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Multiple solutions for *p*-Laplacian systems with critical homogeneous nonlinearity

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Abstract

In this article, we deal with existence and multiplicity of solutions to the *p*-Laplacian system of the type

$$\begin{cases} -\Delta_p u = \dfrac{1}{p^*}\dfrac{\partial F(x,u,v)}{\partial u} + \lambda|u|^{q-2}u, & x \in \Omega, \\ -\Delta_p v = \dfrac{1}{p^*}\dfrac{\partial F(x,u,v)}{\partial v} + \delta|v|^{q-2}v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the *p*-Laplacian operator, $N \geq p^2, 2 \leq p \leq q < p^*, p^* = \frac{Np}{N-p}$ denotes the Sobolev critical exponent, $F \in C^1(\bar{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ is a homogeneous function of degree p^*. By using the variational method and Ljusternik-Schnirelmann theory, we prove that the system has at least $\operatorname{cat}_\Omega(\Omega)$ distinct nonnegative solutions.

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1 Introduction and main results

In this article, we consider the existence and multiplicity of solutions for the following critical *p*-Laplacian system:

$$\begin{cases} -\Delta_p u = \dfrac{1}{p^*}\dfrac{\partial F(x,u,v)}{\partial u} + \lambda|u|^{q-2}u, & x \in \Omega, \\ -\Delta_p v = \dfrac{1}{p^*}\dfrac{\partial F(x,u,v)}{\partial v} + \delta|v|^{q-2}v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the *p*-Laplacian operator, $N \geq p^2, 2 \leq p \leq q < p^*, p^* = \frac{Np}{N-p}$ denotes the Sobolev critical exponent, $F \in C^1(\bar{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ is a homogeneous function of degree $p^*, \left(\dfrac{\partial F(x,u,v)}{\partial u}, \dfrac{\partial F(x,u,v)}{\partial v}\right) = \nabla F$ and λ, δ are positive parameters.

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The starting point on the study of the system (1.1) is its scalar version:

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda |u|^{q-2}u, \; x \in \Omega \\ u = 0, \qquad\qquad\qquad\qquad x \in \partial\Omega, \end{cases} \tag{1.2}$$

with $2 \le p \le q < p^*$. In a pioneer work Brezis and Nirenberg [1] showed that, if $p = q = 2$, the equation (1.2) has at least one positive solution provided $N \ge 4$ and $0 < \lambda < \lambda_1$, where λ_1 is the first eigenvalue of the operator $\left(-\Delta, H_0^1(\Omega)\right)$. In particular, the first multiplicity result for (1.2) has been achieved by Rey [2] in the semilinear case. Precisely Rey proved that if $N \ge 5$, $p = q = 2$, for λ small enough equation (1.2) has at least $\text{cat}_\Omega(\Omega)$ solutions, where $\text{cat}_\Omega(\Omega)$ denotes the Ljusternik-Schnirelmann category of Ω in itself. Furthermore, Alves and Ding [3] obtained the existence of $\text{cat}_\Omega(\Omega)$ positive solutions to equation (1.2) with $p \ge 2$, $p \le q < p^*$.

In recent years, more and more attention have been paid to the elliptic systems. In particular, Ding and Xiao [4] concerned the case $F(x, u, v) = 2|u|^\alpha |v|^\beta, \alpha > 1$, $\beta > 1$ satisfying $\alpha + \beta = p^*$, i.e., the following elliptic system

$$\begin{cases} -\Delta_p u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta + \lambda |u|^{q-2}u, \; x \in \Omega, \\ -\Delta_p v = \dfrac{2\beta}{\alpha+\beta}|u|^\alpha |v|^{\beta-2}v + \delta |v|^{q-2}v, \;\; x \in \Omega, \\ u = v = 0, \qquad\qquad\qquad\qquad\qquad x \in \partial\Omega. \end{cases} \tag{1.3}$$

Using standard tools of the variational theory and the Ljusternik-Schnirelmann category theory, Ding and Xiao [4] have proved that system (1.3) has at least $\text{cat}_\Omega(\Omega)$ positive solutions if λ, δ satisfied a certain condition. Hsu [5] obtained the existence of two positive solutions of system (1.3) with the sublinear perturbation of $1 < q < p < N$. Recently, Shen and Zhang [6] extended the results in [5] to the case (1.1) with $1 < q < p < N$ and obtained similar results. In this article, we study (1.1) and complement the results of [5,6] to the case $2 \le p \le q < p^*$, also extend the results of [4,7]. To the best of our knowledge, problem (1.1) has not been considered before. Thus it is necessary for us to investigate the critical p-Laplacian systems (1.1) deeply. For more similar problems, we refer to [8-17], and references therein.

Before stating our results, we need the following assumptions:

(F_0) $F \in C^1\left(\bar\Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+\right)$ and $F(x, tu, tv) = t^{p^*}F(x, u, v)(t > 0)$ holds for all $(x, u, v) \in \bar\Omega \times \mathbb{R}^+ \times \mathbb{R}^+$;

(F_1) $F(x, u, 0) = F(x, 0, v) = \frac{\partial F(x,u,0)}{\partial u} = \frac{\partial F(x,0,v)}{\partial v} = 0$, where $u, v \in \mathbb{R}^+$;

(F_2) $\frac{\partial F(x,u,v)}{\partial u}$, $\frac{\partial F(x,u,v)}{\partial v}$ are strictly increasing functions about u and v for all $u, v > 0$.

The main results we get are the following:

Theorem 1.1. *Suppose $N \ge p^2$ and F satisfies (F_0)-(F_2), then the problem (1.1) has at least one nonnegative solution for $2 \le p < q < p^*$ and λ, $\delta > 0$, or $q = p$ and λ, $\delta \in (0, \Lambda_1)$, where Λ_1 is the first eigenvalue of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$.*

Theorem 1.2. *Suppose $N \ge p^2$, $2 \le p \le q < p^*$ and F satisfies (F_0)-(F_2), then there exists $\Lambda > 0$ such that the problem (1.1) has at least $\text{cat}_\Omega(\Omega)$ distinct nonnegative solutions for λ, $\delta \in (0,\Lambda)$.*

Remark 1.1. *Theorem 1 in* [4]*is the special case of our Theorem 1.2 corresponding to* $F(x,u,v) = 2|u|^\alpha|v|^\beta, \alpha > 1, \beta > 1, \alpha + \beta = p^*$. *There are functions $F(x,u,v)$ satisfying the conditions of our Theorems 1.1 and 1.2. Some typical examples are:*

(i) $F(x, u, v) = \sum_{i=1}^{k} f_i(x)|u|^{\alpha_i}|v|^{\beta_i}$;

(ii) $F(x, u, v) = \begin{cases} f_1(x)|u|^{\frac{3}{2}}|v|^{\frac{5}{2}} + f_2(x)\dfrac{u^3 v^3}{u^2 + v^2}, & (u, v) \neq (0, 0), \\ 0, & (u, v) = (0, 0), \end{cases}$

where $f_i(x) \geq 0, f_i(x) \not\equiv 0, f_i(x) \in C(\bar{\Omega}) \cap L^\infty(\Omega), \alpha_i, \beta_i > 1, \alpha_i + \beta_i = p^*$. *Obviously, F(x, u, v) satisfies* (F_0)-(F_2).

This article is organized as follows. In Section 2, some notations and the Mountain-Pass levels are established and the Theorem 1.1 is proved. We present some technical lemmas which are crucial in the proof of the Theorem 1.2 in Section 3. Theorem 1.2 is proved in Section 4.

2 Notations and proof of Theorem 1.1

Throughout this article, C, C_i will denote various positive constants whose exact values are not important, $\rightarrow$ (respectively $\rightharpoonup$) denotes strong (respectively weak) convergence. $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)|/\varepsilon^t \leq C$, $o_m(1)$ denotes $o_m(1) \rightarrow 0$ as $m \rightarrow \infty$. $L^s(\Omega)(1 \leq s < +\infty)$ denotes Lebesgue spaces, the norm L^s is denoted by $|\cdot|_s$ for $1 \leq s < +\infty$. Let $B_r(x)$ denotes a ball centered at x with radius r, the dual space of a Banach space E will be denoted by E^{-1}. We define the product space $E := W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ endowed with the norm $\|(u, v)\|_E = \left(\|u\|_{W_0^{1,p}(\Omega)}^p + \|v\|_{W_0^{1,p}(\Omega)}^p\right)^{\frac{1}{p}}$, and the norm $\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}$.

Using assumption of (F_1), we have the so-called Euler identity

$$\frac{\partial F(x, u, v)}{\partial u}u + \frac{\partial F(x, u, v)}{\partial v}v = p^* F(x, u, v). \tag{2.1}$$

In addition, we can extend the function $F(x,u,v)$ to the whole $\bar{\Omega} \times \mathbb{R}^2$ by considering $\tilde{F}(x, u, v) = F(x, u^+, v^+)$, where $u^+ = \max\{u, 0\}$. It is easy to check that $\tilde{F}(x, u, v)$ is of class C^1 and its restriction to $\bar{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^+$ coincides with $F(x,u,v)$. In order to simplify the notation we shall write, from now on, only $F(x,u,v)$ to denote the above extension.

A pair of functions $(u, v) \in E$ is said to be a weak solution of problem (1.1) if

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u\nabla\varphi_1 + |\nabla v|^{p-2}\nabla v\nabla\varphi_2\right)dx - \frac{1}{p^*}\int_\Omega \left(\frac{\partial F(x, u, v)}{\partial u}\varphi_1 + \frac{\partial F(x, u, v)}{\partial v}\varphi_2\right)dx$$

$$- \int_\Omega \left(\lambda|u|^{q-2}u\varphi_1 + \delta|v|^{q-2}v\varphi_2\right)dx = 0, \quad \forall(\varphi_1, \varphi_2) \in E.$$

Thus, by (2.1) the corresponding energy functional of problem (1.1) is defined on E by

$$I_{\lambda,\delta}(u, v) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p + |\nabla v|^p\right)dx - \frac{1}{p^*}\int_\Omega F(x, u, v)dx - \frac{1}{q}\int_\Omega \left(\lambda|u|^q + \delta|v|^q\right)dx.$$

Using (F_0)-(F_2), we can verify $I_{\lambda,\ \delta}(u,\ v) \in C^1(E,\ \mathbb{R})$ (see [6]). It is well known that the weak solutions of problem (1.1) are the critical points of the energy functional $I_{\lambda,\ \delta}(u,\ v)$.

The functional $I \in C^1(E,\ \mathbb{R})$ is said to satisfy the $(PS)_c$ condition if any sequence $\{u_m\} \subset E$ such that as $m \to \infty$, $I(u_m) \to c$, $I'(u_m) \to 0$ strongly in E^{-1} contains a subsequence converging in E to a critical point of I. In this article, we will take $I = I_{\lambda,\ \delta}(u,\ v)$ and $E := W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.

As the energy functional $I_{\lambda,\delta}$ is not bounded below on E, we need to study $I_{\lambda,\delta}$ on the Nehari manifold

$$\mathcal{N}_{\lambda,\delta} = \left\{ (u,v) \in E \backslash \{(0,0)\} : \langle I'_{\lambda,\delta}(u,v), (u,v) \rangle = 0 \right\}.$$

Note that $\mathcal{N}_{\lambda,\delta}$ contains every nonzero solution of problem (1.1), and define the minimax $c_{\lambda,\delta}$ as

$$c_{\lambda,\delta} = \inf_{(u,v) \in \mathcal{N}_{\lambda,\delta}} I_{\lambda,\delta}(u,v).$$

Next, we present some properties of $c_{\lambda,\delta}$ and $\mathcal{N}_{\lambda,\delta}$. Its proofs can be done as [18, Theorem 4.2]. First of all, we note that there exists $\rho > 0$, such that

$$\left\| (u,v) \right\|_E \geq \rho > 0, \quad \forall (u,v) \in \mathcal{N}_{\lambda,\delta}.$$

It is standard to check that $I_{\lambda,\delta}$ satisfies Mountain-Pass geometry, so we can use the homogeneity of F to prove that $c_{\lambda,\delta}$ can be alternatively characterized by

$$c_{\lambda,\delta} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{\lambda,\delta}(\gamma(t)) = \inf_{(u,v) \in E \backslash \{(0,0)\}} \max_{t \geq 0} I_{\lambda,\delta}(t(u,v)) > 0, \tag{2.2}$$

where $\Gamma = \{\gamma \in C([0,\ 1],E) : \gamma(0) = 0, I_{\lambda,\delta}(\gamma(1)) < 0\}$. Moreover, for each $(u,\ v) \in E \backslash \{(0,0)\}$, there exists a unique $t^* > 0$ such that $t^*(u,v) \in \mathcal{N}_{\lambda,\delta}$. The maximum of the function $t \mapsto I_{\lambda,\delta}(t(u,\ v))$, for $t \geq 0$, is achieved at $t = t^*$.

In this section, we will find the range of c where the $(PS)_c$ condition holds for the functional $I_{\lambda,\delta}$. First let us define

$$S_F = \inf_{(u,v) \in E \backslash \{(0,0)\}} \left\{ \frac{\int_\Omega |\nabla u|^p + |\nabla v|^p \, dx}{\left(\int_\Omega F(x,u,v) \, dx \right)^{\frac{p}{p^*}}} : \int_\Omega F(x,u,v) \, dx > 0 \right\}. \tag{2.3}$$

Lemma 2.1. *If $N \geq p^2$ and F satisfies (F_0)-(F_2), then the functional $I_{\lambda,\delta}$ satisfies the $(PS)_c$ condition for all* $c < \frac{1}{N} S_F^{\frac{N}{p}}$, *provide one of the following conditions holds*

(i) $2 \leq p < q < p^*$ and $\lambda,\ \delta > 0$;

(ii) $q = p$, and $\lambda,\ \delta \in (0,\ \Lambda_1)$, where $\Lambda_1 > 0$ denotes the first eigenvalue of $\left(-\Delta_p, W_0^{1,p}(\Omega) \right)$.

Proof. Let $\{(u_m,\ v_m)\} \subset E$ such that $I'_{\lambda,\delta}(u_m, v_m) \to 0$ and $I_{\lambda,\delta}(u_m, v_m) \to c < \frac{1}{N} S_F^{\frac{N}{p}}$.

Now, we first prove that $\{(u_m,\ v_m)\}$ is bounded in E. If the above item (i) is true it suffices to use the definition of $I_{\lambda,\delta}$ to obtain $C_1 > 0$ such that

$$c + C_1 \left\|(u_m, v_m)\right\|_E + o_m(1) \geq I_{\lambda,\delta}(u_m, v_m) - \frac{1}{q}\left\langle I'_{\lambda,\delta}(u_m, v_m), (u_m, v_m)\right\rangle$$

$$= \left(\frac{1}{p} - \frac{1}{q}\right)\left\|(u_m, v_m)\right\|_E^p + \left(\frac{1}{q} - \frac{1}{p^*}\right)\int_\Omega F(x, u_m, v_m)dx$$

$$\geq \frac{q-p}{pq}\left\|(u_m, v_m)\right\|_E^p.$$

The above expression implies that $\{(u_m, v_m)\} \subset E$ is bounded. When (ii) occurs, in this case, it follows that

$$\int_\Omega \left(\lambda|u_m|^p + \delta|v_m|^p\right)dx \leq \max\{\lambda, \delta\}\int_\Omega \left(|u_m|^p + |v_m|^p\right)dx \leq \frac{\max\{\lambda, \delta\}}{\Lambda_1}\left\|(u_m, v_m)\right\|_E^p,$$

and therefore we get

$$c + C_1\left\|(u_m, v_m)\right\|_E + o_m(1) \geq I_{\lambda,\delta}(u_m, v_m) - \frac{1}{p^*}\left\langle I'_{\lambda,\delta}(u_m, v_m), (u_m, v_m)\right\rangle$$

$$= \left(\frac{1}{p} - \frac{1}{p^*}\right)\left\|(u_m, v_m)\right\|_E^p + \left(\frac{1}{p^*} - \frac{1}{p}\right)\int_\Omega \left(\lambda|u_m|^p + \delta|v_m|^p\right)dx$$

$$\geq \frac{1}{N}\left(1 - \frac{\max\{\lambda, \delta\}}{\Lambda_1}\right)\left\|(u_m, v_m)\right\|_E^p.$$

Since $\lambda, \delta \in (0, \Lambda_1)$ the boundedness of $\{(u_m, v_m)\}$ follows as the first case.

So, $\{(u_m, v_m)\}$ is bounded in E. Going if necessary to a subsequence, we can assume that

$$\begin{cases} (u_m, v_m) \rightharpoonup (u, v), \text{ in } E, \\ (u_m, v_m) \to (u, v), \text{ a.e. in } \Omega \\ (u_m, v_m) \to (u, v), \text{ in } L^s(\Omega) \times L^s(\Omega), 1 \leq s < p^*, \end{cases}$$

as $m \to \infty$. Clearly, we have

$$\int_\Omega \left(\lambda|u_m|^q + \delta|v_m|^q\right)dx = \int_\Omega \left(\lambda|u|^q + \delta|v|^q\right)dx + o_m(1). \tag{2.4}$$

Moreover, a standard argument shows that $I'_{\lambda,\delta}(u, v) = 0$. Thus we get

$$I_{\lambda,\delta}(u, v) = \frac{1}{p}\left\|(u, v)\right\|_E^p - \frac{1}{p^*}\int_\Omega F(x, u, v)dx - \frac{1}{q}\int_\Omega \left(\lambda|u|^q + \delta|v|^q\right)dx$$

$$= \left(\frac{1}{p} - \frac{1}{q}\right)\left\|(u, v)\right\|_E^p + \left(\frac{1}{q} - \frac{1}{p^*}\right)\int_\Omega F(x, u, v)dx \tag{2.5}$$

$$\geq 0.$$

Let $(\tilde{u}_m, \tilde{v}_m) = (u_m - u, v_m - v)$, then by Brezis-Lieb Lemma in [19] implies

$$\left\|(u_m, v_m)\right\|_E^p = \left\|(u, v)\right\|_E^p + \left\|(\tilde{u}_m, \tilde{v}_m)\right\|_E^p + o_m(1). \tag{2.6}$$

By the same method of [8, Lemma 5] (or [6, Lemma 3.4]), we obtain

$$\int_\Omega F(x, u_m, v_m)dx = \int_\Omega F(x, u, v)dx + \int_\Omega F(x, \tilde{u}_m, \tilde{v}_m)dx + o_m(1). \tag{2.7}$$

By (2.4)-(2.7) and the weak convergence of (u_m, v_m), we have

$$c + o_m(1) = I_{\lambda,\delta}(u,v) + \frac{1}{p}\left\|(\tilde{u}_m, \tilde{v}_m)\right\|_E^p - \frac{1}{p^*}\int_\Omega F(x, \tilde{u}_m, \tilde{v}_m)dx$$

$$\geq \frac{1}{p}\left\|(\tilde{u}_m, \tilde{v}_m)\right\|_E^p - \frac{1}{p^*}\int_\Omega F(x, \tilde{u}_m, \tilde{v}_m)dx. \qquad (2.8)$$

By using $I'_{\lambda,\delta}(u_m, v_m) \to 0$ and (2.4), (2.6), and (2.7), we get

$$o_m(1) = \left\langle I'_{\lambda,\delta}(u_m, v_m), (u_m, v_m)\right\rangle$$

$$= \left\|(u_m, v_m)\right\|_E^p - \int_\Omega F(x, u_m, v_m)dx - \int_\Omega \left(\lambda|u_m|^q + \delta|v_m|^q\right)dx$$

$$= \left\langle I'_{\lambda,\delta}(u,v), (u,v)\right\rangle + \left\|(\tilde{u}_m, \tilde{v}_m)\right\|_E^p - \int_\Omega F(x, \tilde{u}_m, \tilde{v}_m)dx.$$

Recalling that $I'_{\lambda,\delta}(u,v) = 0$, we can use the above equality and (2.8) to obtain

$$\lim_{m\to\infty}\left\|(\tilde{u}_m, \tilde{v}_m)\right\|_E^p = k = \lim_{m\to\infty}\int_\Omega F(x, \tilde{u}_m, \tilde{v}_m)dx, \quad c \geq \left(\frac{1}{p} - \frac{1}{p^*}\right)k = \frac{1}{N}k,$$

where k is a nonnegative number.

In view of the definition of S_F, we have that

$$\left\|(\tilde{u}_m, \tilde{v}_m)\right\|_E^p \geq S_F\left(\int_\Omega F(x, \tilde{u}_m, \tilde{v}_m)dx\right)^{\frac{p}{p^*}}.$$

Taking the limit we get $k \geq S_F k^{\frac{p}{p^*}}$. So, if $k > 0$, we conclude that $k \geq S_F^{\frac{N}{p}}$ and therefore

$$\frac{1}{N}S_F^{\frac{N}{p}} \leq \frac{1}{N}k \leq c < \frac{1}{N}S_F^{\frac{N}{p}},$$

which is a contradiction. Hence $k = 0$ and therefore $(u_m, v_m) \to (u, v)$ strongly in E.

Before presenting our next result we recall that, for each $\varepsilon > 0$, the function

$$U_\varepsilon(x) = \frac{C_N \cdot \varepsilon^{\frac{N-p}{p^2}}}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}}, \quad C_N = \left(N\left(\frac{N-p}{p-1}\right)^{p-1}\right)^{\frac{N-p}{p^2}}, \quad x \in \mathbb{R}^N \qquad (2.9)$$

satisfies

$$\left|\nabla U_\varepsilon(x)\right|_p^p = \left|U_\varepsilon(x)\right|_{p^*}^{p^*} = S^{\frac{N}{p}}, \qquad (2.10)$$

where S is the best constant of the Sobolev embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$. Thus, using [8, Lemma 3] and the homogeneity of F, we obtain $A, B > 0$ such that

$$S_F = \frac{\left\| (AU_\varepsilon, BU_\varepsilon) \right\|_E^p}{\left(\int_{\mathbb{R}^N} F(x, AU_\varepsilon BU_e) dx \right)^{\frac{p}{p^*}}} = \frac{A^p + B^p}{(F(x, A, B))^{\frac{p}{p^*}}} \cdot \frac{S^{\frac{N}{p}}}{|U_\varepsilon|_{p^*}^p},$$

from which and (2.10) it follows that

$$S_F = \frac{A^p + B^p}{(F(x, A, B))^{\frac{p}{p^*}}} S. \tag{2.11}$$

We define a cut-off function $\phi(x) \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi(x) = 1$ if $|x| \leq R$; $\varphi(x) = 0$ if $|x| \geq 2R$ and $0 \leq \varphi(x) \leq 1$, where $B_{2R}(0) \subset \Omega$, set $u_\varepsilon = \dfrac{\phi(x) U_\varepsilon}{|\phi U_\varepsilon|_{p^*}}$, where U_ε was defined in (2.9). So that $|u_\varepsilon|_{p^*} = 1$. Then, we can get the following results from [[20], Lemma 11.1]:

$$\|u_\varepsilon\|_{W_0^{1,p}(\Omega)}^p = S + O(\varepsilon^{\frac{N-p}{p}}), \tag{2.12}$$

$$\int_\Omega |u_\varepsilon|^\xi dx \approx \begin{cases} \varepsilon^{\frac{N-p}{p^2}\xi}, & \text{if } 1 < \xi < p^* \left(1 - \frac{1}{p} \right), \\ \varepsilon^{\frac{N-p}{p^2}\xi} |\ln \varepsilon|, & \text{if } \xi = p^* \left(1 - \frac{1}{p} \right), \\ \varepsilon^{\frac{(p-1)(Np - \xi(N-p))}{p^2}}, & \text{if } p^* \left(1 - \frac{1}{p} \right) < \xi < p^*, \end{cases} \tag{2.13}$$

where $A \approx B$ means $C_1 B \leq A \leq C_2 B$.

Lemma 2.2. *Suppose that F satisfies (F_0)-(F_2), $2 \leq p < q < p^*$ and $\lambda > 0$, $\delta > 0$, then $c_{\lambda,\delta} < \frac{1}{N} S_F^{\frac{N}{p}}$. The same result holds if $q = p$ and $\lambda, \delta \in (0, \Lambda_1)$, where $\Lambda_1 > 0$ denotes the first eigenvalue of $\left(-\Delta_p, W_0^{1,p}(\Omega) \right)$.*

Proof. We can use the homogeneity of F to get, for any $t \geq 0$,

$$I_{\lambda,\delta}(tAu_\varepsilon, tBu_\varepsilon) = \frac{t^p}{p} \left(A^p + B^p \right) \|u_\varepsilon\|_{W_0^{1,p}(\Omega)}^p - \frac{t^{p^*}}{p^*} F(x, A, B) - \frac{t^q}{q} (\lambda A^q + \delta B^q) |u_\varepsilon|_q^q.$$

We shall denote by $h(t)$ the right-hand side of the above equality and consider two distinct cases.

Case 1. $2 \leq p < q < p^*$.

From the fact that $\lim\limits_{t \to +\infty} h(t) = -\infty$ and $h(t) > 0$ when t is close to 0, there exists $t_\varepsilon > 0$ such that

$$h(t_\varepsilon) = \max_{t \geq 0} h(t). \tag{2.14}$$

Let

$$g(t) = \frac{t^p}{p} \left(A^p + B^p \right) \|u_\varepsilon\|_{W_0^{1,p}(\Omega)}^p - \frac{t^{p^*}}{p^*} F(x, A, B), \quad t \geq 0,$$

and notice that the maximum value of $g(t)$ occurs at the point

$$\tilde{t}_\varepsilon = \left(\frac{(A^p + B^p)\, \|u_\varepsilon\|^p_{W_0^{1,p}(\Omega)}}{F(x, A, B)} \right)^{\frac{1}{p^* - p}}.$$

So, for each $t \geq 0$,

$$g(t) \leq g(\tilde{t}_\varepsilon) = \frac{1}{N} \left(\frac{(A^p + B^p)\, \|u_\varepsilon\|^p_{W_0^{1,p}(\Omega)}}{(F(x, A, B))^{\frac{p}{p^*}}} \right)^{\frac{N}{p}},$$

and therefore

$$h(t_\varepsilon) \leq \frac{1}{N} \left(\frac{(A^p + B^p)\, \|u_\varepsilon\|^p_{W_0^{1,p}(\Omega)}}{(F(x, A, B))^{\frac{p}{p^*}}} \right)^{\frac{N}{p}} - \frac{t_\varepsilon^q}{q} \left(\lambda A^q + \delta B^q \right) |u_\varepsilon|^q_q. \tag{2.15}$$

We claim that, for some $C_2 > 0$, there holds

$$t_\varepsilon^q \left(\lambda A^q + \delta B^q \right) \geq C_2.$$

Indeed, if this is not the case, we have that $t_{\varepsilon_m} \to 0$ for some sequence $\varepsilon_m \to 0^+$, then,

$$0 < c_{\lambda,\delta} \leq \sup_{t \geq 0} I_{\lambda,\delta} \left(tAu_{\varepsilon_m}, tBu_{\varepsilon_m} \right) = I_{\lambda,\delta} \left(t_{\varepsilon_m} Au_{\varepsilon_m}, t_{\varepsilon_m} Bu_{\varepsilon_m} \right) \to 0,$$

which is a contradiction. So, the claim holds and we infer from (2.15) and (2.11)-(2.13) that

$$\begin{aligned}
h(t_\varepsilon) &\leq \frac{1}{N} \left(\frac{A^p + B^p}{(F(x, A, B))^{\frac{p}{p^*}}} \left(S + O\left(\varepsilon^{\frac{N-p}{p}} \right) \right) \right)^{\frac{N}{p}} - C_3\, |u_\varepsilon|^q_q \\
&\leq \frac{1}{N} S_F^{\frac{N}{p}} + O\left(\varepsilon^{\frac{N-p}{p}} \right) - C_3\, |u_\varepsilon|^q_q \\
&\leq \frac{1}{N} S_F^{\frac{N}{p}} + O\left(\varepsilon^{\frac{N-p}{p}} \right) - O\left(\varepsilon^{\frac{(p-1)(Np - q(N-p))}{p^2}} \right),
\end{aligned} \tag{2.16}$$

where $C_3 = \frac{C_2}{q}$. We know $p^* \left(1 - \frac{1}{p} \right) \leq p < q < p^*$ if $N \geq p^2$. By $N \geq p^2$ and $2 \leq p < q < p^*$ we obtain $\frac{N-p}{p} > \frac{(p-1)(Np - q(N-p))}{p^2}$. Thus from the above inequality we conclude that, for each $\varepsilon > 0$ small, there holds

$$c_{\lambda,\delta} \leq \sup_{t \geq 0} I_{\lambda,\delta} \left(tAu_{\varepsilon_m}, tBu_{\varepsilon_m} \right) = h(t_\varepsilon) < \frac{1}{N} S_F^{\frac{N}{p}}.$$

Case 2. $q = p$.

In this case, we have that $h'(t) = 0$ if and only if,

$$(A^p + B^p)\, \|u_\varepsilon\|^p_{W_0^{1,p}(\Omega)} - \left(\lambda A^p - \delta B^p \right) |u_\varepsilon|^p_p = t^{p^* - p} F(x, A, B).$$

Since we suppose $\lambda, \delta \in (0, \Lambda_1)$, we can use Poincaré's inequality to obtain

$$
\begin{aligned}
\left(\lambda A^p + \delta B^p\right) |u_\varepsilon|_p^p &\leq \max\{\lambda, \delta\} \left(A^p + B^p\right) |u_\varepsilon|_p^p \\
&< \Lambda_1 \left(A^p + B^p\right) |u_\varepsilon|_p^p \\
&\leq \left(A^p + B^p\right) \|u_\varepsilon\|_{W_0^{1,p}(\Omega)}^p .
\end{aligned}
$$

Thus, there exists $t_\varepsilon > 0$ satisfying (2.14).

Arguing as in the first case we conclude that, from (2.16) for $\varepsilon > 0$ small, there holds

$$
\begin{aligned}
h(t_\varepsilon) &\leq \frac{1}{N} S_F^{\frac{N}{p}} + O(\varepsilon^{\frac{N-p}{p}}) - C_3 |u_\varepsilon|_p^p \\
&= \begin{cases} \dfrac{1}{N} S_F^{\frac{N}{p}} + O(\varepsilon^{p-1}) - O(\varepsilon^{p-1} |\ln \varepsilon|), & N = p^2, \\[2mm] \dfrac{1}{N} S_F^{\frac{N}{p}} + O(\varepsilon^{\frac{N-p}{p}}) - O(\varepsilon^{p-1}), & N > p^2. \end{cases}
\end{aligned}
$$

Because $p^* \left(1 - \frac{1}{p}\right) < p$ if $N > p^2$ and $p^* \left(1 - \frac{1}{p}\right) = p$ if $N = p^2$, then $\varepsilon^{p-1} = o(\varepsilon^{p-1} | \ln \varepsilon|)$. If $N > p^2$, then $\frac{N-p}{p} > p - 1$, so $\varepsilon^{\frac{N-p}{p}} = o(\varepsilon^{p-1})$. Choosing $\varepsilon > 0$ small enough, we have

$$
c_{\lambda,\delta} \leq \sup_{t \geq 0} I_{\lambda,\delta}(tAu_\varepsilon, tBu_\varepsilon) = h(t_\varepsilon) < \frac{1}{N} S_F^{\frac{N}{p}} .
$$

This concludes the proof.

By Lemmas 2.1 and 2.2 we can prove our first result.

Proof of Theorem 1.1.

Since $I_{\lambda,\delta}$ satisfies the geometric conditions of the Mountain-Pass theorem, there exists $\{(u_m, v_m)\} \subset E$ such that $I_{\lambda,\delta}(u_m, v_m) \to c_{\lambda,\delta}, I'_{\lambda,\delta}(u_m, v_m) \to 0$. It follows from Lemmas 2.1 and 2.2 that $\{(u_m, v_m)\}$ converges, along a subsequence, to a nonzero critical point $(u,v) \in E$ of $I_{\lambda,\delta}$. Then, if we denote by $u^- = \max\{-u, 0\}$ and $v^- = \max\{-v, 0\}$ the negative part of u and v, respectively, we get

$$
\begin{aligned}
0 = \left\langle I'_{\lambda,\delta}(u, v), (u^-, v^-) \right\rangle &= - \left\| (u^-, v^-) \right\|_E^p - \frac{1}{p^*} \int_\Omega \left(\frac{\partial F(x, u, v)}{\partial u} u^- + \frac{\partial F(x, u, v)}{\partial v} v^- \right) dx \\
&\quad - \int_\Omega \left(\lambda |u|^{q-2} u u^- + \delta |v|^{q-2} v v^- \right) dx \\
&\leq - \left\| (u^-, v^-) \right\|_E^p ,
\end{aligned}
$$

it follows that $(u^-, v^-) = (0, 0)$. Hence, $u, v \geq 0$ in Ω. The Theorem 1.1 is proved.

We finalize this section with the study of the asymptotic behavior of the minimax level $c_{\lambda,\delta}$ as both the parameters λ, δ approach zero.

Lemma 2.3. $\displaystyle \lim_{\lambda,\delta \to 0^+} c_{\lambda,\delta} = c_{0,0} = \frac{1}{N} S_F^{\frac{N}{p}}$:

Proof. We first prove the second equality. It follows from $\lambda = \delta = 0$ that $\lambda |u|^q + \delta |v|^q \equiv 0$. If $A, B, u_\varepsilon, g_\varepsilon$, and t_ε are the same as those in the proof of Lemma 2.2, we have that $(t_\varepsilon Au_\varepsilon, t_\varepsilon Bu_\varepsilon) \in \mathcal{N}_{0,0}$. Thus

$$c_{0,0} \leq I_{0,0}\left(t_\varepsilon A u_\varepsilon, t_\varepsilon B u_\varepsilon\right)$$

$$= \frac{1}{N}\left(\frac{A^p + B^p}{(F(x,A,B))^{\frac{p}{p^*}}} \|u_\varepsilon\|^p_{W_0^{1,p}(\Omega)}\right)^{\frac{N}{p}}$$

$$= \frac{1}{N}\left(\frac{A^p + B^p}{(F(x,A,B))^{\frac{p}{p^*}}} \left(S + O\left(\varepsilon^{\frac{N-p}{p}}\right)\right)\right)^{\frac{N}{p}}.$$

Taking the limit as $\varepsilon \to 0^+$ and using (2.11), we conclude that $c_{0,0} \leq \frac{1}{N} S_F^{\frac{N}{p}}$.

In order to obtain the reverse inequality we consider $\{(u_m, v_m)\} \subseteq E$ such that $I_{0,0}$ $(u_m, v_m) \to c_{0,0}$ and $I_{0,0}'(u_m, v_m) \to 0$. It is easy to show that the sequence $\{(u_m, v_m)\}$ is bounded in E and therefore $\langle I_{0,0}'(u_m, v_m), (u_m, v_m)\rangle = \|(u_m, v_m)\|^p_E - \int_\Omega F(x, u_m, v_m)dx = o_m(1)$. It follows that

$$\lim_{m \to \infty} \|(u_m, v_m)\|^p_E = l = \lim_{m \to \infty} \int_\Omega F(x, u_m, v_m)dx.$$

Taking the limit in the inequality $S_F\left(\int_\Omega F(x, u_m, v_m)dx\right)^{\frac{p}{p^*}} \leq \|(u_m, v_m)\|^p_E$ we conclude, as in the proof of Lemma 2.1, that $Nc_{0,0} = l \geq S_F^{\frac{N}{p}}$. Hence,

$$c_{0,0} = \lim_{m \to \infty} I_{0,0}(u_m, v_m) = \lim_{m \to \infty}\left(\frac{1}{p}\|(u_m, v_m)\|^p_E - \frac{1}{p^*}\int_\Omega F(x, u_m, v_m)dx\right) = \frac{1}{N}l \geq \frac{1}{N} S_F^{\frac{N}{p}},$$

and therefore $c_{0,0} = \frac{1}{N} S_F^{\frac{N}{p}}$.

We proceed now with the calculation of $\lim_{\lambda,\delta \to 0^+} c_{\lambda,\delta}$. Let $\{\lambda_m\}, \{\delta_m\} \subseteq \mathbb{R}^+$ such that λ_m, $\delta_m \to 0^+$. Since λ_m, δ_m are positive, we have that $\int_\Omega \left(\lambda_m |u|^q + \delta_m |v|^q\right)dx \geq 0$ whenever (u, v) is nonnegative. Thus, for this kind of function, we have that $I_{\lambda_m, \delta_m}(u, v) \leq I_{0,0}(u, v)$.

It follows that

$$c_{\lambda_m, \delta_m} = \inf_{(u,v) \neq (0,0)} \max_{t \geq 0} I_{\lambda_m, \delta_m}(t(u, v))$$

$$\leq \inf_{\substack{(u,v) \neq (0,0), \\ (u,v) \geq 0}} \max_{t \geq 0} I_{\lambda_m, \delta_m}(t(u, v))$$

$$\leq \inf_{\substack{(u,v) \neq (0,0), \\ (u,v) \geq 0}} \max_{t \geq 0} I_{0,0}(t(u, v)) = c_{0,0},$$

in the last equality, we have used the infimum $c_{0,0}$ which can be attained at a nonnegative solution. The above inequality implies that

$$\limsup_{m \to \infty} c_{\lambda_m, \delta_m} \leq c_{0,0}. \tag{2.17}$$

On the other hand, it follows from Theorem 1.1 that there exists $\{(u_m, v_m)\} \subseteq E$ such that

$$I_{\lambda_m, \delta_m}(u_m, v_m) = c_{\lambda_m, \delta_m}, \quad I'_{\lambda_m, \delta_m}(u_m, v_m) \to 0.$$

Since c_{λ_m, δ_m} is bounded, the same argument performed in the proof of Lemma 2.1 implies that $\{(u_m, v_m)\}$ is bounded in E. Since

$$\lim_{m \to \infty} \int_\Omega \left(\lambda_m |u_m|^q + \delta_m |v_m|^q \right) dx = 0. \tag{2.18}$$

Let $t_m > 0$ be such that $t_m(u_m, v_m) \in \mathcal{N}_{0,0}$. Since $(u_m, v_m) \in \mathcal{N}_{\lambda_m, \delta_m}$, we have that

$$c_{0,0} \leq I_{0,0}(t_m(u_m, v_m))$$

$$= I_{\lambda_m, \delta_m}(t_m(u_m, v_m)) + \frac{t_m^q}{q} \int_\Omega \left(\lambda_m |u_m|^q + \delta_m |v_m|^q \right) dx$$

$$\leq I_{\lambda_m, \delta_m}(u_m, v_m) + \frac{t_m^q}{q} \int_\Omega \left(\lambda_m |u_m|^q + \delta_m |v_m|^q \right) dx$$

$$= c_{\lambda_m, \delta_m} + \frac{t_m^q}{q} \int_\Omega \left(\lambda_m |u_m|^q + \delta_m |v_m|^q \right) dx.$$

If $\{t_m\}$ is bounded, we can use the above estimate and (2.18) to get

$$c_{0,0} \leq \liminf_{m \to \infty} c_{\lambda_m, \delta_m}.$$

This and (2.17) we get

$$c_{0,0} \leq \liminf_{m \to \infty} c_{\lambda_m, \delta_m} \leq \limsup_{m \to \infty} c_{\lambda_m, \delta_m} \leq c_{0,0},$$

that is $c_{0,0} = \lim_{m \to \infty} c_{\lambda_m, \delta_m}$.

It remains to check that $\{t_m\}$ is bounded. A straightforward calculation shows that

$$t_m = \left(\frac{\|(u_m, v_m)\|_E^p}{\int_\Omega F(x, u_m, v_m) dx} \right)^{\frac{1}{p^* - p}}. \tag{2.19}$$

Since $(u_m, v_m) \in \mathcal{N}_{\lambda_m, \delta_m}$, we obtain

$$\|(u_m, v_m)\|_E^p = \int_\Omega F(x, u_m, v_m) dx + \int_\Omega \left(\lambda_m |u_m|^q + \delta_m |v_m|^q \right) dx \leq S_F^{-\frac{p^*}{p}} \|(u_m, v_m)\|_E^{p^*} + o_m(1).$$

Hence $\|(u_m, v_m)\|_E^p \geq C_4 > 0$, and therefore from the above expression it follows that $\int_\Omega F(x, u_m, v_m) dx \geq C_5 > 0$. Thus, the boundedness of $\{(u_m, v_m)\}$ and (2.19) imply that $\{t_m\}$ is bounded. This completes the proof.

3 Some technical lemmas

In this section, we denote by $\mathcal{M}(\Omega)$ the Banach space of finite Radon measures over Ω equipped with the norm $\|\sigma\| = \sup_{\substack{\varphi \in C_0(\Omega) \\ |\varphi|_\infty = 1}} |\sigma(\varphi)|$. A sequence $\{\sigma_m\} \subset \mathcal{M}(\Omega)$ is said to converge weakly to $\sigma \in \mathcal{M}(\Omega)$ provided $\sigma_m(\phi) \to \sigma(\phi)$ for all $\phi \in C_0(\Omega)$. By [18, Theorem 1.39], every bounded sequence $\{\sigma_m\} \subset \mathcal{M}(\Omega)$ contains a weakly convergent subsequence.

The next lemma is a version of the second concentration-compactness lemma of Lions [21]. It is also inspired by [18, Lemma 1.40] and [[22], Lemma 2.4].

Lemma 3.1. *Suppose that the sequence* $\{(u_m, v_m)\} \subset D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N)$ *satisfies*

$$(u_m, v_m) \rightharpoonup (u, v) \text{ in } D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N),$$

$$(u_m, v_m) \to (u, v) \text{ a.e. } x \in \mathbb{R}^N,$$

$$(\nabla u_m, \nabla v_m) \to (\nabla u, \nabla v) \text{ a.e. } x \in \mathbb{R}^N,$$

$$\left|\nabla(u_m - u)\right|^p \rightharpoonup \mu, \quad \left|\nabla(v_m - v)\right|^p \rightharpoonup \sigma \text{ in } \mathcal{M}(\mathbb{R}^N),$$

$$F(x, u_m - u, v_m - v) \rightharpoonup \nu \text{ in } \mathcal{M}(\mathbb{R}^N),$$

and define

$$\begin{cases} \mu_\infty = \lim\limits_{R \to \infty} \limsup\limits_{m \to \infty} \int\limits_{|x|>R} |\nabla u_m|^p dx, \\ \sigma_\infty = \lim\limits_{R \to \infty} \limsup\limits_{m \to \infty} \int\limits_{|x|>R} |\nabla v_m|^p dx, \\ \nu_\infty = \lim\limits_{R \to \infty} \limsup\limits_{m \to \infty} \int\limits_{|x|>R} F(x, u_m, v_m) dx, \end{cases} \tag{3.1}$$

then it follows that

$$\limsup_{m \to \infty} \int\limits_{\mathbb{R}^N} |\nabla u_m|^p dx = \|\mu\| + \mu_\infty + \int\limits_{\mathbb{R}^N} |\nabla u|^p dx, \tag{3.2}$$

$$\limsup_{m \to \infty} \int\limits_{\mathbb{R}^N} |\nabla v_m|^p dx = \|\sigma\| + \sigma_\infty + \int\limits_{\mathbb{R}^N} |\nabla u|^p dx, \tag{3.3}$$

$$\limsup_{m \to \infty} \int\limits_{\mathbb{R}^N} F(x, u_m, v_m) dx = \|\nu\| + \nu_\infty + \int\limits_{\mathbb{R}^N} F(x, u, v) dx, \tag{3.4}$$

$$\|\nu\|^{\frac{p}{p^*}} \leq S_F^{-1}(\|\mu\| + \|\sigma\|), \quad \nu_\infty^{\frac{p}{p^*}} \leq S_F^{-1}(\mu_\infty + \sigma_\infty). \tag{3.5}$$

Moreover, if $(u, v) = (0, 0)$ *and* $\|\nu\|^{\frac{p}{p^*}} = S_F^{-1}(\|\mu\| + \|\sigma\|)$, *then the measures* μ, ν, *and* σ *are concentrated at a single point, respectively.*

Proof. We first recall that, in view of the definition of S_F, for each nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^N)$ we have

$$\left(\int\limits_{\mathbb{R}^N} |\varphi|^{p^*} F(x, u_m, v_m) dx\right)^{\frac{p}{p^*}} = \left(\int\limits_{\mathbb{R}^N} F(x, \varphi u_m, \varphi v_m) dx\right)^{\frac{p}{p^*}} \leq S_F^{-1} \|(\varphi u_m, \varphi v_m)\|_E^p.$$

Moreover, arguing as [8, Lemma 5], we have that

$$\int\limits_{\mathbb{R}^N} \varphi F(x, u_m - u, v_m - v) dx = \int\limits_{\mathbb{R}^N} \varphi F(x, u_m, v_m) dx - \int\limits_{\mathbb{R}^N} \varphi F(x, u, v) dx + o_m(1).$$

Since F is p^*-homogeneous, we can use the two above expressions and argue along the same line of the proof of Lemma 1.40 in [18] to conclude that (3.2)-(3.5) hold. If

$(u, v) = (0,0)$ and $\|v\|^{\frac{p}{p^*}} = S_F^{-1}(\|\mu\| + \|\sigma\|)$, the same argument of step 3 of the proof of Lemma 1.40 in [18] implies that the measures μ, v and σ are concentrated at a single point, respectively.

Remark 3.1. *We notice that the last conclusion of the above result holds even if (u, v) ⊠ $(0,0)$. Indeed, in this case we can define $(\tilde{u}_m, \tilde{v}_m) = (u_m - u, v_m - v)$ and notice that*

$$(\tilde{u}_m, \tilde{v}_m) \rightharpoonup (\tilde{u}, \tilde{v}) \ in \ D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N),$$
$$(\tilde{u}_m, \tilde{v}_m) \to (0, 0) \ a.e. \ x \in \mathbb{R}^N,$$
$$(\nabla\tilde{u}_m, \nabla\tilde{v}_m) \to (\nabla\tilde{u}, \nabla\tilde{v}) \ a.e. \ x \in \mathbb{R}^N,$$
$$\left|\nabla(\tilde{u}_m - \tilde{u})\right|^p \rightharpoonup \tilde{\mu}, \quad \left|\nabla(\tilde{v}_m - \tilde{v})\right|^p \rightharpoonup \tilde{\sigma} \ in \ \mathcal{M}(\mathbb{R}^N),$$
$$F(x, \tilde{u}_m - \tilde{u}, \tilde{v}_m - \tilde{v}) \rightharpoonup \tilde{v} \ in \ \mathcal{M}(\mathbb{R}^N),$$

Since $(\tilde{u}_m - \tilde{u}, \tilde{v}_m - \tilde{v}) = (u_m - u, v_m - v)$ and therefore $\tilde{\mu} = \mu$, $\tilde{\sigma} = \sigma$, and $\tilde{v} = v$, where μ,σ, and v are the same as those in Lemma 3.1. Thus, if $\|v\|^{\frac{p}{p^*}} = S_F^{-1}(\|\mu\| + \|\sigma\|)$ we also have that $\|\tilde{v}\|^{\frac{p}{p^*}} = S_F^{-1}(\|\tilde{\mu}\| + \|\tilde{\sigma}\|)$ and the result follows from the last part of Lemma 3.1.

Now, we introduce the following Lemma.

Lemma 3.2. *Suppose $\{(u_m, v_m)\} \subset E$ such that $\int_\Omega F(x, u_m, v_m)dx = 1$ and $\lim_{m\to\infty} \|(u_m, v_m)\|_E^p = S_F$. Then there exist $\{r_m\} \subset (0, +\infty)$ and $\{y_m\} \subset \mathbb{R}^N$ such that*

$$(\omega_m^1(x), \omega_m^2(x)) = r_m^{\frac{N-p}{p}} (u_m(r_m x + y_m), v_m(r_m x + y_m)) \tag{3.6}$$

contains a convergent subsequence denoted again by $\{(\omega_m^1(x), \omega_m^2(x))\}$ such that $(\omega_m^1(x), \omega_m^2(x)) \to (\omega_1, \omega_2)$ in $D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N)$. Moreover, as $m \to \infty$, we have $r_m \to 0$ and $y_m \to y \in \bar{\Omega}$.

Proof. For each $r > 0$, we consider the Lévy concentration functions

$$H_m(r) = \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} F(x, u_m, v_m)dx.$$

Since for every m,

$$\lim_{r\to 0^+} H_m(r) = 0, \quad \lim_{r\to\infty} H_m(r) = 1,$$

there exist $r_m > 0$ and a sequence $\{y_m^k\} \subset \mathbb{R}^N$ satisfying

$$\frac{1}{2} = H_m(r_m) = \lim_{k\to\infty} \int_{B_{r_m}(y_m^k)} F(x, u_m, v_m)dx.$$

Recalling that $\lim_{|y|\to\infty} \int_{B_{r_m}(y)} F(x, u_m, v_m)dx = 0$, we conclude that $\{y_m^k\}$ is bounded.

Hence, up to a subsequence, $\lim_{k\to\infty} y_m^k = y_m \in \mathbb{R}^N$ and we obtain

$$\frac{1}{2} = \int_{B_{r_m}(y_m)} F(x, u_m, v_m)dx.$$

We shall prove that the above sequences $\{r_m\}$ and $\{y_m\}$ satisfy the statements of the lemma. First notice that

$$\frac{1}{2} = \int_{B_{r_m}(y_m)} F(x, u_m, v_m)dx = \int_{B_1(0)} F(x, \omega_m^1, \omega_m^2)dx = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} F(x, \omega_m^1, \omega_m^2)dx. \tag{3.7}$$

By (3.6), a straightforward calculation provides

$$\lim_{m \to \infty} \left\| (\omega_m^1, \omega_m^2) \right\|_E^p = \lim_{m \to \infty} \left\| (u_m, v_m) \right\|_E^p = S_F, \int_{\mathbb{R}^N} F(x, \omega_m^1, \omega_m^2)dx = 1.$$

Hence, we can apply Lemma 3.1 to obtain $(\omega_1, \omega_2) \in D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N)$ satisfying

$$S_F = \|\mu\| + \mu_\infty + \|\sigma\| + \sigma_\infty + \left\| (\omega_1, \omega_2) \right\|_E^p, \quad 1 = \|v\| + v_\infty + \int_{\mathbb{R}^N} F(x, \omega_1, \omega_2)dx, \tag{3.8}$$

$$\|v\|^{\frac{p}{p^*}} \leq S_F^{-1}(\|\mu\| + \|\sigma\|), \quad v_\infty^{\frac{p}{p^*}} \leq S_F^{-1}(\mu_\infty + \sigma_\infty). \tag{3.9}$$

The second equality in (3.8) implies that $\|v\|, v_\infty, \int_{\mathbb{R}^N} F(x, \omega_1, \omega_2)dx \in [0, 1]$. If one of these values belongs to the open interval $(0,1)$, we can use (3.8), $\frac{p}{p^*} < 1$, $\left(\int_{\mathbb{R}^N} F(x, \omega_1, \omega_2)dx \right)^{\frac{p}{p^*}} \leq S_F^{-1} \left\| (\omega_1, \omega_2) \right\|_E^p$ and (3.9) to get

$$S_F = S_F \left(\|v\| + v_\infty + \int_{\mathbb{R}^N} F(x, \omega_1, \omega_2)dx \right)$$

$$< S_F \left(\|v\|^{\frac{p}{p^*}} + v_\infty^{\frac{p}{p^*}} + \left(\int_{\mathbb{R}^N} F(x, \omega_1, \omega_2)dx \right)^{\frac{p}{p^*}} \right)$$

$$\leq S_F,$$

which is a contradiction. Thus $\|v\|, v_\infty, \int_{\mathbb{R}^N} F(x, \omega_1, \omega_2)dx \in \{0, 1\}$. Actually, it follows from (3.7) that $\int_{|x|>R} F(x, \omega_m^1, \omega_m^2)dx \leq \frac{1}{2}$ for any $R > 1$. Thus, we conclude that $v_\infty = 0$.

Let us prove that $||v|| = 0$. Arguing by contradiction, then $||v|| = 1$. It follows from the first equality in (3.8) that $S_F \geq ||\mu|| + ||\sigma||$. On the other hand, the first inequality in (3.9) provides $||\mu|| + ||\sigma|| \geq S_F$. Hence, we conclude that $||\mu|| + ||\sigma|| = S_F$. Since we suppose that $||v|| = 1$ we obtain $\|v\|^{\frac{p}{p^*}} = S_F^{-1}(\|\mu\| + \|\sigma\|)$. It follows from Remark 3.1 that $v = \delta_{x_0}$ for some $x_0 \in \mathbb{R}^N$. Thus, from (3.7), we get

$$\frac{1}{2} \geq \lim_{m \to \infty} \int_{B_1(x_0)} F(x, \omega_m^1, \omega_m^2)dx = \int_{B_1(x_0)} dv = \|v\| = 1.$$

This contradiction proves that $\|v\| = 0$.

Since $\|v\| = v_\infty = 0$, we have that $\int_{\mathbb{R}^N} F(x, \omega_1, \omega_2)dx = 1$. This and (3.8) provide

$$\lim_{m \to \infty} \left\|(\omega_m^1, \omega_m^2)\right\|_E^p = S_F \geq \left\|(\omega_1, \omega_2)\right\|_E^p \geq S_F \left(\int_{\mathbb{R}^N} F(x, \omega_1, \omega_2)dx\right)^{\frac{p}{p^*}} = S_F.$$

So, $\left\|(\omega_1, \omega_2)\right\|_E^p = S_F$ and therefore $(\omega_m^1, \omega_m^2) \to (\omega_1, \omega_2) \not\equiv (0, 0)$ strongly in $D^{1,p}(\mathbb{R}^N)$ $\times D^{1,p}(\mathbb{R}^N)$ and $(\omega_m^1(x), \omega_m^2(x)) \to (\omega_1(x), \omega_2(x))$ for a.e. $x \in \mathbb{R}^N$. In order to conclude the proof we notice that

$$\left\|(\omega_m^1, \omega_m^2)\right\|_{L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)} = \frac{1}{r_m^{\frac{p}{}}} \left\|(u_m, v_m)\right\|_{L^p(\Omega) \times L^p(\Omega)}.$$

Since $\{(u_m, v_m)\}$ is bounded and $(\omega_1, \omega_2) \boxtimes (0,0)$, we infer from the above equality that, up to a subsequence, $r_m \to r_0 \geq 0$. If $|y_m| \to \infty$, for each fixed $x \in \mathbb{R}^N$, we have that there exists $m_x \in N$ such that $r_m x + y_m \notin \Omega$ for $m \geq m_x$. For such values of m we have that $(\omega_m^1(x), \omega_m^2(x)) = (0, 0)$. Taking the limit and recalling that $x \in \mathbb{R}$ is arbitrary, we conclude that $(\omega_1, \omega_2) = (0,0)$, which is a contradiction. So, along a subsequence, $y_m \to y \in \mathbb{R}^N$.

We claim that $r_0 = 0$. Indeed, suppose by contradiction that $r_0 > 0$. Then, as m becomes large, the set $\Omega_m = (\Omega - y_m)/r_m$ approaches $\Omega_0 = (\Omega - y)/r_0 \neq \mathbb{R}^N$. This implies that ω_1, ω_2 has compact support in $\mathbb{R}^N$. On the other hand, since (ω_1, ω_2) achieves the infimum in (2.3) and F is homogeneous, we can use the Lagrange Multiplier Theorem to conclude that (ω_1, ω_2) satisfies

$$-\Delta_p u = \theta \frac{\partial F(x, u, v)}{\partial u}, \quad -\Delta_p v = \theta \frac{\partial F(x, u, v)}{\partial u}, x \in \mathbb{R}^N,$$

for $\theta = \frac{S_F}{p^*} > 0$. It follows from (F_2) and the maximum principle that at least one of the functions ω_1, ω_2 is positive in $\mathbb{R}^N$. But this contradicts supp $(\omega_1, \omega_2) \subset \Omega_0$. Hence, we conclude that $r_0 = 0$. Finally, if $y \notin \bar{\Omega}$ we obtain $r_m x + y_m \notin \Omega$ for large values of m, and therefore we should have $(\omega_1, \omega_2) \equiv (0, 0)$ again. Thus, $y \in \bar{\Omega}$ and the proof is completed.

Up to translations, we may assume that $0 \in \Omega$, since Ω is a smooth bounded domain of $\mathbb{R}^N$, we can choose $r > 0$ small enough such that $B_r = B_r(0) = \{x \in \mathbb{R}^N : d(x, 0) < r\} \subset \Omega$ and the sets

$$\Omega_r^+ = \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) < r\}, \quad \Omega_r^- = \{x \in \mathbb{R}^N : \text{dist}(x, \partial\Omega) > r\},$$

are homotopically equivalent to Ω. Let

$$W_{0,\text{rad}}^{1,p}(B_r) = \{u \in W_0^{1,p}(B_r) : u \text{ is radial}\}$$

and

$$E_{\text{rad}}(B_r) = W_{0,\text{rad}}^{1,p}(B_r) \times W_{0,\text{rad}}^{1,p}(B_r).$$

We define the functional

$$I_{B_r}(u,v) = \frac{1}{p}\int_{B_r}(|\nabla u|^p + |\nabla v|^p)dx - \frac{1}{p^*}\int_{B_r}F(x,u,v)dx - \frac{1}{q}\int_{B_r}(\lambda|u|^q + \delta|v|^q)dx, \quad (u,v)\in E_{\mathrm{rad}}(B_r),$$

and set

$$m_{\lambda,\delta} = \inf_{(u,v)\in\mathcal{N}^{B_r}_{\lambda,\delta}} I_{B_r}(u,v),$$

where

$$\mathcal{N}^{B_r}_{\lambda,\delta} := \{(u,v)\in E_{\mathrm{rad}}(B_r)\backslash\{(0,0)\} : \langle I'_{B_r}(u,v),(u,v)\rangle = 0\}.$$

Clearly, $m_{\lambda,\delta}$ is nonincreasing in λ, δ. Note that $m_{\lambda,\delta} > 0$ for all λ, $\delta > 0$.

Arguing as in the proof of Lemma 2.3 and Theorem 1.1, we obtain the following result.

Lemma 3.3. *Suppose F satisfies* (F_0)-(F_2), *then the infimum* $m_{\lambda,\delta}$ *is attained by a non-neg-ative radial function* $(u_{\lambda,\delta}, v_{\lambda,\delta}) \in E_{\mathrm{rad}}$ *whenever* $2 \le p < q < p^*$ *and* $\lambda,\delta > 0$, *or* $q = p$ *and* $\lambda,\delta \in (0,\Lambda_{1,\mathrm{rad}})$, *where* $\Lambda_{1,\mathrm{rad}} > 0$ *is the first eigenvalue of the operator* $(-\Delta_p, W^{1,p}_{0,\mathrm{rad}}(B_r))$. *Moreover,*

$$m_{\lambda,\delta} < \frac{1}{N}S_F^{\frac{N}{p}}, \quad \lim_{\lambda,\delta\to 0^+} m_{\lambda,\delta} = \frac{1}{N}S_F^{\frac{N}{p}}.$$

We introduce the barycenter map $\beta : \mathcal{N}_{\lambda,\delta} \to \mathbb{R}^N$ as follows

$$\beta(u,v) = S_F^{-\frac{N}{p}}\int_{\Omega}F(x,u,v)x\,dx.$$

This map has the following property.

Lemma 3.4. *If* $N \ge p^2, 2 \le p \le q < p^*$ *and F satisfies* (F_0)-(F_2), *then there exists* $\lambda^* > 0$ *such that* $\beta(u,v) \in \Omega_r^+$ *whenever* $(u,v) \in \mathcal{N}_{\lambda,\delta}, \lambda, \delta \in (0,\lambda^*)$ *and* $I_{\lambda,\delta}(u,v) \le m_{\lambda,\delta}$.

Proof. By way of contradiction, we suppose that there exist $\{\lambda_m\}$, $\{\delta_m\} \subset \mathbb{R}^+$ and $\{(u_m,v_m)\} \subset \mathcal{N}_{\lambda_m,\delta_m}$ such that λ_m, $\delta_m \to 0^+$ as $m \to \infty, I_{\lambda_m,\delta_m}(u_m,v_m) \le m_{\lambda_m,\delta_m}$ but $\beta(u_m,v_m) \notin \Omega_r^+$.

From $\{(u_m,v_m)\} \subset \mathcal{N}_{\lambda_m,\delta_m}$ and $I_{\lambda_m,\delta_m}(u_m,v_m) \le m_{\lambda_m,\delta_m}$ we have that $\{(u_m, v_m)\}$ is bounded in E. Moreover,

$$0 = \langle I'_{\lambda_m,\delta_m}(u_m,v_m),(u_m,v_m)\rangle = \|(u_m,v_m)\|^p_E - \int_{\Omega}F(x,u_m,v_m)dx - \int_{\Omega}(\lambda_m|u_m|^q + \delta_m|v_m|^q)dx.$$

Since λ_m, $\delta_m \to 0^+$, we can use the boundedness of $\{(u_m,v_m)\}$ to get

$$0 \le \int_{\Omega}(\lambda_m|u_m|^q + \delta_m|v_m|^q)dx \to 0,$$

from which it follows that

$$\lim_{m\to\infty}\|(u_m,v_m)\|^p_E = \lim_{m\to\infty}\int_{\Omega}F(x,u_m,v_m)dx = k \ge 0.$$

Notice that

$$
\begin{aligned}
c_{\lambda_m,\delta_m} &\le I_{\lambda_m,\delta_m}(u_m, v_m) \\
&= \frac{1}{p}\left\|(u_m, v_m)\right\|_E^p - \frac{1}{p^*}\int_\Omega F(x, u_m, v_m)dx - \frac{1}{q}\int_\Omega (\lambda_m|u_m|^q + \delta_m|v_m|^q)dx \\
&\le m_{\lambda_m,\delta_m}.
\end{aligned}
$$

Recalling that c_{λ_m,δ_m} and m_{λ_m,δ_m} both converge to $\frac{1}{N}S_F^{\frac{N}{p}}$, we can use the above expression and $\int_\Omega(\lambda_m|u_m|^q + \delta_m|v_m|^q)dx \to 0$ again to conclude that $k = S_F^{\frac{N}{p}}$, that is,

$$
\lim_{m\to\infty}\left\|(u_m, v_m)\right\|_E^p = S_F^{\frac{N}{p}} = \lim_{m\to\infty}\int_\Omega F(x, u_m, v_m)dx. \tag{3.10}
$$

Let $t_m = \left(\int_\Omega F(x, u_m, v_m)dx\right)^{-\frac{1}{p^*}} > 0$ and notice that $t_m(u_m, v_m)$ satisfies the hypotheses of Lemma 3.2. Using Lemma 3.2, there exist sequences $\{r_m\} \subset (0,+\infty)$ and $\{y_m\} \subset \mathbb{R}^N$ satisfying $r_m \to 0, y_m \to y \in \bar\Omega$ we have that $(\omega_m^1, \omega_m^2) \to (\omega_1, \omega_2)$ in $D^{1,p}(\mathbb{R}^N) \times D^{1,p}(\mathbb{R}^N)$.

The definition of $\beta(u, v)$, (3.10), the strong convergence of $\{(\omega_m^1, \omega_m^2)\}$ and Lebesgue's theorem provide

$$
\begin{aligned}
\beta(u_m, v_m) &= t_m^{-p^*}S_F^{-\frac{N}{p}}\int_\Omega F(x, t_m(u_m, v_m))x\,dx \\
&= (1 + o_m(1))\int_\Omega F(x, t_m(u_m, v_m))x\,dx \\
&= (1 + o_m(1))\int_\Omega F(x, \omega_m^1, \omega_m^2)(r_m x + y_m)dx \\
&= (1 + o_m(1))\left(\int_\Omega F(x, \omega_1, \omega_2)\bar y\,dx + o_m(1)\right).
\end{aligned}
$$

Since $\bar y \in \bar\Omega$ and $\int_\Omega F(x,\omega_1,\omega_2)dx = 1$, the above expression implies that

$$
\lim_{m\to\infty}\operatorname{dist}(\beta(u_m, v_m), \bar\Omega) = 0,
$$

which contradicts $\beta(u_m, v_m) \notin \Omega_r^+$.

According to Lemma 3.3, for each $\lambda, \delta > 0$ small the infimum $m_{\lambda,\delta}$ is attained by a nonnegative radial function $\sigma_{\lambda,\delta} = (u_{\lambda,\delta}, v_{\lambda,\delta}) \in \mathcal{N}_{\lambda,\delta}^{B_r}$. We consider

$$
I_{\lambda,\delta}^{m_{\lambda,\delta}} = \{(u, v) \in E : I_{\lambda,\delta}(u, v) \le m_{\lambda,\delta}\}
$$

and define the function $\gamma : \Omega_r^- \to I_{\lambda,\delta}^{m_{\lambda,\delta}}$ by setting, for each $y \in \Omega_r^-$,

$$
\gamma(y) = \begin{cases} \sigma_{\lambda,\delta}(x - y), & \text{if } x \in B_r(y), \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}
$$

A change of variables and straightforward calculations show that the map γ is well defined. Since $\sigma_{\lambda,\delta}$ is radial, we have that $\int_{B_r} F(x, u_{\lambda,\delta}, v_{\lambda,\delta})x dx = 0$. Hence, for each $\gamma \in \Omega_r^-$, we obtain

$$
\begin{aligned}
(\beta \circ \gamma)(\gamma) &= S_F^{-\frac{N}{p}} \int_\Omega F(x - \gamma, u_{\lambda,\delta}(x - \gamma), v_{\lambda,\delta}(x - \gamma))x dx \\
&= S_F^{-\frac{N}{p}} \int_\Omega F(t, u_{\lambda,\delta}(t), v_{\lambda,\delta}(t))(t + \gamma) dt \\
&= S_F^{-\frac{N}{p}} \int_\Omega F(t, u_{\lambda,\delta}(t), v_{\lambda,\delta}(t))\gamma dt \\
&= \gamma \alpha_{\lambda,\delta},
\end{aligned}
$$

where $\alpha_{\lambda,\delta} = S_F^{-\frac{N}{p}} \int_\Omega F(t, u_{\lambda,\delta}(t), v_{\lambda,\delta}(t)) dt.$

Along the way of proving Lemma 3.4 we can check easily the following

Lemma 3.5. *If $\lambda, \delta \to 0^+$, $\alpha_{\lambda,\delta} \to 1$.*

Proof. By Lemma 3.3, we have

$$
m_{\lambda,\delta} = \frac{1}{p} \int_{B_r} (|\nabla u_{\lambda,\delta}|^p + |\nabla v_{\lambda,\delta}|^p) dx - \frac{1}{p^*} \int_{B_r} F(x, u_{\lambda,\delta}, v_{\lambda,\delta}) dx - \frac{1}{q} \int_{B_r} (\lambda |u_{\lambda,\delta}|^q + \delta |v_{\lambda,\delta}|^q) dx < \frac{1}{N} S_F^{\frac{N}{p}}.
$$

As before $\int_{B_r} (\lambda |u_{\lambda,\delta}|^q + \delta |v_{\lambda,\delta}|^q) dx \to 0$. Thus, $I'_{B_r}(u_{\lambda,\delta}, v_{\lambda,\delta}) = 0$, the above expression and the same arguments used in the proof of Lemma 3.3 imply that

$$
\int_\Omega F(x, u_{\lambda,\delta}, v_{\lambda,\delta}) dx \to S_F^{\frac{N}{p}}.
$$

The above equality and the definition of $\alpha_{\lambda,\delta}$ imply that $\alpha_{\lambda,\delta} \to 1$. The lemma is proved.

Next we define $H_{\lambda,\delta} : [0, 1] \times (\mathcal{N}_{\lambda,\delta} \cap I_{\lambda,\delta}^{m_{\lambda,\delta}}) \to \mathbb{R}^N$ by

$$
H_{\lambda,\delta}(t, (u, v)) = \left(t + \frac{1 - t}{\alpha_{\lambda,\delta}} \right) \beta(u, v).
$$

We have the following

Lemma 3.6. *if F satisfies (F_0)-(F_2), then there exists $\lambda^{**} > 0$ such that*

$$
H_{\lambda,\delta}([0, 1] \times (\mathcal{N}_{\lambda,\delta} \cap I_{\lambda,\delta}^{m_{\lambda,\delta}})) \subset \Omega_r^+, \tag{3.12}
$$

for all $\lambda, \delta \in (0, \lambda^{**})$.

Proof. Arguing by contradiction, we suppose that there exist sequences $\{\lambda_m\}, \{\delta_m\} \subset \mathbb{R}^+$ and $t_m \in [0, 1], (u_m, v_m) \in (\mathcal{N}_{\lambda,\delta} \cap I_{\lambda,\delta}^{m_{\lambda,\delta}})$ such that $\lambda_m, \delta_m \to 0^+$, as $m \to \infty$, and $H_{\lambda_m,\delta_m}(t_m, (u_m, v_m)) \notin \Omega_r^+$ for all m. Up to a subsequence $t_m \to t_0 \in [0, 1]$. Moreover, the compactness of $\bar{\Omega}$ and Lemma 3.4 imply that, up to a subsequence, $\beta(u_m, v_m) \to \gamma \in \bar{\Omega}$. From Lemma 3.5 $\alpha_{\lambda_m,\delta_m} \to 1$. So, we can use the definition of $H_{\lambda,\delta}$ to conclude that $H_{\lambda_m,\delta_m}(t_m, (u_m, v_m)) \to \gamma \in \bar{\Omega}$, which is a contradiction. The lemma is proved.

4 Proof of Theorem 1.2

We begin with the following lemma.

Lemma 4.1. *If (u, v) is a critical point of $I_{\lambda,\delta}$ on $\mathcal{N}_{\lambda,\delta}$, then it is a critical point of $I_{\lambda,\delta}$ in E.*

Proof. The proof is almost the same as that [4, Lemma 4.1] and is omitted here.

Lemma 4.2. *Suppose F satisfies (F_0)-(F_2), then any sequence $\{(u_m, v_m)\} \subset \mathcal{N}_{\lambda,\delta}$ such that $I_{\lambda,\delta}(u_m, v_m) \to c < \frac{1}{N} S_F^{\frac{N}{p}}$ and $I'_{\lambda,\delta}(u_m, v_m) \to 0$ contains a convergent subsequence for $\lambda, \delta > 0$ if $q > p$ and $\lambda, \delta \in (0, \lambda^*)$ if $q = p$ for some small $\lambda^* > 0$.*

Proof. By hypothesis there exists a sequence $\theta_m \in \mathbb{R}$ such that $\left\| I'_{\lambda,\delta}(u_m, v_m) - \theta_m J'_{\lambda,\delta}(u_m, v_m) \right\|_E \to 0$ as $m \to \infty$, where $J_{\lambda,\delta}(u, v) = \langle I'_{\lambda,\delta}(u, v), (u, v) \rangle$. Thus

$$I'_{\lambda,\delta}(u_m, v_m) = \theta_m J'_{\lambda,\delta}(u_m, v_m) + o_m(1).$$

Recall that

$$\langle J'_{\lambda,\delta}(u_m, v_m), (u_m, v_m) \rangle \le 0 \ \text{ for all } (u_m, v_m) \in \mathcal{N}_{\lambda,\delta.}$$

If $\langle J'_{\lambda,\delta}(u_m, v_m), (u_m, v_m) \rangle \to 0$, we have

$$\int_\Omega \left(\lambda |u_m|^q + \delta |v_m|^q \right) dx \to 0, \quad \int_\Omega F(x, u_m, v_m) dx \to 0.$$

Consequently $\|(u_m, v_m)\|_E \to 0$.

On the other hand, if $(u_m, v_m) \subset \mathcal{N}_{\lambda,\delta}$ it follows that

$$1 \le C \left(\lambda \left\| (u_m, v_m) \right\|_E^{q-p} + \delta \left\| (u_m, v_m) \right\|_E^{q-p} + \left\| (u_m, v_m) \right\|_E^{p^*-p} \right)$$

for some $C > 0$. Hence we arrive at a contradiction if $\lambda, \delta > 0$ and $q > p$ or $\lambda, \delta \in (0, \lambda^*)$ for small $\lambda^* > 0$ when $q = p$. Thus we may assume that $\langle J'_{\lambda,\delta}(u_m, v_m), (u_m, v_m) \rangle \to \ell < 0$. Since $\langle I'_{\lambda,\delta}(u_m, v_m), (u_m, v_m) \rangle = 0$, we conclude that $\theta_m = 0$, consequently, $I'_{\lambda,\delta}(u_m, v_m) \to 0$. Using this information we have

$$I'_{\lambda,\delta}(u_m, v_m) \to c < \frac{1}{N} S_F^{\frac{N}{p}} \text{ and } I'_{\lambda,\delta}(u_m, v_m) \to 0,$$

so by Lemma 2.1 the proof is completed.

Below we denote by $I_{\mathcal{N},\delta}$ the restriction of $I_{\lambda,\delta}$ on $\mathcal{N}_{\lambda,\delta}$.

Lemma 4.3. *Suppose $N \ge p^2, 2 \le p \le q < p^*$ and F satisfies (F_0)-(F_2), let $\Lambda = \min\{\lambda^*, \lambda^{**}\} > 0$, $\lambda, \delta \in (0, \Lambda)$, then $\mathrm{cat}_{I_{\mathcal{N}_{\lambda,\delta}}^{m_{\lambda,\delta}}} \left(I_{\mathcal{N}_{\lambda,\delta}}^{m_{\lambda,\delta}} \right) \ge \mathrm{cat}_\Omega(\Omega)$, where λ^*, λ^{**} given by Lemmas 34 and 3.6, respectively.*

Proof. Assume that $I_{\mathcal{N}_{\lambda,\delta}}^{m_{\lambda,\delta}} = A_1 \cup A_2 \cup \cdots \cup A_m$, where $A_j, j = 1, 2, \ldots, m$, are closed and contractible sets in $I_{\mathcal{N}_{\lambda,\delta}}^{m_{\lambda,\delta}}$, i.e., there exists $h_j \in C\left([0,1] \times A_j, I_{\mathcal{N}_{\lambda,\delta}}^{m_{\lambda,\delta}}\right)$ such that

$$h_j(0, z) = z, \quad h_j(1, z) = \vartheta, \ \text{ for all } z \in A_j,$$

where $\vartheta \in A_j$ is fixed. Consider $B_j = \gamma\text{-}1(A_j)$, $1 \leq j \leq m$. The sets B_j are closed and

$$\Omega_r^- = B_1 \cup B_2 \cup \cdots \cup B_m.$$

We define the deformation $g_j : [0, 1] \times B_j$ by setting

$$g_j(t, \gamma) = H_{\lambda,\delta}\left(t, h_j(t, \gamma(\gamma))\right),$$

for $\lambda, \delta \in (0, \Lambda)$. Note that

$$g_j(0, \gamma) = H_{\lambda,\delta}\left(0, h_j(0, \gamma(\gamma))\right) = \frac{(\beta o \gamma)(\gamma)}{\alpha_{\lambda,\delta}}$$

implies

$$g_j(0, \gamma) = \frac{\alpha_{\lambda,\delta}\gamma}{\alpha_{\lambda,\delta}} = \gamma, \quad \text{for all } \gamma \in B_j,$$

and $g_j(1, y) = H_{\lambda,\delta}(1, h_j(1, \gamma(y))) = \beta(h_j(1, \gamma(y)))$ implies

$$g_j(1, \gamma) = \beta(\vartheta) \in \Omega_r^+.$$

Thus the sets B_j are contractible in Ω_r^+. It follows that $\mathrm{cat}_\Omega(\Omega) = \mathrm{cat}_{\Omega_r^+}\left(\Omega_r^+\right) \leq m$.

Proof of Theorem 1.2.

Using Lemmas 2.1, 2.2, and 3.3 we know that $c_{\lambda,\delta}, m_{\lambda,\delta} < \frac{1}{N}S_F^{\frac{N}{p}}$ for $\lambda, \delta \in (0, \Lambda)$. Moreover, by Lemma 4.2, $I_{\mathcal{N}_{\lambda,\delta}}$ satisfies the $(PS)_c$ condition for all $c < \frac{1}{N}S_F^{\frac{N}{p}}$. Therefore, by Lemma 4.3, a standard deformation argument implies that, for $\lambda, \delta \in (0, \Lambda)$, $I_{\mathcal{N}_{\lambda,\delta}}$ contains at least $\mathrm{cat}_\Omega(\Omega)$ critical points of the restriction of $I_{\lambda,\delta}$ on $\mathcal{N}_{\lambda,\delta}$. Now Lemma 4.1 implies that $I_{\lambda,\delta}$ has at least $\mathrm{cat}_\Omega(\Omega)$ critical points, and therefore at least $\mathrm{cat}_\Omega(\Omega)$ nontrivial solutions of (1.1). As Theorem 1.1, the obtained solutions are nonnegative in Ω. The proof is completed.

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