$\|\nabla v\|_2^2 - (N-1)\int_{\partial\Omega} v^2H(x)\mathrm{d}S \geq \lambda_1\,\|v\|_{1,2}^2, \quad \forall v \in W^{1,2}(\Omega). \tag{4.8}$$

Hence, by (4.7) and (4.8), we see that there exists C_1 and C_2 such that

$$\frac{d}{\mathrm{d}t}\left(\|\nabla u(t)\|_k^k\right) + C_1\left\||\nabla u(t)|^{\frac{k+m}{2}}\right\|_{1,2}^2 \leq Ck^3h_1(t)(1 + \|\nabla u(t)\|_k^k). \tag{4.9}$$

Let $k_1 = m + 2$, $R > m + 1$, $k_n = R^{n-2}(R-1-m) + m(R-1)^{-1}$, $\theta_n = RN(1 - k_{n-1}k_n^{-1})(R(N-1)+2)^{-1}$, $n = 2, 3, \ldots$. Then, the application of Lemma 2.9 gives

$$\|\nabla u\|_{k_n} \leq C^{\frac{2}{k_n+m}}\,\|\nabla u\|_{k_{n-1}}^{1-\theta_n}\left\||\nabla u|^{\frac{k_n+m}{2}}\right\|_{1,2}^{\frac{2\theta_n}{k_{n+m}}}. \tag{4.10}$$

Inserting this into (4.9)($k = k_n$), we get

$$\frac{d}{\mathrm{d}t}\left(\|\nabla u\|_{k_n}^{k_n}\right) + C_1C^{-2/\theta_n}\,\|\nabla u(t)\|_{k_{n-1}}^{(k_n+m)(1-1/\theta_n)}\,\|\nabla u(t)\|_{k_n}^{(k_n+m)/\theta_n} \\ \leq C_2k_n^3h_1(t)(1 + \|\nabla u(t)\|_{k_n}^{k_n}). \tag{4.11}$$

By (3.28), we take $y_1 = \max\{1, C_0\}$, $z_1 = (1 + \lambda)/(m + 2)$. As the Proof of Lemma 3.3, we can show that there exist bounded sequences y_n and z_n such that

$$\left\|\nabla u(t)\right\|_{k_n} \le \gamma_n t^{-z_n}, \quad 0 < t \le T, \tag{4.12}$$

in which $z_n \to \sigma = (2 + 2\lambda + N)(mN + 2m + 4)^{-1}$. Letting $n \to \infty$ in (4.12), we have the estimate (2.6). This completes the Proof of Theorem 2.5.

5 Asymptotic behavior of solution

In this section, we will prove that the problem (1.1) admits a global solution if the initial data $u_0(x)$ is small under the assumptions of Theorems 2.7 and 2.8. Also, we derive the asymptotic behavior of solution $u(t)$.

Proof of Theorem 2.7 The existence of solution for (1.1) in small u_0 can be obtained by a similar argument as the Proof of Theorem 2.4. So, it is sufficient to derive the estimate (2.8).

Multiplying the equation in (2.7) by u and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|u(t)\right\|_2^2 + C_1\left\|\nabla u(t)\right\|_{m+2}^{m+2} \le \int_\Omega |u|^\alpha |\nabla u|^p dx \tag{5.1}$$

with $C_1 = \left(\frac{m+2}{4}\right)^{m+2}$.

Since $p < m + 2 < p + \alpha$, it follows from Lemma 2.9 that

$$\begin{aligned}\int_\Omega |u|^\alpha |\nabla u|^p dx &\le \left\|\nabla u(t)\right\|_{m+2}^p \|u\|_s^\alpha \le C_0 \|\nabla u\|_{m+2}^p \|u\|_r^{\alpha(1-\theta)} \|\nabla u\|_{m+2}^{\alpha\theta} \\ &\le C_0 \left\|\nabla u(t)\right\|_{m+2}^{m+2} \left\|u(t)\right\|_r^{p1}\end{aligned} \tag{5.2}$$

with

$$s = \frac{\alpha(m+2)}{m+2-p}, \theta = \left(\frac{1}{r} - \frac{1}{s}\right)\left(\frac{1}{N} + \frac{1}{r} - \frac{1}{m+2}\right)^{-1}, r = \frac{Np_1}{m+2-p}, p_1 = p+\alpha-m-2.$$

The assumption on α shows that $r \le 2$. Then, (5.1) can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\left\|u(t)\right\|_2^2 + \left\|\nabla u(t)\right\|_{m+2}^{m+2}(C_1 - C_0\left\|u(t)\right\|_2^{p_1}) \le 0. \tag{5.3}$$

By the Sobolev embedding theorem,

$$\left\|\nabla u(t)\right\|_{m+2}^{m+2} \ge C_2\left\|u(t)\right\|_{m+2}^{m+2} \ge C_2\left\|u(t)\right\|_2^{m+2}, \tag{5.4}$$

we obtain from (5.3) and (5.4) that $\exists d_0 > 0$, $\lambda_0 > 0$, such that $\|u_0\|_2 < d_0$ and

$$\phi'(t) + \lambda_0\phi^{1+m/2}(t) \le 0, \quad t \ge 0 \tag{5.5}$$

with $\phi(t) = \left\|u(t)\right\|_2^2$. This implies that

$$\left\|u(t)\right\|_2 \le C(1+t)^{-1/m}, \quad t \ge 0, \tag{5.6}$$

where the constant C depends only $\|u_0\|_2$. This completes the Proof of Theorem 2.7.

Proof of Theorem 2.8 Multiplying the equation in (2.9) by u and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|u(t)\right\|_2^2 + \gamma\int_\Omega |u|^\beta|\nabla u|^q dx \le \int_\Omega |u|^\alpha|\nabla u|^p dx. \tag{5.7}$$

Since $p < q$, $q + \beta < p + \alpha$, it follows from the Hölder inequality that

$$\begin{aligned}\int_\Omega |u|^\alpha|\nabla u|^p dx &\le \left\|\nabla u^{1+\beta/q}\right\|_q^p \|u\|_\mu^{\mu(1-p/q)} \\ &\le C_1\left\|\nabla u^{1+\beta/q}\right\|_q^p \|u\|_\tau^{\mu_1(1-p/q)}\left\|\nabla u^{1+\beta/q}\right\|_q^{\mu_2(1-p/q)} \\ &\le C_1\left\|\nabla u^{1+\beta/q}\right\|_q^q \|u\|_\tau^{\mu_1(1-p/q)} \le C_1\left\|\nabla u^{1+\beta/q}\right\|_q^q \|u\|_2^{\mu_3}\end{aligned} \tag{5.8}$$

with $\mu_2 = q$, $\mu_1 = \mu - q$, $\mu_3 = \mu_1(1 - p/q)$ and $\tau = N(\mu - q)(q + \beta)(q^2 + N\beta)^{-1} \le 2$. Then (5.7) becomes

$$\frac{d}{dt}\left\|u(t)\right\|_2^2 + \left\|\nabla u^{1+\beta/q}\right\|_q^q (C_0 - C_1\left\|u(t)\right\|_2^{\mu_3}) \le 0. \tag{5.9}$$

This implies that $\exists d_0 > 0$, $\lambda_1 > 0$, such that $\|u_0\|_2 < d_0$ and

$$\phi'(t) + \lambda_1\phi^{(q+\beta)/2}(t) \le 0, \quad t \ge 0 \tag{5.10}$$

with $\phi(t) = \left\|u(t)\right\|_2^2$. This implies that

$$\left\|u(t)\right\|_2 \le C(1+t)^{-1/(q+\beta-2)}, \quad t \ge 0. \tag{5.11}$$

This is the estimate (2.10) and we finish the Proof of Theorem 2.8.

Acknowledgements
The authors wish to express their gratitude to the referees for useful comments and suggestions.

Authors' contributions
CC proposed the topic and the main ideas. The main results in this article were derived by CC. FY and ZY participated in the discussion of topic. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 10 August 2011 Accepted: 15 February 2012 Published: 15 February 2012

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doi:10.1186/1687-2770-2012-19
Cite this article as: Chen *et al.*: L^∞ estimates of solutions for the quasilinear parabolic equation with nonlinear gradient term and L^1 data. *Boundary Value Problems* 2012 **2012**:19.