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Multiple positive solutions of semilinear elliptic equations involving concave and convex nonlinearities in $\mathbb{R}^N$

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Abstract

In this article, we investigate the effect of the coefficient *f*(*z*) of the sub-critical nonlinearity. For sufficiently large $\lambda > 0$, there are at least $k + 1$ positive solutions of the semilinear elliptic equations

$$\begin{cases} -\Delta v + \lambda v = f(z)v^{p-1} + h(z)v^{q-1} \text{ in } \mathbb{R}^N; \\ v \in H^1(\mathbb{R}^N), \end{cases}$$

where $1 \le q < 2 < p < 2^* = 2N/(N - 2)$ for $N \ge 3$.
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1 Introduction

For $N \ge 3$, $1 \le q < 2 < p < 2^* = 2N/(N - 2)$, we consider the semilinear elliptic equations

$$\begin{cases} -\Delta v + \lambda v = f(z)v^{p-1} + h(z)v^{q-1} \text{ in } \mathbb{R}^N; \\ v \in H^1(\mathbb{R}^N), \end{cases} \qquad (E_\lambda)$$

where $\lambda > 0$.

Let f and h satisfy the following conditions:

(f 1) f is a positive continuous function in $\mathbb{R}^N$ and $\lim_{|z| \to \infty} f(z) = f_\infty > 0$.

(f2) there exist k points $a^1, a^2, \dots, a^k$ in $\mathbb{R}^N$ such that

$$f(a^i) = f_{\max} = \max_{z \in \mathbb{R}^N} f(z) \text{ for } 1 \le i \le k,$$

and $f_\infty < f_{\max}$.

(h 1) $h \in L^{\frac{p}{p-q}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $h \gneqq 0$.

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$$\begin{cases} -\Delta u = ch(z)|u|^{q-2}u + |u|^{p-2}u \text{ in } \Omega; \\ u \in H_0^1(\Omega), \end{cases} \qquad (E_c)$$

have been studied by Ambrosetti et al. [1] ($h \equiv 1$, and $1 < q < 2 < p \leq 2^* = 2N/(N-2)$) and Wu [2]$h \in C(\bar{\Omega})$ and changes sign, $1 < q < 2 < p < 2^*$). They proved that this equation has at least two positive solutions for sufficiently small $c > 0$. More general results of Equation (E_c) were done by Ambrosetti et al. [3], Brown and Zhang [4], and de Figueiredo et al. [5].

In this article, we consider the existence and multiplicity of positive solutions of Equation (E_λ) in $\mathbb{R}^N$. For the case $q = \lambda = 1$ and $f(z) \equiv 1$ for all $z \in \mathbb{R}^N$, suppose that h is nonnegative, small, and exponential decay, Zhu [6] showed that Equation (E_λ) admits at least two positive solutions in $\mathbb{R}^N$. Without the condition of exponential decay, Cao and Zhou [7] and Hirano [8] proved that Equation (E_λ) admits at least two positive solutions in $\mathbb{R}^N$. For the case $q = \lambda = 1$, by using the idea of category and Bahri-Li's minimax argument, Adachi and Tanaka [9] asserted that Equation (E_λ) admits at least four positive solutions in $\mathbb{R}^N$, where $f(z) ⊠ 1$, $f(z) \geq 1 - C\exp((-(2+\delta)\ |z|)$ for some C, $\delta > 0$, and sufficiently small $\|h\|_{H^{-1}} > 0$. Similarly, in Hsu and Lin [10], they have studied that there are at least four positive solutions of the general case $-\Delta u + u = f(z)v^{p-1} + \lambda h(z)\ v^{q-1}$ in $\mathbb{R}^N$ for sufficiently small $\lambda > 0$.

By the change of variables

$$\varepsilon = \lambda^{-\frac{1}{2}} \text{ and } u(z) = \varepsilon^{\frac{2}{p-2}} v(\varepsilon z),$$

Equation (E_λ) is transformed to

$$\begin{cases} -\Delta u + u = f(\varepsilon z)u^{p-1} + \varepsilon^{\frac{2(p-q)}{p-2}} h(\varepsilon z)u^{q-1} \text{ in } \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \qquad (E_\varepsilon)$$

Associated with Equation (E_ε), we consider the C^1-functional J_ε, for $u \in H^1(\mathbb{R}^N)$,

$$J_\varepsilon(u) = \frac{1}{2}\|u\|_H^2 - \frac{1}{p}\int_{\mathbb{R}^N} f(\varepsilon z)u_+^p dz - \frac{1}{q}\int_{\mathbb{R}^N} \varepsilon^{\frac{2(p-q)}{p-2}} h(\varepsilon z)u_+^q dz,$$

where $\|u\|_H^2 = \int_{\mathbb{R}^N} \left(|\Delta u|^2 + |u|^2\right) dz$ is the norm in $H^1(\mathbb{R}^N)$ and $u_+ = \max\{u, 0\} \geq 0$. We know that the nonnegative weak solutions of Equation (E_ε) are equivalent to the critical points of J_ε. This article is organized as follows. First of all, we use the argument of Tarantello [11] to divide the Nehari manifold $\mathbf{M}_\varepsilon$ into the two parts $\mathbf{M}_\varepsilon^+$ and $\mathbf{M}_\varepsilon^-$. Next, we prove that the existence of a positive ground state solution $u_0 \in \mathbf{M}_\varepsilon^+$ of Equation (E_ε). Finally, in Section 4, we show that the condition ($f2$) affects the number of positive solutions of Equation (E_ε), that is, there are at least k critical points $u_1, \ldots, u_k \in \mathbf{M}_\varepsilon^-$ of J_ε such that $J_\varepsilon(u_i) = \beta_\varepsilon^i((\text{PS}) - \text{value})$ for $1 \leq i \leq k$.

Let

$$S = \sup_{\substack{u \in H^1(\mathbb{R}^N) \\ \|u\|_H = 1}} \|u\|_{L^p},$$

then

$$\|u\|_{L^p} \leq S\|u\|_H \quad \text{for any } u \in H^1(\mathbb{R}^N)\backslash\{0\}. \qquad (1.1)$$

For the semilinear elliptic equations

$$\begin{cases} -\Delta u + u = f(\varepsilon z)u^{p-1} \text{ in } \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{E0}$$

we define the energy functional $I_\varepsilon(u) = \frac{1}{2}\|u\|_H^2 - \frac{1}{p}\int_{\mathbb{R}^N} f(\varepsilon z)u_+^p dz$, and

$$\gamma_\varepsilon = \inf_{u \in \mathbf{N}_\varepsilon} I_\varepsilon(u),$$

where $\mathbf{N}_\varepsilon = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid u_+ \boxtimes 0 \text{ and } \langle I'_\varepsilon(u), u\rangle = 0\}$. Note that

(*i*) if $f \equiv f_\infty$, we define $I_\infty(u) = \frac{1}{2}\|u\|_H^2 - \frac{1}{p}\int_{\mathbb{R}^N} f_\infty u_+^p d_z$ and

$$\gamma_\infty = \inf_{u \in \mathbf{N}_\infty} I_\infty(u),$$

where $\mathbf{N}_\infty = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid u_+ \boxtimes 0 \text{ and } \langle I'_\infty(u), u\rangle = 0\}$;

(*ii*) if $f \equiv f_{\max}$, we define $I_{\max}(u) = \frac{1}{2}\|u\|_H^2 - \frac{1}{p}\int_{\mathbb{R}^N} f_{\max} u_+^p dz$ and

$$\gamma_{\max} = \inf_{u \in \mathbf{N}_{\max}} I_{\max}(u),$$

where $\mathbf{N}_{\max} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid u_+ \boxtimes 0 \text{ and } \langle I'_{\max}(u), u\rangle = 0\}$.

Lemma 1.1

$$\gamma_{\max} = \frac{p-2}{2p}(f_{\max}S^p)^{-2/(p-2)} > 0.$$

Proof. It is similar to Theorems 4.12 and 4.13 in Wang [[12], p. 31].

Our main results are as follows.

(*I*) Let $\Lambda = \varepsilon^{2(p-q)/(p-2)}$. Under assumptions ($f$ 1) and (h1), if

$$0 < \Lambda < \Lambda_0 = (p-2)\left(\frac{2-q}{f_{\max}}\right)^{\frac{2-q}{p-2}}\left[(p-q)S^2\right]^{\frac{q-p}{p-2}}\|h\|_\#^{-1},$$

where $\|h\|_\#$ is the norm in $L^{\frac{p}{p-q}}(\mathbb{R}^N)$, then Equation ($E_\varepsilon$) admits at least a positive ground state solution. (See Theorem 3.4)

(*II*) Under assumptions (f1) - (f2) and (h1), if λ is sufficiently large, then Equation (E_λ) admits at least $k + 1$ positive solutions. (See Theorem 4.8)

2 The Nehari manifold

First of all, we define the Palais-Smale (denoted by (PS)) sequences and (PS)-conditions in $H^1(\mathbb{R}^N)$ for some functional J.

Definition 2.1 (*i*) *For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_\beta$-sequence in $H^1(\mathbb{R}^N)$ for J if $J(u_n) = \beta + o_n(1)$ and $J'(u_n) = o_n(1)$ strongly in $H^{-1}(\mathbb{R}^N)$ as $n \to \infty$, where $H^{-1}(\mathbb{R}^N)$ is the dual space of $H^1(\mathbb{R}^N)$;*

(*ii*) *J satisfies the* $(PS)_\beta$*-condition in* $H^1(\mathbb{R}^N)$ *if every* $(PS)_\beta$*-sequence in* $H^1(\mathbb{R}^N)$ *for J contains a convergent subsequence.*

Next, since J_ε is not bounded from below in H^1 $(\mathbb{R}^N)$, we consider the Nehari manifold

$$\mathbf{M}_\varepsilon = \left\{u \in H^1(\mathbb{R}^N)\backslash\{0\} \,\middle|\, u_+ \not\equiv 0 \text{ and } \langle J'_\varepsilon(u), u\rangle = 0\right\}, \tag{2.1}$$

where

$$\langle J'_\varepsilon(u), u\rangle = \|u\|_H^2 - \int_{\mathbb{R}^N} f(\varepsilon z)u_+^p dz - \int_{\mathbb{R}^N} \varepsilon^{\frac{2(p-q)}{p-2}} h(\varepsilon z)u_+^q dz.$$

Note that $\mathbf{M}_\varepsilon$ contains all nonnegative solutions of Equation (E_ε). From the lemma below, we have that J_ε is bounded from below on $\mathbf{M}_\varepsilon$.

Lemma 2.2 *The energy functional* J_ε *is coercive and bounded from below on* $\mathbf{M}_\varepsilon$.

Proof. For $u \in \mathbf{M}_\varepsilon$, by (2.1), the Hölder inequality $\left(p_1 = \frac{p}{p-q}, p_2 = \frac{p}{q}\right)$ and the Sobolev embedding theorem (1.1), we get

$$\begin{aligned} J_\varepsilon(u) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_H^2 - \left(\frac{1}{q} - \frac{1}{p}\right)\int_{\mathbb{R}^N} \varepsilon^{\frac{2(p-q)}{p-2}} h(\varepsilon z)u_+^q dz \\ &\geq \frac{\|u\|_H^q}{p}\left[\frac{p-2}{2}\|u\|_H^{2-q} - \frac{p-q}{q}\varepsilon^{\frac{2(p-q)}{p-2}}\|h\|_\# S^q\right]. \end{aligned}$$

Hence, we have that J_ε is coercive and bounded from below on $\mathbf{M}_\varepsilon$.

Define

$$\psi_\varepsilon(u) = \langle J'_\varepsilon(u), u\rangle.$$

Then for $u \in \mathbf{M}_\varepsilon$, we get

$$\begin{aligned} \langle \psi'_\varepsilon(u), u\rangle &= 2\|u\|_H^2 - p\int_{\mathbb{R}^N} f(\varepsilon z)u_+^p dz - q\int_{\mathbb{R}^N} \varepsilon^{\frac{2(p-q)}{p-2}} h(\varepsilon z)u_+^q dz \\ &= (p-q)\int_{\mathbb{R}^N} \varepsilon^{\frac{2(p-q)}{p-2}} h(\varepsilon z)u_+^q dz - (p-2)\|u\|_H^2 \end{aligned} \tag{2.2}$$

$$= (2-q)\|u\|_H^2 - (p-q)\int_{\mathbb{R}^N} f(\varepsilon z)u_+^p dz. \tag{2.3}$$

We apply the method in Tarantello [11], let

$$\mathbf{M}_\varepsilon^+ = \{u \in \mathbf{M}_\varepsilon \,|\, \langle \psi'_\varepsilon(u), u\rangle > 0\};$$
$$\mathbf{M}_\varepsilon^0 = \{u \in \mathbf{M}_\varepsilon \,|\, \langle \psi'_\varepsilon(u), u\rangle = 0\};$$
$$\mathbf{M}_\varepsilon^- = \{u \in \mathbf{M}_\varepsilon \,|\, \langle \psi'_\varepsilon(u), u\rangle < 0\}.$$

Lemma 2.3 *Under assumptions (f1) and (h1), if* $0 < \Lambda \ (= \varepsilon^{2(p-q)/(p-2)}) < \Lambda_0$, *then* $\mathbf{M}_\varepsilon^0 = \emptyset$.

Proof. See Hsu and Lin [[10], Lemma 5].

Lemma 2.4 *Suppose that u is a local minimizer for* J_ε *on* $\mathbf{M}_\varepsilon$ *and* $u \notin \mathbf{M}_\varepsilon^0$. *Then* $J'_\varepsilon(u) = 0$ *in* $H^{-1}(\mathbb{R}^N)$.

Proof. See Brown and Zhang [[4], Theorem 2.3].

Lemma 2.5 *We have the following inequalities.*

(*i*) $\int_{\mathbb{R}^N} h(\varepsilon z)u_+^q dz > 0$ *for each* $u \in \mathbf{M}_\varepsilon^+$;

(*ii*) $\|u\|_H < \left(\frac{p-q}{p-2}\Lambda\|h\|_\# S^q\right)^{1/(2-q)}$ *for each* $u \in \mathbf{M}_\varepsilon^+$;

(*iii*) $\|u\|_H > \left[\frac{2-q}{(p-q)f_{\max}S^p}\right]^{1/(p-2)}$ *for each* $u \in \mathbf{M}_\varepsilon^-$;

(*iv*) *If* $0 < \Lambda \left(= \varepsilon^{2(p-q)/(p-2)}\right) < \frac{q\Lambda_0}{2}$, *then* $J_\varepsilon(u) > 0$ *for each* $u \in \mathbf{M}_\varepsilon^-$.

Proof. (*i*) It can be proved by using (2.2).

(*ii*) For any $u \in \mathbf{M}_\varepsilon^+ \subset \mathbf{M}_\varepsilon$, by (2.2), we apply the Hölder inequality $(p_1 = \frac{p}{p-q}, p_2 = \frac{p}{q})$ to obtain that

$$\begin{aligned} 0 &< (p-q)\int_{\mathbb{R}^N} \Lambda h(\varepsilon z)u_+^q dz - (p-2)\|u\|_H^2 \\ &\leq (p-q)\Lambda\|h\|_\# S^q \|u\|_H^q - (p-2)\|u\|_H^2. \end{aligned}$$

(*iii*) For any $u \in \mathbf{M}_\varepsilon^-$, by (2.3), we have that

$$\|u\|_H^2 < \frac{p-q}{2-q}\int_{\mathbb{R}^N} f(\varepsilon z)u_+^p dz \leq \frac{p-q}{2-q} S^p \|u\|_H^p f_{\max}.$$

(*iv*) For any $u \in \mathbf{M}_\varepsilon^- \subset \mathbf{M}_\varepsilon$, by (*iii*), we get that

$$\begin{aligned} J_\varepsilon(u) &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|_H^2 - \left(\frac{1}{q}-\frac{1}{p}\right)\int_{\mathbb{R}^N}\Lambda h(\varepsilon z)u_+^q dz \\ &\geq \frac{\|u\|_H^q}{p}\left[\frac{p-2}{2}\|u\|_H^{2-q} - \frac{p-q}{q}\Lambda\|h\|_\# S^q\right] \\ &> \frac{1}{p}\left[\frac{2-q}{(p-q)f_{\max}S^p}\right]^{\frac{q}{p-2}}\left[\frac{p-2}{2}\left[\frac{2-q}{(p-q)f_{\max}S^p}\right]^{\frac{2-q}{p-2}} - \frac{p-q}{q}\Lambda\|h\|_\# S^q\right]. \end{aligned}$$

Thus, if $0 < \Lambda < \frac{q}{2}(p-2)\left(\frac{2-q}{f_{\max}}\right)^{\frac{2-q}{p-2}}\left[(p-q)S^2\right]^{\frac{q-p}{p-2}}\|h\|_\#^{-1}$, we get that $J_\varepsilon(u) \geq d_0 > 0$ for some constant $d_0 = d_0(\varepsilon, p, q, S, \|h\|_\#, f_{\max})$.

For $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ and u_+ ⊠ 0, let

$$\bar{t} = \bar{t}(u) = \left[\frac{(2-q)\|u\|_H^2}{(p-q)\int_{\mathbb{R}^N} f(\varepsilon z)u_+^p dz}\right]^{1/(p-2)} > 0.$$

Lemma 2.6 *For each* $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ *and* $u_+ \not\equiv 0$, *we have that*

(*i*) *if* $\int_{\mathbb{R}^N} h(\varepsilon z) u_+^q dz = 0$, *then there exists a unique positive number* $t^- = t^-(u) > \bar{t}$ *such that* $t^- u \in \mathbf{M}_\varepsilon^-$ *and* $J_\varepsilon(t^- u) = \sup_{t \geq 0} J_\varepsilon(tu)$;

(*ii*) *if* $0 < \Lambda\ (= \varepsilon^{2(p-q)/(p-2)}) < \Lambda_0$ *and* $\int_{\mathbb{R}^N} h(\varepsilon z) u_+^q dz > 0$, *then there exist unique positive numbers* $t^+ = t^+(u) < \bar{t} < t^- = t^-(u)$ *such that* $t^+ u \in \mathbf{M}_\varepsilon^+$, $t^- u \in \mathbf{M}_\varepsilon^-$ *and*

$$J_\varepsilon(t^+ u) = \inf_{0 \leq t \leq \bar{t}} J_\varepsilon(tu), \quad J_\varepsilon(t^- u) = \sup_{t \geq \bar{t}} J_\varepsilon(tu).$$

Proof. See Hsu and Lin [[10], Lemma 7].

Applying Lemma 2.3 $(\mathbf{M}_\varepsilon^0 = \emptyset \text{ for } 0 < \Lambda < \Lambda_0)$, we write $\mathbf{M}_\varepsilon = \mathbf{M}_\varepsilon^+ \cup \mathbf{M}_\varepsilon^-$, where

$$\mathbf{M}_\varepsilon^+ = \left\{ u \in \mathbf{M}_\varepsilon \,|\, (2-q)\, \|u\|_H^2 - (p-q) \int_{\mathbb{R}^N} f(\varepsilon z) u_+^p dz > 0 \right\},$$

$$\mathbf{M}_\varepsilon^- = \left\{ u \in \mathbf{M}_\varepsilon \,|\, (2-q)\, \|u\|_H^2 - (p-q) \int_{\mathbb{R}^N} f(\varepsilon z) u_+^p dz < 0 \right\}.$$

Define

$$\alpha_\varepsilon = \inf_{u \in \mathbf{M}_\varepsilon} J_\varepsilon(u); \quad \alpha_\varepsilon^+ = \inf_{u \in \mathbf{M}_\varepsilon^+} J_\varepsilon(u); \quad \alpha_\varepsilon^- = \inf_{u \in \mathbf{M}_\varepsilon^-} J_\varepsilon(u).$$

Lemma 2.7 (*i*) *If* $0 < \Lambda\ (= \varepsilon^{2(p-q)/(p-2)}) < \Lambda_0$, *then* $\alpha_\varepsilon \leq \alpha_\varepsilon^+ < 0$;

(*ii*) *If* $0 < \Lambda < q\Lambda_0/2$, *then* $\alpha_\varepsilon^- \geq d_0 > 0$ *for some constant* $d_0 = d_0(\varepsilon, p, q, S, \|h\|_\#, f_{\max})$.

Proof. (*i*) Let $u \in \mathbf{M}_\varepsilon^+$, by (2.2), we get

$$(p-2)\, \|u\|_H^2 < (p-q) \int_{\mathbb{R}^N} \Lambda h(\varepsilon z) u_+^q dz.$$

Then

$$\begin{aligned} J_\varepsilon(u) &= \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|_H^2 - \left(\frac{1}{q} - \frac{1}{p}\right) \int_{\mathbb{R}^N} \Lambda h(\varepsilon z) u_+^q dz \\ &< \left[\left(\frac{1}{2} - \frac{1}{p}\right) - \left(\frac{1}{q} - \frac{1}{p}\right) \frac{p-2}{p-q}\right] \|u\|_H^2 \\ &= -\frac{(2-q)(p-2)}{2pq} \|u\|_H^2 < 0. \end{aligned}$$

By the definitions of α_ε and α_ε^+, we deduce that $\alpha_\varepsilon \leq \alpha_\varepsilon^+ < 0$.

(*ii*) See the proof of Lemma 2.5 (*iv*).

Applying Ekeland's variational principle and using the same argument in Cao and Zhou [7] or Tarantello [11], we have the following lemma.

Lemma 2.8 (*i*) *There exists a* $(PS)_{\alpha_\varepsilon}$*-sequence* $\{u_n\}$ *in* $\mathbf{M}_\varepsilon$ *for* J_ε;

(*ii*) *There exists a* $(PS)_{\alpha_\varepsilon^+}$*-sequence* $\{u_n\}$ *in* $\mathbf{M}_\varepsilon^+$ *for* J_ε;

(*iii*) *There exists a* $(PS)_{\alpha_\varepsilon^-}$*-sequence* $\{u_n\}$ *in* $\mathbf{M}_\varepsilon^-$ *for* J_ε.

3 Existence of a ground state solution

In order to prove the existence of positive solutions, we claim that J_ε satisfies the $(\mathrm{PS})_\beta$-condition in $H^1(\mathbb{R}^N)$ for $\beta \in \left(-\infty, \gamma_\infty - C_0\Lambda^{\frac{2}{2-q}}\right)$, where $\Lambda = \varepsilon^{2(p-q)/(p-2)}$ and C_0 is defined in the following lemma.

Lemma 3.1 *Assume that h satisfies (h1) and $0 < \Lambda\ (= \varepsilon^{2(p-q)/(p-2)}) < \Lambda_0$. If $\{u_n\}$ is a $(PS)_\beta$-sequence in $H^1(\mathbb{R}^N)$ for J_ε with $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, then $J'_\varepsilon(u) = 0$ in $H^{-1}(\mathbb{R}^N)$ and* $J_\varepsilon(u) \ge -C_0\Lambda^{\frac{2}{2-q}} \ge -C'_0$, *where*

$$C_0 = (2-q)\left[(p-q)\|h\|_\# S^q\right]^{\frac{2}{2-q}} \Big/ \left[2pq(p-2)^{\frac{q}{2-q}}\right],$$

and

$$C'_0 = \left[(p-2)(2-q)^{\frac{p}{p-2}}\right] \Big/ \left\{2pq\left[f_{\max}(p-q)\right]^{\frac{2}{p-2}} S^{\frac{2p}{p-2}}\right\}.$$

Proof. Since $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H^1(\mathbb{R}^N)$ for J_ε with $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, it is easy to check that $J'_\varepsilon(u) = 0$ in $H^{-1}(\mathbb{R}^N)$ and $u \ge 0$. Then we have $\langle J'_\varepsilon(u), u\rangle = 0$, that is, $\int_{\mathbb{R}^N} f(\varepsilon z)u^p dz = \|u\|_H^2 - \int_{\mathbb{R}^N} \Lambda h(\varepsilon z)u^q dz$. Hence, by the Young inequality $\left(p_1 = \frac{2}{q} \text{ and } p_2 = \frac{2}{2-q}\right)$

$$\begin{aligned}
J_\varepsilon(u) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_H^2 - \left(\frac{1}{q} - \frac{1}{p}\right)\int_{\mathbb{R}^N} \Lambda h(\varepsilon z)u^q dz \\
&\ge \frac{p-2}{2p}\|u\|_H^2 - \frac{p-q}{pq}\Lambda\|h\|_\# S^q\|u\|_H^q \\
&\ge \frac{p-2}{2p}\|u\|_H^2 - \frac{p-2}{pq}\left[\frac{q\|u\|_H^2}{2} + \left(\frac{p-q}{p-2}\Lambda\|h\|_\# S^q\right)^{\frac{2}{2-q}}\frac{2-q}{2}\right] \\
&\ge -\frac{(p-2)(2-q)^{\frac{p}{p-2}}}{2pq\left[f_{\max}(p-q)\right]^{\frac{2}{p-2}} S^{\frac{2p}{p-2}}}.
\end{aligned}$$

Lemma 3.2 *Assume that f and h satisfy (f1) and (h1). If $0 < \Lambda\ (= \varepsilon^{2(p-q)/(p-2)}) < \Lambda_0$, then J_ε satisfies the $(PS)_\beta$-condition in $H^1(\mathbb{R}^N)$ for $\beta \in \left(-\infty, \gamma_\infty - C_0\Lambda^{\frac{2}{2-q}}\right)$.*

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_\beta$-sequence in $H^1(\mathbb{R}^N)$ for J_ε such that $J_\varepsilon(u_n) = \beta + o_n(1)$ and $J_\varepsilon'(u_n) = o_n(1)$ in $H^{-1}(\mathbb{R}^N)$. Then

$$\begin{aligned}|\beta| + c_n + \frac{d_n\|u_n\|_H}{p} &\geq J_\varepsilon(u_n) - \frac{1}{p}\langle J'_\varepsilon(u_n),(u_n)\rangle\\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_H^2 - \left(\frac{1}{q}-\frac{1}{p}\right)\int_{\mathbb{R}^N} \varepsilon^{\frac{2(p-q)}{p-2}} h(\varepsilon z)(u_n)_+^q dz\\ &\geq \frac{p-2}{2p}\|u_n\|_H^2 - \frac{p-q}{pq}\Lambda\|h\|_\# S^q\|u_n\|_H^q,\end{aligned}$$

where $c_n = o_n(1)$, $d_n = o_n(1)$ as $n \to \infty$. It follows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Hence, there exist a subsequence $\{u_n\}$ and a nonnegative $u \in H^1(\mathbb{R}^N)$ such that $J_\varepsilon'(u) = 0$ in $H^{-1}(\mathbb{R}^N)$, $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $u_n \rightharpoonup u$ a.e. in $\mathbb{R}^N$, $u_n \rightharpoonup u$ strongly in $L^s_{loc}(\mathbb{R}^N)$ for any $1 \leq s < 2^*$. Using the Brézis-Lieb lemma to get (3.1) and (3.2) below.

$$\int_{\mathbb{R}^N} f(\varepsilon z)(u_n-u)_+^p dz = \int_{\mathbb{R}^N} f(\varepsilon z)(u_n)_+^p dz - \int_{\mathbb{R}^N} f(\varepsilon z)u^p dz + o_n(1); \tag{3.1}$$

$$\int_{\mathbb{R}^N} h(\varepsilon z)(u_n-u)_+^q dz = \int_{\mathbb{R}^N} h(\varepsilon z)(u_n)_+^q dz - \int_{\mathbb{R}^N} h(\varepsilon z)u^q dz + o_n(1). \tag{3.2}$$

Next, claim that

$$\int_{\mathbb{R}^N} h(\varepsilon z)|u_n-u|^q dz \to 0 \text{ as } n \to \infty. \tag{3.3}$$

For any $\sigma > 0$, there exists $r > 0$ such that $\int_{[B^N(0;r)]^c} h(\varepsilon z)^{\frac{p}{p-q}} dz < \sigma$. By the Hölder inequality and the Sobolev embedding theorem, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^N} h(\varepsilon z)|u_n-u|^q dz\right| &\leq \int_{B^N(0;r)} h(\varepsilon z)\,|u_n-u|^q dz\\ &\quad+ \int_{[B^N(0;r)]^c} h(\varepsilon z)|u_n-u|^q dz\\ &\leq \|h\|_\#\left(\int_{B^N(0;r)} |u_n-u|^p dz\right)^{q/p}\\ &\quad+ S^q\left(\int_{[B^N(0;r)]^c} h(\varepsilon z)^{\frac{p}{p-q}} dz\right)^{\frac{p-q}{p}} \|u_n-u\|_H^q\\ &\leq C'\sigma + o_n(1).\end{aligned}$$

$(\because \{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N))$

Applying ($f1$) and $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N)$, we get that

$$\int_{\mathbb{R}^N} f(\varepsilon z)(u_n - u)_+^p dz = \int_{\mathbb{R}^N} f_\infty (u_n - u)_+^p dz + o_n(1). \tag{3.4}$$

Let $p_n = u_n - u$. Suppose $p_n \nrightarrow 0$ strongly in $H^1(\mathbb{R}^N)$. By (3.1)-(3.4), we deduce that

$$\begin{aligned}
\|p_n\|_H^2 &= \|u_n\|_H^2 - \|u\|_H^2 + o_n(1) \\
&= \int_{\mathbb{R}^N} f(\varepsilon z)(u_n)_+^p dz - \int_{\mathbb{R}^N} \varepsilon^{\frac{2(p-q)}{p-2}} h(\varepsilon z)(u_n)_+^q dz \\
&\quad - \int_{\mathbb{R}^N} f(\varepsilon z) u^p dz + \int_{\mathbb{R}^N} \varepsilon^{\frac{2(p-q)}{p-2}} h(\varepsilon z) u^q dz + o_n(1) \\
&= \int_{\mathbb{R}^N} f(\varepsilon z)(u_n - u)_+^p dz + o_n(1) = \int_{\mathbb{R}^N} f_\infty (p_n)_+^p dz + o_n(1).
\end{aligned}$$

Then

$$\begin{aligned}
I_\infty(p_n) &= \frac{1}{2}\|p_n\|_H^2 - \frac{1}{p}\int_{\mathbb{R}^N} f_\infty (p_n)_+^p dz \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\|p_n\|_H^2 + o_n(1) > 0.
\end{aligned}$$

By Theorem 4.3 in Wang [12], there exists a sequence $\{s_n\} \subset \mathbb{R}^+$ such that $s_n = 1 + o_n(1)$, $\{s_n p_n\} \subset \mathbf{N}_\infty$ and $I_\infty(s_n p_n) = I_\infty(p_n) + o_n(1)$. It follows that

$$\begin{aligned}
\gamma_\infty &\le I_\infty(s_n p_n) = I_\infty(p_n) + o_n(1) \\
&= J_\varepsilon(u_n) - J_\varepsilon(u) + o_n(1) \\
&= \beta - J_\varepsilon(u) + o_n(1) < \gamma_\infty,
\end{aligned}$$

which is a contradiction. Hence, $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$.

Remark 3.3 *By Lemma 1.1, we obtain*

$$C_0' = \frac{2-q}{q}\left(\frac{2-q}{p-q}\right)^{\frac{2}{p-2}} \gamma_{\max} < \gamma_{\max} < \gamma_\infty,$$

and $\gamma_\infty - C_0 \Lambda^{\frac{2}{2-q}} > 0$ *for* $0 < \Lambda < \Lambda_0$.

By Lemma 2.8 (i), there is a $(\mathrm{PS})_{\alpha_\varepsilon}$-sequence $\{u_n\}$ in $\mathbf{M}_\varepsilon$ for J_ε. Then we prove that Equation (E_ε) admits a positive ground state solution u_0 in $\mathbb{R}^N$.

Theorem 3.4 *Under assumptions* ($f1$), ($h1$), *if* $0 < \Lambda\ (= \varepsilon^{2(p-q)/(p-2)}) < \Lambda_0$, *then there exists at least one positive ground state solution* u_0 *of Equation* (E_ε) *in* $\mathbb{R}^N$. *Moreover, we have that* $u_0 \in \mathbf{M}_\varepsilon^+$ *and*

$$J_\varepsilon(u_0) = \alpha_\varepsilon = \alpha_\varepsilon^+ \ge -C_0 \Lambda^{\frac{2}{2-q}}. \tag{3.5}$$

Proof. By Lemma 2.8 (*i*), there is a minimizing sequence $\{u_n\} \subset \mathbf{M}_\varepsilon$ for J_ε such that $J_\varepsilon(u_n) = \alpha_\varepsilon + o_n(1)$ and $J'_\varepsilon(u_n) = o_n(1)$ in $H^{-1}(\mathbb{R}^N)$. Since $\alpha_\varepsilon < 0 < \gamma^\infty - C_0\Lambda^{\frac{2}{2-q}}$, by Lemma 3.2, there exist a subsequence $\{u_n\}$ and $u_0 \in H^1(\mathbb{R}^N)$ such that $u_n \to u_0$ strongly in $H^1(\mathbb{R}^N)$. It is easy to see that $u_0 \gneqq 0$ is a solution of Equation (E_ε) in $\mathbb{R}^N$ and $J_\varepsilon(u_0) = \alpha_\varepsilon$. Next, we claim that $u_0 \in \mathbf{M}_\varepsilon^+$. On the contrary, assume that $u_0 \in \mathbf{M}_\varepsilon^-$ ($\mathbf{M}_\varepsilon^0 = \emptyset$ for $0 < \Lambda(= \varepsilon^{2(p-q)/(p-2)}) < \Lambda_0$).

We get that

$$\int_{\mathbb{R}^N} \Lambda h(\varepsilon z)(u_0)_+^q dz > 0.$$

Otherwise,

$$\begin{aligned} 0 = \int_{\mathbb{R}^N} \Lambda h(\varepsilon z)(u_0)_+^q dz &= \int_{\mathbb{R}^N} \Lambda h(\varepsilon z)(u_n)_+^q dz + o_n(1) \\ &= \|u_n\|_H^2 - \int_{\mathbb{R}^N} f(\varepsilon z)(u_n)_+^p dz + o_n(1). \end{aligned}$$

It follows that

$$\alpha_\varepsilon + o_n(1) = J_\varepsilon(u_n) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_H^2 + o_n(1),$$

which contradicts to $\alpha_\varepsilon < 0$. By Lemma 2.6 (*ii*), there exist positive numbers $t^+ < \bar{t} < t^- = 1$ such that $t^+u_0 \in \mathbf{M}_\varepsilon^+$, $t^-u_0 \in \mathbf{M}_\varepsilon^-$ and

$$J_\varepsilon(t^+u_0) < J_\varepsilon(t^-u_0) = J_\varepsilon(u_0) = \alpha_\varepsilon,$$

which is a contradiction. Hence, $u_0 \in \mathbf{M}_\varepsilon^+$ and

$$-C_0\Lambda^{\frac{2}{2-q}} \le J_\varepsilon(u_0) = \alpha_\varepsilon = \alpha_\varepsilon^+.$$

By Lemma 2.4 and the maximum principle, then u_0 is a positive solution of Equation (E_ε) in $\mathbb{R}^N$.

4 Existence of k + 1 solutions

From now, we assume that f and h satisfy (f1)-(f2) and (h1). Let $w \in H^1(\mathbb{R}^N)$ be the unique, radially symmetric, and positive ground state solution of Equation (E0) in $\mathbb{R}^N$ for $f = f_{\max}$. Recall the facts (or see Bahri and Li [13], Bahri and Lions [14], Gidas et al. [15], and Kwong [16]).

(*i*) $w \in L^\infty(\mathbb{R}^N) \cap C_{loc}^{2,\theta}(\mathbb{R}^N)$ for some $0 < \theta < 1$ and $\lim_{|z|\to\infty} w(z) = 0$;

(*ii*) for any $\varepsilon > 0$, there exist positive numbers C_1, C_1, C_2^ε, and C_3^ε such that for all $z \in \mathbb{R}^N$

$$C_2^\varepsilon \exp\left(-(1-\varepsilon)|z|\right) \le w(z) \le C_1 \exp\left(-|z|\right)$$

and

$$\left|\nabla w(z)\right| \le C_3^\varepsilon \exp\left(-(1-\varepsilon)|z|.\right)$$

For $1 \le i \le k$, we define

$$w_\varepsilon^i(z) = w\left(z - \frac{a^i}{\varepsilon}\right), \text{ where } f(a^i) = f_{\max}.$$

Clearly, $w_\varepsilon^i(z) \in H^1(\mathbb{R}^N)$. By Lemma 2.6 (*ii*), there is a unique number $\left(t_\varepsilon^i\right)^- > 0$ such that $\left(t_\varepsilon^i\right)^- w_\varepsilon^i \in \mathrm{M}_\varepsilon^- \subset \mathrm{M}_\varepsilon$, where $1 \le i \le k$.

We need to prove that

$$\lim_{\varepsilon\to 0+} J_\varepsilon\left(\left(t_\varepsilon^i\right)^- w_\varepsilon^i\right) \le \gamma_{\max} \text{ uniformly in } i.$$

Lemma 4.1 (*i*) *There exists a number $t_0 > 0$ such that for $0 \le t \le t_0$ and any $\varepsilon > 0$, we have that*

$$J_\varepsilon\left(tw_\varepsilon^i\right) < \gamma_{\max} \textit{ uniformly in } i;$$

(*ii*) *There exist positive numbers t_1 and ε_1 such that for any $t > t_1$ and $\varepsilon < \varepsilon_1$, we have that*

$$J_\varepsilon\left(tw_\varepsilon^i\right) < 0 \textit{ uniformly in } i.$$

Proof. (*i*) Since J_ε is continuous in $H^1\left(\mathbb{R}^N\right)$, $\{w_\varepsilon^i\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$ for any $\varepsilon > 0$, and $\gamma_{\max} > 0$, there is $t_0 > 0$ such that for $0 \le t \le t_0$ and any $\varepsilon > 0$

$$J_\varepsilon\left(tw_\varepsilon^i\right) < \gamma_{\max}.$$

(*ii*) There is an $r_0 > 0$ such that $f(z) \ge f_{\max}/2$ for $z \in B^N(a^i; r_0)$ uniformly in i. Then there exists $\varepsilon_1 > 0$ such that for $\varepsilon < \varepsilon_1$

$$\begin{aligned} J_\varepsilon\left(tw_\varepsilon^i\right) &= \frac{t^2}{2}\|w_\varepsilon^i\|_H^2 - \frac{t^p}{p}\int_{\mathbb{R}^N} f(\varepsilon z)(w_\varepsilon^i)^p dz - \frac{t^q}{q}\int_{\mathbb{R}^N} \Lambda h(\varepsilon z)(w_\varepsilon^i)^q dz \\ &\le \frac{t^2}{2}\int_{\mathbb{R}^N}[|\nabla w|^2 w^2]dz - \frac{t^p}{2p}\int_{B^{N(0;1)}} f_{\max} w^p dz. \end{aligned}$$

Thus, there is $t_1 > 0$ such that for any $t > t_1$ and $\varepsilon < \varepsilon_1$

$$J_\varepsilon\left(tw_\varepsilon^i\right) < 0 \text{ uniformly in } i.$$

Lemma 4.2 *Under assumptions* (f1), (f2), *and* (h1). *If $0 < \Lambda$ ($= \varepsilon^{2(p-q)/(p-2)}$) $< q\,\Lambda_0/2$, then*

$$\lim_{\varepsilon\to 0+} \sup_{t\to 0} J_\varepsilon\left(tw_\varepsilon^i\right) \le \gamma_{\max} \textit{ uniformly in } i.$$

Proof. By Lemma 4.1, we only need to show that

$$\lim_{\varepsilon\to 0+} \sup_{t_0\le t\le t_1} J_\varepsilon\left(tw_\varepsilon^i\right) \le \gamma_{\max} \text{ uniformly in } i.$$

We know that $\sup_{t\ge 0} I_{\max}(tw) = \gamma_{\max}$. For $t_0 \le t \le t_1$, we get

$$\begin{aligned} J_\varepsilon\left(tw_\varepsilon^i\right) &= \frac{1}{2}\|tw_\varepsilon^i\|_H^2 - \frac{1}{p}\int_{\mathbb{R}^N} f(\varepsilon z)\left(tw_\varepsilon^i\right)^p dz - \frac{1}{q}\int_{\mathbb{R}^N} \Lambda h(\varepsilon z)\left(tw_\varepsilon^i\right)^q dz \\ &= \frac{t^2}{2}\int_{\mathbb{R}^N}\left[\left|\nabla w\left(z - \frac{a^i}{\varepsilon}\right)\right|^2 + w\left(z - \frac{a^i}{\varepsilon}\right)\right]dz \\ &\quad - \frac{t^p}{p}\int_{\mathbb{R}^N} f(\varepsilon z) w\left(z - \frac{a^i}{\varepsilon}\right)^p dz - \frac{t^q}{q}\int_{\mathbb{R}^N} \Lambda h(\varepsilon z) w\left(z - \frac{a^i}{\varepsilon}\right)^q dz \\ &= \left\{\frac{t^2}{p}\int_{\mathbb{R}^N}[|\nabla w|^2 + w^2]dz - \frac{t^p}{p}\int_{\mathbb{R}^N} f_{\max} w^p dz\right\} \\ &\quad + \frac{t^p}{p}\int_{\mathbb{R}^N}(f_{\max} - f(\varepsilon z)) w\left(z - \frac{a^i}{\varepsilon}\right)^p dz - \frac{t^q}{q}\Lambda\int_{\mathbb{R}^N} h(\varepsilon z) w\left(z - \frac{a^i}{\varepsilon}\right)^q dz \\ &\le \gamma_{\max} + \frac{t_1^p}{p}\int_{\mathbb{R}^N}(f_{\max} - f(\varepsilon z)) w\left(z - \frac{a^i}{\varepsilon}\right)^p dz - \frac{t_0^q}{q}\Lambda\int_{\mathbb{R}^N} h(\varepsilon z) w\left(z - \frac{a^i}{\varepsilon}\right) dz. \end{aligned}$$

Since

$$\int_{\mathbb{R}^N} (f_{\max} - f(\varepsilon z)) w\left(z - \frac{a^i}{\varepsilon}\right)^p dz$$
$$= \int_{\mathbb{R}^N} \left[f_{\max} - f(\varepsilon z + a^i)\right] w^p dz = o(1) \text{ as } \varepsilon \to 0^+ \text{uniformly in } i,$$

and

$$\Lambda \int_{\mathbb{R}^N} h(\varepsilon z) w\left(z - \frac{a^i}{\varepsilon}\right)^q dz \leq \varepsilon^{\frac{2(p-q)}{p-2}} \|h\|_{\#} S^q \|w\|_H^q = o(1) \text{ as } \varepsilon \to 0^+,$$

then $\lim_{\varepsilon \to 0+} \sup_{t_0 \leq t \leq t_1} J_\varepsilon \left(t w_\varepsilon^i\right) \leq \gamma_{\max}$, that is, $\lim_{\varepsilon \to 0+} \sup_{t \geq 0} J_\varepsilon \left(t w_\varepsilon^i\right) \leq \gamma_{\max}$ uniformly in i.

Applying the results of Lemmas 2.6, 2.7(*ii*), and 4.2, we can deduce that

$$0 < d_0 \leq \alpha_\varepsilon^- \leq \gamma_{\max} + o(1) \text{ as } \varepsilon \to 0^+.$$

Since $\gamma_{\max} < \gamma_\infty$, there exists $\varepsilon_0 > 0$ such that

$$\gamma_{\max} < \gamma_\infty - C_0 \Lambda^{\frac{2}{2-q}} \text{ for any } \varepsilon < \varepsilon_0. \tag{4.1}$$

Choosing $0 < \rho_0 < 1$ such that

$$\overline{B_{\rho 0}^N(a^i)} \cap \overline{B_{\rho 0}^N(a^j)} = \emptyset \text{ for } i \neq j \text{ and } 1 \leq i, j \leq k,$$

where $\overline{B_{\rho 0}^N(a^i)} = \left\{z \in \mathbb{R}^N \mid \left|z - a^i\right| \leq \rho 0\right\}$ and $f(a^i) = f_{\max}$. Define $\mathbf{K} = \{a^i \mid 1 \leq i \leq k\}$ and $\mathbf{K}_{\rho_0/2} = \cup_{i=1}^k \overline{B_{\rho_0/2}^N(a^i)}$. Suppose $\cup_{i=1}^k \overline{B_{\rho_0}^N(a^i)} \subset B_{r_0}^N(0)$ for some $r_0 > 0$.

Let $Q_\varepsilon : H^1(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ be given by

$$Q_\varepsilon(u) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon z) |u|^p dz}{\int_{\mathbb{R}^N} |u|^p dz},$$

where $\chi : \mathbb{R}^N \to \mathbb{R}^N$, $\chi(z) = z$ for $|z| \leq r_0$ and $\chi(z) = r_0 z / |z|$ for $|z| > r_0$.

Lemma 4.3 *There exists $0 < \varepsilon^0 \leq \varepsilon_0$ such that if $\varepsilon < \varepsilon^0$, then $Q_\varepsilon\left(\left(t_\varepsilon^i\right)^- w_\varepsilon^i\right) \in \mathbf{K}_{\rho_0/2}$ for each $1 \leq i \leq k$.*

Proof. Since

$$Q_\varepsilon\left(\left(t_\varepsilon^i\right)^- w_\varepsilon^i\right) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon z) \left|w\left(z - \frac{a^i}{\varepsilon}\right)\right|^p dz}{\int_{\mathbb{R}^N} \left|w\left(z - \frac{a^i}{\varepsilon}\right)\right|^p dz}$$
$$= \frac{\int_{\mathbb{R}^N} \chi(\varepsilon z + a^i) |w(z)|^p dz}{\int_{\mathbb{R}^N} |w(z)|^p dz}$$
$$\to a^i \text{ as } \varepsilon \to 0^+,$$

there exists $\varepsilon^0 > 0$ such that

$$Q_\varepsilon\left(\left(t_\varepsilon^i\right)^- w_\varepsilon^i\right) \in \mathbf{K}_{\rho_0/2} \text{ for any } \varepsilon < \varepsilon^0 \text{ and each } 1 \leq i \leq k.$$

Lemma 4.4 *There exists a number* $\bar{\delta} > 0$*such that if* $u \in \mathbf{N}_\varepsilon$ *and* $I_\varepsilon(u) \leq \gamma_{\max} + \bar{\delta}$, *then* $Q_\varepsilon(u) \in \mathbf{K}_{\rho_0/2}$*for any* $0 < \varepsilon < \varepsilon^0$.

Proof. On the contrary, there exist the sequences $\{\varepsilon_n\} \subset \mathbb{R}^+$ and $\{u_n\} \subset \mathbf{N}_{\varepsilon_n}$ such that $\varepsilon_n \to 0, I_{\varepsilon_n}(u_n) = \gamma_{\max}(> 0) + o_n(1)$ as $n \to \infty$ and $Q_{\varepsilon_n}(u_n) \notin \mathbf{K}_{\rho_0/2}$ for all $n \in \mathbb{N}$. It is easy to check that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Suppose $u_n \to 0$ strongly in $L^p(\mathbb{R}^N)$. Since

$$\|u_n\|_H^2 = \int_{\mathbb{R}^N} f(\varepsilon_n z)(u_n)_+^p dz \text{ for each } n \in \mathbb{N},$$

and

$$I_{\varepsilon_n}(u_n) = \frac{1}{2}\|u_n\|_H^2 - \frac{1}{p}\int_{\mathbb{R}^N} f(\varepsilon_n z)(u_n)_+^p dz = \gamma_{\max} + o_n(1),$$

then

$$\gamma_{\max} + o_n(1) = I_{\varepsilon_n}(u_n) = \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^N} f(\varepsilon_n z)(u_n)_+^p dz = o_n(1),$$

which is a contradiction. Thus, $u_n \nrightarrow 0$ strongly in $L^p(\mathbb{R}^N)$. Applying the concentration-compactness principle (see Lions [17] or Wang [[12], Lemma 2.16]), then there exist a constant $d_0 > 0$ and a sequence $\{\widetilde{z_n}\} \subset \mathbb{R}^N$ such that

$$\int_{B^N(\widetilde{z_n};1)} |u_n(z)|^2 dz \geq d_0 > 0. \tag{4.2}$$

Let $v_n(z) = u_n(z + \widetilde{z_n})$, there are a subsequence $\{v_n\}$ and $v \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$. Using the similar computation in Lemma 2.6, there is a sequence $\{s_{\max}^n\} \subset \mathbb{R}^+$ such that $\widetilde{v_n} = s_{\max}^n v_n \in \mathbf{N}_{\max}$ and

$$\begin{aligned} 0 < \gamma_{\max} &\leq I_{\max}(\widetilde{v_n}) \leq I_{\varepsilon_n}(s_{\max}^n u_n) \\ &\leq I_{\varepsilon_n}(u_n) = \gamma_{\max} + o_n(1) \text{ as } n \to \infty. \end{aligned}$$

We deduce that a convergent subsequence $\{s_{\max}^n\}$ satisfies $s_{\max}^n \to s_0 > 0$. Then there are subsequences $\{\widetilde{v_n}\}$ and $\tilde{v} \in H^1(\mathbb{R}^N)$ such that $\tilde{v}_n \rightharpoonup \tilde{v}(= s_0 v)$ weakly in $H^1(\mathbb{R}^N)$. By (4.2), then $\tilde{v} \neq 0$. Moreover, we can obtain that $\widetilde{v_n} \to \tilde{v}$ strongly in $H^1(\mathbb{R}^N)$ and $I_{\max}(\tilde{v}) = \gamma_{\max}$. Now, we want to show that there exists a subsequence $\{z_n\} = \{\varepsilon_n \widetilde{z_n}\}$ such that $z_n \to z_0 \in \mathbf{K}$.

(*i*) Claim that the sequence $\{z_n\}$ is bounded in $\mathbb{R}^N$. On the contrary, assume that $|z_n| \to \infty$, then

$$\begin{aligned} \gamma_{\max} &= I_{\max}(\tilde{v}) < I_\infty(\tilde{v}) \\ &\leq \liminf_{n\to\infty}\left[\frac{1}{2}\|\widetilde{v_n}\|_H^2 - \frac{1}{p}\int_{\mathbb{R}^N} f(\varepsilon_n z + z_n)(\widetilde{v_n})_+^p dz\right] \\ &= \liminf_{n\to\infty}\left[\frac{(s_{\max}^n)^2}{2}\|u_n\|_H^2 - \frac{(s_{\max}^n)^p}{p}\int_{\mathbb{R}^N} f(\varepsilon_n z)(u_n)_+^p dz\right] \\ &= \liminf_{n\to\infty} I_{\varepsilon_n}(s_{\max}^n u_n) \leq \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) = \gamma_{\max}, \end{aligned}$$

which is a contradiction.

(*ii*) Claim that $z_0 \in \mathbf{K}$. On the contrary, assume that $z_0 \notin \mathbf{K}$, that is, $f(z_0) < f_{\max}$. Then using the above argument to obtain that

$$\begin{aligned}\gamma_{\max} = I_{\max}(\tilde{v}) &< \frac{1}{2}\left\|\tilde{v}\right\|_H^2 - \frac{1}{p}\int_{\mathbb{R}^N} f(z_0)(\tilde{v})_+^p dz \\ &\le \liminf_{n\to\infty}\left[\frac{1}{2}\|\tilde{v_n}\|_H^2 - \frac{1}{p}\int_{\mathbb{R}^N} f(\varepsilon_n z + z_n)(\tilde{v_n})_+^p dz\right] \\ &= \gamma_{\max},\end{aligned}$$

which is a contradiction. Since $v_n \to v \neq 0$ *in* $H^1(\mathbb{R}^N)$, we have that

$$\begin{aligned}Q_{\varepsilon_n}(u_n) &= \frac{\int_{\mathbb{R}^N} \chi(\varepsilon_n z)\left|v_n(z-\tilde{z_n})\right|^p dz}{\int_{\mathbb{R}^N}\left|v_n(z-\tilde{z_n})\right|^p dz} \\ &= \frac{\int_{\mathbb{R}^N} \chi(\varepsilon_n z + \varepsilon_n \tilde{z_n})|v_n|^p dz}{\int_{\mathbb{R}^N}|v_n|^p dz} \to z_0 \subset \mathbf{K}_{\rho_0/2} \text{ as } n\to\infty,\end{aligned}$$

which is a contradiction.

Hence, there exists a number $\bar{\delta} > 0$ such that if $u \in \mathbf{N}_\varepsilon$ and $I_\varepsilon(u) \le \gamma_{\max} + \bar{\delta}$, then $Q_\varepsilon(u) \in \mathbf{K}_{\rho_0/2}$ for any $0 < \varepsilon < \varepsilon^0$.

From (4.1), choosing $0 < \delta_0 < \bar{\delta}$ such that

$$\gamma_{\max} + \delta_0 < \gamma_\infty - C_0\Lambda^{\frac{2}{2-q}} \text{ for any } 0 < \varepsilon < \varepsilon^0. \tag{4.3}$$

For each $1 \le i \le k$, define

$$\begin{aligned}O_\varepsilon^i &= \{u \in \mathbf{M}_\varepsilon^- \mid \left|Q_\varepsilon(u) - a^i\right| < \rho_0\}, \\ \partial O_\varepsilon^i &= \{u \in \mathbf{M}_\varepsilon^- \mid \left|Q_\varepsilon(u) - a^i\right| = \rho_0\},\end{aligned}$$

$\beta_\varepsilon^i = \inf_{u\in O_\varepsilon^i} J_\varepsilon(u)$ and $\tilde{\beta}_\varepsilon^i = \inf_{u\in\partial O_\varepsilon^i} J_\varepsilon(u)$.

Lemma 4.5 *If* $u \in \mathbf{M}_\varepsilon^-$ *and* $J_\varepsilon(u) \le \gamma_{\max} + \delta_0/2$, *then there exists a number* $0 < \bar{\varepsilon} < \varepsilon^0$ *such that* $Q_\varepsilon(u) \in \mathbf{K}_{\rho_0/2}$ *for any* $0 < \varepsilon < \bar{\varepsilon}$.

Proof. We use the similar computation in Lemma 2.6 to get that there is a unique positive number

$$s_\varepsilon^u = \left(\frac{\|u\|_H^2}{\int_{\mathbb{R}^N} f(\varepsilon z)u_+^p dz}\right)^{1/(p-2)}$$

such that $s_\varepsilon^u u \in \mathbf{N}_\varepsilon$. We want to show that $s_\varepsilon^u < c$ for some constant $c > 0$ (independent of *u*). First, since $u \in \mathbf{M}_\varepsilon^- \subset \mathbf{M}_\varepsilon$,

$$0 < d_0 \le \alpha_\varepsilon^- \le J_\varepsilon(u) \le \gamma_{\max} + \delta_0/2,$$

and J_ε is coercive on $\mathbf{M}_\varepsilon$, then $0 < c_2 < \|u\|_H^2 < c_1$ for some constants c_1 and c_2 (independent of u). Next, we claim that $\|u\|_{L^p}^p > c_3 > 0$ for some constant $c_3 > 0$

(independent of u). On the contrary, there exists a sequence $\{u_n\} \subset \mathbf{M}_\varepsilon^-$ such that

$$\|u_n\|_{L^p}^p = o_n(1) \text{ as } n \to \infty.$$

By (2.3),

$$\frac{2-q}{p-q} < \frac{\int_{\mathbb{R}^N} f(\varepsilon z)(u_n)_+^p dz}{\|u_n\|_H^2} \le \frac{f_{\max} \|u_n\|_{L^p}^p}{c_2} = o_n(1),$$

which is a contradiction. Thus, $s_\varepsilon^u < c$ for some constant $c > 0$ (independent of u). Now, we get that

$$\begin{aligned}\gamma_{\max} + \delta_0/2 \ge J_\varepsilon(u) = \sup_{t\ge 0} J_\varepsilon(tu) &\ge J_\varepsilon(s_\varepsilon^u u) \\ &= \frac{1}{2}\left\|s_\varepsilon^u u\right\|_H^2 - \frac{1}{p}\int_{\mathbb{R}^N} f(\varepsilon z)(s_\varepsilon^u u)_+^p dz - \frac{1}{q}\int_{\mathbb{R}^N} \Lambda h(\varepsilon z)(s_\varepsilon^u u)_+^q dz \\ &\ge I_\varepsilon(s_\varepsilon^u u) - \frac{1}{q}\int_{\mathbb{R}^N} \Lambda h(\varepsilon z)(s_\varepsilon^u u)_+^q dz.\end{aligned}$$

From the above inequality, we deduce that

$$\begin{aligned}I_\varepsilon(s_\varepsilon^u u) &\le \gamma_{\max} + \delta_0/2 + \frac{1}{q}\int_{\mathbb{R}^N} \Lambda h(\varepsilon z)(s_\varepsilon^u u)_+^q dz \\ &\le \gamma_{\max} + \delta_0/2 + \Lambda \|h\|_\# S^q \left\|s_\varepsilon^u u\right\|_H^q \\ &< \gamma_{\max} + \delta_0/2 + \Lambda c^q (c_1)^{q/2} \|h\|_\# S^q, \text{ where } \Lambda = \varepsilon^{2(p-q)/(p-2)}.\end{aligned}$$

Hence, there exists $0 < \bar{\varepsilon} < \varepsilon^0$ such that for $0 < \varepsilon < \bar{\varepsilon}$

$$I_\varepsilon(s_\varepsilon^u u) \le \gamma_{\max} + \delta_0, \text{ where } s_\varepsilon^u u \in \mathbf{N}_\varepsilon.$$

By Lemma 4.4, we obtain

$$Q_\varepsilon(s_\varepsilon^u u) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon z)\left|s_\varepsilon^u u(z)\right|^p dz}{\int_{\mathbb{R}^N} \left|s_\varepsilon^u u(z)\right|^p dz} \in \mathbf{K}_{\rho_0/2} \text{ for any } 0 < \varepsilon < \bar{\varepsilon},$$

or $Q_\varepsilon(u) \in \mathbf{K}_{\rho_0/2}$ for any $0 < \varepsilon < \bar{\varepsilon}$.
Applying the above lemma, we get that

$$\tilde{\beta}_\varepsilon^i \ge \gamma_{\max} + \delta_0/2 \text{ for any } 0 < \varepsilon < \bar{\varepsilon}. \tag{4.4}$$

By Lemmas 4.2, 4.3, and Equation (4.3), there exists $0 < \varepsilon^* \le \bar{\varepsilon}$ such that

$$\beta_\varepsilon^i \le J_\varepsilon\left(\left(t_\varepsilon^i\right)^- w_\varepsilon^i\right) \le \gamma_{\max} + \delta_0/3 < \gamma_\infty - C_0 \Lambda^{\frac{2}{2-q}} \text{ for any } 0 < \varepsilon < \varepsilon^*. \tag{4.5}$$

Lemma 4.6 *Given $u \in O_\varepsilon^i$, then there exist an $\eta > 0$ and a differentiable functional l : $B(0;\eta) \subset H^1(\mathbb{R}^N) \to \mathbb{R}^+$ such that $l(0) = 1$, $l(v)(u - v) \in O_\varepsilon^i$ for any $v \in B(0;\eta)$ and*

$$\langle l'(v), \phi\rangle|_{(l,v)=(1,0)} = \frac{\langle \psi'_\varepsilon(u), \phi\rangle}{\langle \psi'_\varepsilon(u), u\rangle} \text{ for any } \phi \in C_c^\infty(\mathbb{R}^N), \tag{4.6}$$

where $\psi_\varepsilon(u) = \langle J_\varepsilon'(u), u\rangle$.

Proof. See Cao and Zhou [7].

Lemma 4.7 *For each* $1 \le i \le k$, *there is a* $(PS)_{\beta_\varepsilon^i}$*-sequence* $\{u_n\} \subset O_\varepsilon^i$ *in* $H^1(\mathbb{R}^N)$ *for* J_ε.

Proof. For each $1 \le i \le k$, by (4.4) and (4.5),

$$\beta_\varepsilon^i < \tilde{\beta}_\varepsilon^i \text{ for any } 0 < \varepsilon < \varepsilon^*. \tag{4.7}$$

Then

$$\beta_\varepsilon^i = \inf_{u \in O_\varepsilon^i \cup \partial O_\varepsilon^i} J_\varepsilon(u) \text{ for any } 0 < \varepsilon < \varepsilon^*.$$

Let $\{u_n^i\} \subset O_\varepsilon^i \cup \partial O_\varepsilon^i$ be a minimizing sequence for β_ε^i. Applying Ekeland's variational principle, there exists a subsequence $\{u_n^i\}$ such that $J_\varepsilon(u_n^i) = \beta_\varepsilon^i + 1/n$ and

$$J_\varepsilon(u_n^i) \le J_\varepsilon(w) + \left\| w - u_n^i \right\|_H / n \text{ for all } w \in O_\varepsilon^i \cup \partial O_\varepsilon^i. \tag{4.8}$$

Using (4.7), we may assume that $u_n^i \in O_\varepsilon^i$ for sufficiently large n. By Lemma 4.6, then there exist an $\eta_n^i > 0$ and a differentiable functional $l_n^i : B(0; \eta_n^i) \subset H^1(\mathbb{R}^N) \to \mathbb{R}^+$ such that $l_n^i(0) = 1$, and $l_n^i(v)\left(u_n^i - v\right) \in O_\varepsilon^i$ for $v \in B(0; \eta_n^i)$. Let $v_\sigma = \sigma v$ with $\left\| v \right\|_H = 1$ and $0 < \sigma < \eta_n^i$. Then $v_\sigma \in B\left(0, \eta_n^i\right)$ and $w_\sigma = l_n^i(v_\sigma)\left(u_n^i - v_\sigma\right) \in O_\varepsilon^i$. From (4.8) and by the mean value theorem, we get that as $\sigma \to 0$

$$\begin{aligned}
\frac{\left\| w_\sigma - u_n^i \right\|_H}{n} &\ge J_\varepsilon(u_n^i) - J_\varepsilon(w_\sigma) \\
&= \langle J'_\varepsilon(t_0 u_n^i + (1 - t_0) w_\sigma), u_n^i - w_\sigma \rangle \text{ where } t_0 \in (0, 1) \\
&= \langle J'_\varepsilon(u_n^i), u_n^i - w_\sigma \rangle + o(\left\| u_n^i - w_\sigma \right\|_H) \ (\because J_\varepsilon \in C^1) \\
&= \sigma l_n^i(v_\sigma) \langle J'_\varepsilon(u_n^i), v \rangle + (1 - l_n^i(v_\sigma)) \langle J'_\varepsilon(u_n^i), u_n^i \rangle + o(\left\| u_n^i - w_\sigma \right\|_H) \\
&\qquad (\because l_n^i(v_\sigma) \to l_n^i(0) = 1 \text{ as } \sigma \to 0) \\
&= \sigma l_n^i(\sigma v) \langle J'_\varepsilon(u_n^i), v \rangle + o(\left\| u_n^i - w_\sigma \right\|_H).
\end{aligned}$$

Hence,

$$\begin{aligned}
\left| \langle J'_\varepsilon\left(u_n^i\right), v \rangle \right| &\le \frac{\left\| w_\sigma - u_n^i \right\|_H \left(\frac{1}{n} + \left| o(1) \right|\right)}{\sigma \left| l_n^i(\sigma v) \right|} \\
&\le \frac{\left\| u_n^i \left(l_n^i(\sigma v) - l_n^i(0)\right) - \sigma v l_n^i(\sigma v) \right\|_H \left(\frac{1}{n} + \left| 0(1) \right|\right)}{\sigma \left| l_n^i(\sigma v) \right|} \\
&\le \frac{\left\| u_n^i \right\|_H \left| l_n^i(\sigma v) - l_n^i(0) \right| + \sigma \left\| v \right\|_H \left| l_n^i(\sigma v) \right|}{\sigma \left| l_n^i(\sigma v) \right|} \left(\frac{1}{n} + \left| o(1) \right|\right) \\
&\le C\left(1 + \left\| (l_n^i)'(0) \right\|\right) \left(\frac{1}{n} + \left| o(1) \right|\right).
\end{aligned}$$

Since we can deduce that $\left\| (l_n^i)'(0) \right\| \le c$ for all n and i from (4.6), then $J'_\varepsilon\left(u_n^i\right) = o_n(1)$ strongly in $H^{-1}(\mathbb{R}^N)$ as $n \to \infty$.

Theorem 4.8 *Under assumptions* (f1), (f_2), *and* (h1), *there exists a positive number* λ^* ($\lambda^* = (\varepsilon^*)^{-2}$) *such that for* $\lambda > \lambda^*$, *Equation* (E_λ) *has* $k + 1$ *positive solutions in* $\mathbb{R}^N$.

Proof. We know that there is a $(PS)_{\beta_\varepsilon^i}$-sequence $\{u_n\} \subset \mathbf{M}_\varepsilon^-$ in $H^1(\mathbb{R}^N)$ for J_ε for each $1 \le i \le k$, and (4.5). Since J_ε satisfies the $(PS)_\beta$-condition for $\beta \in \left(-\infty, \gamma_\infty - C_0 \Lambda^{\frac{2}{2-q}}\right)$, then J_ε has at least k critical points in $\mathbf{M}_\varepsilon^-$ for $0 < \varepsilon < \varepsilon^*$. It follows that Equation (E_λ)

has k nonnegative solutions in $\mathbb{R}^N$. Applying the maximum principle and Theorem 3.4, Equation (E_λ) has $k + 1$ positive solutions in $\mathbb{R}^N$.

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Competing interests
The author declares that he has no competing interests.

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References

1. Ambrosetti, A, Brezis, H, Cerami, G: Combined effects of concave and convex nonlinearities in some elliptic problems. J Funct Anal. **122**, 519–543 (1994). doi:10.1006/jfan.1994.1078
2. Wu, TF: On semilinear elliptic equations involving concave-convex non-linearities and sign-changing weight function. J Math Anal Appl. **318**, 253–270 (2006). doi:10.1016/j.jmaa.2005.05.057
3. Ambrosetti, A, Garcia Azorero, J, Peral Alonso, I: Multiplicity results for some nonlinear elliptic equations. J Funct Anal. **137**, 219–242 (1996). doi:10.1006/jfan.1996.0045
4. Brown, KJ, Zhang, Y: The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function. J Diff Equ. **193**, 481–499 (2003). doi:10.1016/S0022-0396(03)00121-9
5. de Figueiredo, DG, Gossez, JP, Ubilla, P: Local superlinearity and sub-linearity for indefinite semilinear elliptic problems. J Funct Anal. **199**, 452–467 (2003). doi:10.1016/S0022-1236(02)00060-5
6. Zhu, XP: A perturbation result on positive entire solutions of a semilinear elliptic equation. J Diff Equ. **92**, 163–178 (1991). doi:10.1016/0022-0396(91)90045-B
7. Cao, DM, Zhou, HS: Multiple positive solutions of nonhomogeneous semi-linear elliptic equations in $\mathbb{R}^N$. Proc Roy Soc Edinburgh, Sect A. **126**, 443–463 (1996). doi:10.1017/S0308210500022836
8. Hirano, N: Existence of entire positive solutions for nonhomogeneous elliptic equations. Nonlinear Anal. **29**, 889–901 (1997). doi:10.1016/S0362-546X(96)00176-9
9. Adachi, S, Tanaka, K: Four positive solutions for the semilinear elliptic equation: $-\Delta u + u = a(x)u^p + f(x)$ in $\mathbb{R}^N$. Calc Var Partial Diff Equ. **11**, 63–95 (2000). doi:10.1007/s005260050003
10. Hsu, TS, Lin, HL: Four positive solutions of semilinear elliptic equations involving concave and convex nonlinearities in $\mathbb{R}^N$. J Math Anal Appl. **365**, 758–775 (2010). doi:10.1016/j.jmaa.2009.12.004
11. Tarantello, G: On nonhomogeneous elliptic involving critical Sobolev exponent. Ann Inst H Poincaré Anal Non Linéaire. **9**, 281–304 (1992)
12. Wang, HC: Palais-Smale approaches to semilinear elliptic equations in unbounded domains. Electron J Diff Equ Monogragh 142 (2004). 06
13. Bahri, A, Li, YY: On a min-max procedure for the existence of a positive solution for certain scalar field equations in $\mathbb{R}^N$. Rev Mat Iberoamericana. **6**, 1–15 (1990)
14. Bahri, A, Lions, PL: On the existence of a positive solution of semilin-ear elliptic equations in unbounded domains. Ann Inst H Poincaré Anal Nonlinéaire. **14**, 365–413 (1997)
15. Gidas, B, Ni, WM, Nirenberg, L: Symmetry and related properties via the maximum principle. Comm Math Phys. **68**, 209–243 (1979). doi:10.1007/BF01221125
16. Kwong, MK: Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$. Arch Ration Mech Anal. **105**, 234–266 (1989)
17. Lions, PL: The concentration-compactness principle in the calculus of variations. The locally compact case. I II Ann Inst H Poincaré Anal Non-linéaire **1**, 109–145 (1984). 223-283

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