Ala et al. Boundary Value Problems 2012, 2012:37
http://www.boundaryvalueproblems.com/content/2012/1/37

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence of positive solutions for variable exponent elliptic systems

Samira Ala[1*], Ghasem Alizadeh Afrouzi[2], Qihu Zhang[3] and Asadollah Niknam[4]

* Correspondence: golestan62@yahoo.com
[1]Department of Mathematics, Sciences and Research, Islamic Azad University (IAU) Tehran, Iran
Full list of author information is available at the end of the article

Abstract

We consider the system of differential equations

$$\begin{cases} -\Delta_{p(x)}u = \lambda^{p(x)}[g(x)a(u) + f(v)] & \text{in } \Omega, \\ -\Delta_{p(x)}v = \lambda^{p(x)}[g(x)b(v) + h(u)] & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega$, $1 < p(x) \in C^1(\bar{\Omega})$ is a function. $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. We discuss the existence of positive solution via sub-super solutions without assuming sign conditions on $f(0)$, $h(0)$.

MSC: 35J60; 35B30; 35B40.

Keywords: positive solutions, $p(x)$-Laplacian problems, sub-supersolution

1. Introduction

The study of diferential equations and variational problems with variable exponent has been a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc., (see[1-3]). Many results have been obtained on this kind of problems, for example [1,3-8]. In [7], Fan gives the regularity of weak solutions for differential equations with variable exponent. On the existence of solutions for elliptic systems with variable exponent, we refer to [8,9]. In this article, we mainly consider the existence of positive weak solutions for the system

$$(P)\begin{cases} -\Delta_{p(x)}u = \lambda^{p(x)}[g(x)a(u) + f(v)] & \text{in } \Omega, \\ -\Delta_{p(x)}v = \lambda^{p(x)}[g(x)b(v) + h(u)] & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega$, $1 < p(x) \in C^1(\bar{\Omega})$ is a function. The operator $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. Especially, if $p(x) \equiv p$ (a constant), (P) is the well-known p-Laplacian system. There are many articles on the existence of solutions for p-Laplacian elliptic systems, for example [5,10]. Owing to the nonhomogeneity of $p(x)$-Laplacian problems are more complicated than those of p-Laplacian, many results and methods for p-Laplacian are invalid for $p(x)$-Laplacian; for example, if Ω is bounded, then the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx}$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ (see [11]), and maybe the first eigenvalue and the first eigenfunction of $p(x)$-Laplacian do not exist, but the fact that the first eigenvalue $\lambda_p > 0$ and the existence of the first eigenfunction are very important in the study of p-Laplacian problems. There are more difficulties in discussing the existence of solutions of variable exponent problems.

Hai and Shivaji [10], consider the existence of positive weak solutions for the following p-Laplacian problems

$$\text{(I)} \begin{cases} -\Delta_p u = \lambda f(v) \text{ in } \Omega, \\ -\Delta_p v = \lambda g(u) \text{ in } \Omega, \\ u = v = 0 \qquad \text{on } \partial\Omega \end{cases}$$

the first eigenfunction is used to construct the subsolution of p-Laplacian problems success-fully. On the condition that λ is large enough and

$$\lim_{u \to +\infty} \frac{f\left[M(g(u))^{\frac{1}{(p-1)}}\right]}{u^{p-1}} = 0, \quad \text{for every} \quad M > 0,$$

the authors give the existence of positive solutions for problem (I).

Chen [5], considers the existence and nonexistence of positive weak solution to the following quasilinear elliptic system:

$$\text{(II)} \begin{cases} -\Delta_p u = \lambda f(u,v) = \lambda u^\alpha v^\gamma \text{ in } \Omega, \\ -\Delta_q v = \lambda g(u,v) = \lambda u^\delta v^\beta \text{ in } \Omega, \\ u = v = 0 \qquad \text{on } \partial\Omega \end{cases}$$

the first eigenfunction is used to construct the subsolution of problem(II), the main results are as following

(*i*) *If* $\alpha, \beta \geq 0$, $\gamma, \delta > 0$, $\theta = (p - 1 - \alpha)(q - 1 - \beta) - \gamma\delta > 0$, then problem (II) has a positive weak solution for each $\lambda > 0$;

(*ii*) *If* $\theta = 0$ *and* $p\gamma = q(p - 1 - \alpha)$, then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, then problem (II) has no nontrivial nonnegative weak solution.

On the $p(x)$-Laplacian problems, maybe the first eigenvalue and the first eigenfunction of $p(x)$-Laplacian do not exist. Even if the first eigenfunction of $p(x)$-Laplacian exist, because of the nonhomogeneity of $p(x)$-Laplacian, the first eigenfunction cannot be used to construct the subsolution of $p(x)$-Laplacian problems. Zhang [12] investigated the existence of positive solutions of the system

$$\begin{cases} -\Delta_{p(x)} u = \lambda^{p(x)} f(v) \text{ in } \Omega, \\ -\Delta_{p(x)} v = \lambda^{p(x)} g(u) \text{ in } \Omega, \\ u = v = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

In this article, we consider the existence of positive solutions of the system

$$\begin{cases} -\Delta_{p(x)}u = \lambda^{p(x)}F(x,u,v) & \text{in } \Omega, \\ -\Delta_{p(x)}v = \lambda^{p(x)}G(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p(x) \in C^1(\bar{\Omega})$ is a function, $F(x, u, v) = [g(x)a(u) + f(v)]$, $G(x, u, v) = [g(x)b(v) + h(u)]$, λ is a positive parameter and $\Omega \subset \mathbb{R}^N$ is a bounded domain.

To study $p(x)$-Laplacian problems, we need some theory on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [6,13]). If $\Omega \subset \mathbb{R}^N$ is an open domain, write

$$C_+(\Omega) = \{h : h \in C(\Omega), h(x) > 1 \text{ for } x \in \Omega\},$$
$$h^+ = \sup_{x\in\Omega} h(x), h^- = \inf_{x\in\Omega} h(x), \text{ for any } h \in C(\Omega).$$

Throughout the article, we will assume that:

(H_1) $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^2 boundary $\partial\Omega$.

(H_2) $p(x) \in C^1(\bar{\Omega})$ and $1 < p^- \le p^+$.

(H_3) $a, b \in C^1([0, \infty))$ are nonnegative, nondecreasing functions such that

$$\lim_{u\to+\infty} \frac{a(u)}{u^{p^- -1}} = 0, \quad \lim_{u\to+\infty} \frac{b(u)}{u^{p^- -1}} = 0.$$

(H_4) $f, h : [0, +\infty) \to R$ are C^1, monotone functions, $\lim_{u\to+\infty} f(u) = +\infty$, $\lim_{u\to+\infty} h(u) = +\infty$, and

$$\lim_{u\to+\infty} \frac{f\left[M(h(u))^{\frac{1}{(p^- -1)}}\right]}{u^{p^- -1}} = 0, \quad \forall M > 0.$$

(H_5) $g : [0, +\infty) \to (0, +\infty)$ is a continuous function such that $L_1 = \min_{x\in\bar{\Omega}} g(x)$, and $L_2 = \max_{x\in\bar{\Omega}} g(x)$.

Denote

$$L^{p(x)}(\Omega) = \left\{ u | u \text{ is a measurable real - valued function}, \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

We introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

and $(L^{p(x)}(\Omega), |.|_{p(x)})$ becomes a Banach space, we call it generalized Lebesgue space. The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is a separable, reflexive, and uniform convex Banach space (see [[6], Theorems 1.10 and 1.14]).

The space $W^{1,p(x)}(\Omega)$ is defined by $W^{1,p(x)}(\Omega) = \{u \in L^{p(x)} : |\nabla u| \in L^{p(x)}\}$, and it is equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by ${W_0}^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive, and uniform convex Banach space (see [[6], Theorem 2.1] We define

$$(L(u), v) = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \quad \forall v, u \in {W_0}^{1,p(x)}(\Omega),$$

then $L : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ is a continuous, bounded, and strictly monotone operator, and it is a homeomorphism (see [[14], Theorem 3.1]).

If $u, v \in W_0^{1,p(x)}(\Omega), (u, v)$ is called a weak solution of (P) if it satisfies

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla q dx = \int_\Omega \lambda^{p(x)} F(x, u, v) q dx, \forall q \in {W_0}^{1,p(x)}(\Omega), \\ \int_\Omega |\nabla v|^{p(x)-2} \nabla v \cdot \nabla q dx = \int_\Omega \lambda^{p(x)} G(x, u, v) q dx, \forall q \in {W_0}^{1,p(x)}(\Omega). \end{cases}$$

Define $A : W^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ as

$$< Au, \ \varphi >= \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla \varphi + l(x, \ u)\varphi) dx,$$
$$\forall u \in W^{1,p(x)}(\Omega), \quad \forall \varphi \in W_0^{1,p(x)}(\Omega),$$

where $l(x, u)$ is continuous on $\bar{\Omega} \times \mathbb{R}$, and $l(x, .)$ is increasing. It is easy to check that A is a continuous bounded mapping. Copying the proof of [15], we have the following lemma.

Lemma 1.1. *(Comparison Principle). Let* $u, v \in W^{1,p(x)}(\Omega)$ *satisfying* $Au - Av \geq 0$ *in* $(W_0^{1,p(x)}(\Omega))^*, \varphi(x) = \min\{u(x) - v(x), 0\}$. *If* $\varphi(x) \in W_0^{1,p(x)}(\Omega)$ *(i.e.,* $u \geq v$ *on* $\partial\Omega$ *), then* $u \geq v$ *a.e. in* Ω.

Here and hereafter, we will use the notation $d(x, \partial\Omega)$ to denote the distance of $x \in \Omega$ to the boundary of Ω.

Denote $d(x) = d(x, \partial\Omega)$ and $\partial\Omega_\epsilon = \{x \in \Omega | d(x, \partial\Omega) < \epsilon\}$. Since $\partial\Omega$ is C^2 regularly, then there exists a constant $\delta \in (0, 1)$ such that $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, and $|\nabla d(x)| \equiv 1$.

Denote

$$v_1(x) = \begin{cases} \gamma d(x), \quad d(x) < \delta, \\ \gamma\delta + \int_\delta^{d(x)} \gamma \left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- - 1}} (L_1 + 1)^{\frac{2}{p^- - 1}} dt, \quad \delta \leq d(x) < 2\delta, \\ \gamma\delta + \int_\delta^{2\delta} \gamma \left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- - 1}} (L_1 + 1)^{\frac{2}{p^- - 1}} dt, \quad 2\delta \leq d(x). \end{cases}$$

Obviously, $0 \leq v_1(x) \in C^1(\bar{\Omega})$. Considering

$$-\Delta_{p(x)} w(x) = \eta \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega, \tag{1}$$

we have the following result

Lemma 1.2. *(see [16]). If positive parameter η is large enough and w is the unique solution of (1), then we have*

(i) For any $\theta \in (0, 1)$ there exists a positive constant C_1 such that

$$C_1 \eta^{\frac{1}{p^+ - 1 + \theta}} \leq \max_{x \in \bar{\Omega}} w(x);$$

(ii) There exists a positive constant C_2 such that

$$\max_{x \in \bar{\Omega}} w(x) \leq C_2 \eta^{\frac{1}{p^- - 1}}.$$

2. Existence results

In the following, when there be no misunderstanding, we always use C_i to denote positive constants.

Theorem 2.1. *On the conditions of (H_1) - (H_5), then (P) has a positive solution when λ is large enough.*

Proof. We shall establish Theorem 2.1 by constructing a positive subsolution (Φ_1, Φ_2) and supersolution (z_1, z_2) of (P), such that $\Phi_1 \leq z_1$ and $\Phi_2 \leq z_2$. That is (Φ_1, Φ_2) and (z_1, z_2) satisfies

$$\begin{cases} \int_\Omega |\nabla \Phi_1|^{p(x)-2} \nabla \Phi_1 \cdot \nabla q dx \leq \int_\Omega \lambda^{p(x)} g(x) a(\Phi_1) q dx + \int_\Omega \lambda^{p(x)} f(\Phi_2) q dx, \\ \int_\Omega |\nabla \Phi_2|^{p(x)-2} \nabla \Phi_2 \cdot \nabla q dx \leq \int_\Omega \lambda^{p(x)} g(x) b(\Phi_2) q dx + \int_\Omega \lambda^{p(x)} h(\Phi_1) q dx, \end{cases}$$

and

$$\begin{cases} \int_\Omega |\nabla z_1|^{p(x)-2} \nabla z_1 \cdot \nabla q dx \geq \int_\Omega \lambda^{p(x)} g(x) a(z_1) q dx + \int_\Omega \lambda^{p(x)} f(z_2) q dx, \\ \int_\Omega |\nabla z_2|^{p(x)-2} \nabla z_2 \cdot \nabla q dx \geq \int_\Omega \lambda^{p(x)} g(x) b(z_2) q dx + \int_\Omega \lambda^{p(x)} h(z_1) q dx, \end{cases}$$

for all $q \in W_0^{1,p(x)}(\Omega)$ with $q \geq 0$. According to the sub-supersolution method for $p(x)$-Laplacian equations (see [16]), then (P) has a positive solution.

Step 1. We construct a subsolution of (P).

Let $\sigma \in (0, \delta)$ is small enough. Denote

$$\phi(x) = \begin{cases} e^{kd(x)} - 1, & d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma} \right)^{\frac{2}{p^- - 1}} dt, & \sigma \leq d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma} \right)^{\frac{2}{p^- - 1}} dt, & 2\delta \leq d(x). \end{cases}$$

It is easy to see that $\phi \in C^1(\bar{\Omega})$. Denote

$$\alpha = \min \left\{ \frac{\inf p(x) - 1}{4(\sup |\nabla p(x)| + 1)}, 1 \right\}, \qquad \zeta = \min\{a(0)L_1 + f(0), b(0)L_1 + h(0), -1\}.$$

By computation

$$-\Delta_{p(x)}\phi = \begin{cases} -k(ke^{kd(x)})^{p(x)-1}\left[(p(x)-1)+(d(x)+\frac{\ln k}{k})\nabla p\nabla d+\frac{\triangle d}{k}\right], \quad d(x)<\sigma, \\ \left\{\frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^{-}-1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^{-}-1}}\right)\nabla p\nabla d+\Delta d\right]\right\} \\ \times(ke^{k\sigma})^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^{-}-1}-1}(L_1+1), \quad \sigma<d(x)<2\delta, \\ 0, \qquad 2\delta<d(x). \end{cases}$$

From (H_3) and (H_4), there exists a positive constant $M>1$ such that

$$f(M-1)\geq 1, \qquad h(M-1)\geq 1.$$

Let $\sigma=\frac{1}{k}\ln M$, then

$$\sigma k=\ln M. \tag{2}$$

If k is sufficiently large, from (2), we have

$$-\Delta_{p(x)}\phi\leq -k^{p(x)}\alpha, \qquad d(x)<\sigma. \tag{3}$$

Let $-\lambda\zeta = k\alpha$, then

$$k^{p(x)}\alpha\geq -\lambda^{p(x)}\zeta,$$

from (3), then we have

$$-\Delta_{p(x)}\phi\leq\lambda^{p(x)}(a(0)L_1+f(0))\leq\lambda^{p(x)}(g(x)a(\phi)+f(\phi)), \qquad d(x)<\sigma. \tag{4}$$

Since $d(x)\in C^2(\overline{\partial\Omega_{3\delta}})$, then there exists a positive constant C_3 such that

$$\begin{aligned}-\Delta_{p(x)}\phi &\leq (ke^{k\sigma})^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^{-}-1}-1} \\ &\cdot\left|\left\{\frac{2(p(x)-1)}{(2\delta-\sigma)(p^{-}-1)}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^{-}-1}}\right)\nabla p\nabla d+\Delta d\right]\right\}\right| \\ &\leq C_3(ke^{k\sigma})^{p(x)-1}\ln k, \qquad \sigma<d(x)<2\delta.\end{aligned}$$

If k is sufficiently large, let $-\lambda\zeta = k\alpha$, we have

$$C_3(ke^{k\sigma})^{p(x)-1}\ln k = C_3(kM)^{p(x)-1}\ln k\leq\lambda^{p(x)},$$

then

$$-\Delta_{p(x)}\phi\leq\lambda^{p(x)}(L_1+1), \qquad \sigma<d(x)<2\delta.$$

Since $\varphi(x)\geq 0$ and a, f are monotone, when λ is large enough, then we have

$$-\Delta_{p(x)}\phi\leq\lambda^{p(x)}(g(x)a(\phi)+f(\phi)), \qquad \sigma<d(x)<2\delta. \tag{5}$$

Obviously

$$-\Delta_{p(x)}\phi = 0 \le \lambda^{p(x)}(L_1 + 1) \le \lambda^{p(x)}(g(x)a(\phi) + f(\phi)), \qquad 2\delta < d(x). \tag{6}$$

Combining (4), (5), and (6), we can conclude that

$$-\Delta_{p(x)}\phi \le \lambda^{p(x)}(g(x)a(\phi) + f(\phi)), \qquad \text{a.e. on } \Omega. \tag{7}$$

Similarly

$$-\Delta_{p(x)}\phi \le \lambda^{p(x)}(g(x)b(\phi) + h(\phi)), \qquad \text{a.e. on } \Omega. \tag{8}$$

From (7) and (8), we can see that $(\varphi_1, \varphi_2) = (\varphi, \varphi)$ is a subsolution of (P).

Step 2. We construct a supersolution of (P).

We consider

$$\begin{cases} -\Delta_{p(x)}z_1 = \lambda^{p^+}\mu(L_2 + 1) & \text{in } \Omega, \\ -\Delta_{p(x)}z_2 = \lambda^{p^+}(L_2 + 1)h(\beta(\lambda^{p^+}(L_2 + 1)\mu)) & \text{in } \Omega, \\ z_1 = z_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\beta = \beta(\lambda^{p^+}(L_2 + 1)\mu) = \max_{x\in\bar{\Omega}} z_1(x)$. We shall prove that (z_1, z_2) is a supersolution for (p).

For $q \in W_0^{1,p(x)}(\Omega)$ with $q \ge 0$, it is easy to see that

$$\begin{aligned} \int_\Omega |\nabla z_2|^{p(x)-2}\nabla z_2 \cdot \nabla q dx &= \int_\Omega \lambda^{p^+}(L_2 + 1)h(\beta(\lambda^{p^+}(L_2 + 1)\mu))q dx \\ &\ge \int_\Omega \lambda^{p^+}L_2 h(\beta(\lambda^{p^+}(L_2 + 1)\mu))q dx + \int_\Omega \lambda^{p^+}h(z_1)q dx. \end{aligned} \tag{9}$$

Since $\lim_{u\to+\infty}\frac{f\left[M(h(u))\frac{1}{(p^- -1)}\right]}{u^{p^- -1}} = 0$, when μ is sufficiently large, combining Lemma 1.2 and (H_3), then we have

$$h(\beta(\lambda^{p^+}(L_2 + 1)\mu)) \ge b\left(C_2\left[\lambda^{p^+}(L_2 + 1)h(\beta(\lambda^{p^+}(L_2 + 1)\mu))\right]^{\frac{1}{p^- -1}}\right) \ge b(z_2) \tag{10}$$

Hence

$$\int_\Omega |\nabla z_2|^{p(x)-2}\nabla z_2 \cdot \nabla q dx \ge \int_\Omega \lambda^{p^+}g(x)b(z_2)q dx + \int_\Omega \lambda^{p^+}h(z_1)q dx. \tag{11}$$

Also

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla q dx = \int_\Omega \lambda^{p^+}(L_2 + 1)\mu q dx$$

By (H_3), (H_4), when μ is sufficiently large, combining Lemma 1.2 and (H_3), we have

$$\begin{aligned}(L_2+1)\mu &\geq \frac{1}{\lambda^{p^+}}\left[\frac{1}{C_2}\beta(\lambda^{p^+}(L_2+1)\mu)\right]^{p^--1}\\ &\geq L_2 a(\beta(\lambda^{p^+}(L_2+1)\mu)) + f\left(C_2\left[\lambda^{p^+}(L_2+1)h(\beta(\lambda^{p^+}(L_2+1)\mu))\right]^{\frac{1}{p^--1}}\right).\end{aligned}$$

Then

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1\cdot\nabla q dx \geq \int_\Omega \lambda^{p^+}g(x)a(z_1)qdx + \int_\Omega \lambda^{p^+}f(z_2)qdx. \tag{12}$$

According to (11) and (12), we can conclude that (z_1, z_2) is a supersolution for (P).
It only remains to prove that $\varphi_1 \leq z_1$ and $\varphi_2 \leq z_2$.
In the definition of $v_1(x)$, let $\gamma = \frac{2}{\delta}\left(\max_{x\in\bar\Omega}\phi(x) + \max_{x\in\bar\Omega}|\nabla\phi(x)|\right)$. We claim that

$$\phi(x) \leq v_1(x), \qquad \forall x\in\Omega. \tag{13}$$

From the definition of v_1, it is easy to see that

$$\phi(x) \leq 2\max_{x\in\bar\Omega}\phi(x) \leq v_1(x), \qquad \text{when } d(x)=\delta,$$

and

$$\phi(x) \leq 2\max_{x\in\bar\Omega}\phi(x) \leq v_1(x), \qquad \text{when } d(x)\geq\delta.$$

It only remains to prove that

$$\phi(x) \leq v_1(x), \qquad \text{when } d(x)<\delta.$$

Since $v_1-\phi\in C^1(\overline{\partial\Omega_\delta})$, then there exists a point $x_0\in\overline{\partial\Omega_\delta}$ such that

$$v_1(x_0)-\phi(x_0) = \min_{x_0\in\overline{\partial\Omega_\delta}}[v_1(x)-\phi(x)].$$

If $v_1(x_0) - \varphi(x_0) <0$, it is easy to see that $0 < d(x_0) < \delta$, and then

$$\nabla v_1(x_0) - \nabla\phi(x_0) = 0.$$

From the definition of v_1, we have

$$|\nabla v_1(x_0)| = \gamma = \frac{2}{\delta}\left(\max_{x\in\bar\Omega}\phi(x) + \max_{x\in\bar\Omega}|\nabla\phi(x)|\right) > |\nabla\phi(x_0)|.$$

It is a contradiction to $\nabla v_1(x_0) - \nabla\varphi(x_0) = 0$. Thus (13) is valid.
Obviously, there exists a positive constant C_3 such that

$$\gamma \leq C_3\lambda.$$

Since $d(x)\in C^2(\overline{\partial\Omega_{3\delta}})$, according to the proof of Lemma 1.2, then there exists a positive constant C_4 such that

$$-\Delta_{p(x)}v_1(x) \leq C_*\gamma^{p(x)-1+\theta} \leq C_4\lambda^{p(x)-1+\theta}, \qquad \text{a.e. in } \Omega, \text{where } \theta\in(0,1).$$

When $\eta \geq \lambda^{p^+}$ is large enough, we have

$$-\Delta_{p(x)} v_1(x) \leq \eta.$$

According to the comparison principle, we have

$$v_1(x) \leq w(x), \qquad \forall x \in \Omega. \tag{14}$$

From (13) and (14), when $\eta \geq \lambda^{p^+}$ and $\lambda \geq 1$ is sufficiently large, we have

$$\phi(x) \leq v_1(x) \leq w(x), \qquad \forall x \in \Omega. \tag{15}$$

According to the comparison principle, when μ is large enough, we have

$$v_1(x) \leq w(x) \leq z_1(x), \qquad \forall x \in \Omega.$$

Combining the definition of $v_1(x)$ and (15), it is easy to see that

$$\phi_1(x) = \phi(x) \leq v_1(x) \leq w(x) \leq z_1(x), \qquad \forall x \in \Omega.$$

When $\mu \geq 1$ and λ is large enough, from Lemma 1.2, we can see that $\beta(\lambda^{p^+}(L_2+1)\mu)$ is large enough, then $\lambda^{p^+}(L_2+1)h(\beta(\lambda^{p^+}(L_2+1)\mu))$ is large enough. Similarly, we have $\varphi_2 \leq z_2$. This completes the proof. □

3. Asymptotic behavior of positive solutions

In this section, when parameter $\lambda \to +\infty$, we will discuss the asymptotic behavior of maximum of solutions about parameter λ, and the asymptotic behavior of solutions near boundary about parameter λ.

Theorem 3.1. *On the conditions of* (H_1)-(H_5), *if* (u, v) *is a solution of* (P) *which has been given in Theorem 2.1, then*

(i) *There exist positive constants* C_1 *and* C_2 *such that*

$$C_1\lambda \leq \max_{x\in\bar{\Omega}} u(x) \leq C_2\left(\lambda^{p^+}(L_2+1)\mu\right)^{\frac{1}{p^- -1}} \tag{16}$$

$$C_1\lambda \leq \max_{x\in\bar{\Omega}} v(x) \leq C_2\left\{\lambda^{p^+}(L_2+1)h\left[C_2(\lambda^{p^+}(L_2+1)\mu)^{\frac{1}{p^- - 1}}\right]\right\}^{\frac{1}{p^- -1}} \tag{17}$$

(ii) *for any* $\theta \in$ (0, 1), *there exist positive constants* C_3 *and* C_4 *such that*

$$C_3\lambda d(x) \leq u(x) \leq C_4(\lambda^{p^+}(L_2+1)\mu)^{1/(p^- -1)}(d(x))^{\theta},\ \textit{as}\ d(x) \to 0, \tag{18}$$

$$C_3\lambda d(x) \leq v(x) \leq C_4\left\{\lambda^{p^+}(L_2+1)h\left[C_2(\lambda^{p^+}(L_2+1)\mu)^{\frac{1}{p^- - 1}}\right]\right\}^{\frac{1}{p^- -1}}(d(x))^{\theta},\ \textit{as}\ d(x) \to 0 \tag{19}$$

where μ *satisfies* (10).

Proof. (*i*) Obviously, when $2\delta \leq d(x)$, we have

$$u(x), v(x) \geq \phi(x) = e^{k\sigma} - 1 + \int_{\sigma}^{2\delta} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} dt \geq -\lambda\frac{\zeta}{\alpha}\int_{\sigma}^{2\delta} M\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} dt,$$

then there exists a positive constant C_1 such that

$$C_1\lambda \leq \max_{x\in\bar{\Omega}} u(x) \quad \text{and} \quad C_1\lambda \leq \max_{x\in\bar{\Omega}} v(x).$$

It is easy to see

$$u(x) \leq z_1(x) \leq \max_{x\in\bar{\Omega}} z_1(x) \leq C_2(\lambda^{p^+}(L_2 + 1)\mu)^{\frac{1}{p^- - 1}},$$

then

$$\max_{x\in\bar{\Omega}} u(x) \leq C_2\left(\lambda^{p^+}(L_2 + 1)\mu\right)^{\frac{1}{p^- - 1}}.$$

Similarly

$$\max_{x\in\bar{\Omega}} v(x) \leq C_2\left\{\lambda^{p^+}(L_2 + 1)h\left[C_2\left(\lambda^{p^+}(L_2 + 1)\mu\right)^{\frac{1}{p^- - 1}}\right]\right\}^{\frac{1}{p^- - 1}}$$

Thus (16) and (17) are valid.

(*ii*) Denote

$$v_3(x) = \alpha\left(d(x)\right)^{\theta}, \quad d(x) \leq \rho,$$

where $\theta \in (0, 1)$ is a positive constant, $\rho \in (0, \delta)$ is small enough.
Obviously, $v_3(x) \in C^1(\Omega_\rho)$, By computation

$$-\Delta_{p(x)}v_3(x) = -(\alpha\theta)^{p(x)-1}(\theta-1)(p(x)-1)(d(x))^{(\theta-1)(p(x)-1)-1}(1+\Pi(x)), \qquad d(x) < \rho,$$

where

$$\Pi(x) = d\frac{\left(\nabla p\nabla d\right) \operatorname{In}\alpha\theta}{(\theta - 1)\left(p(x) - 1\right)} + d\frac{\left(\nabla p\nabla d\right) \operatorname{In} d}{\left(p(x) - 1\right)} + d\frac{\Delta d}{(\theta - 1)\left(p(x) - 1\right)}.$$

Let $\alpha = \frac{1}{\rho}C_2\left(\lambda^{p^+}(L_2 + 1)\mu\right)^{1/(p^- - 1)}$, where $\rho > 0$ is small enough, it is easy to see that

$$(\alpha)^{p(x)^{-1}} \geq \lambda^{p^+}\mu(L_2 + 1) \text{ and } |\Pi(x)| \leq \frac{1}{2}.$$

where $\rho > 0$ is small enough, then we have

$$-\Delta_{p(x)}v_3(x) \geq \lambda^{p^+}\mu(L_2 + 1).$$

Obviously $v_3(x) \geq z_1(x)$ on $\partial\Omega_\rho$. According to the comparison principle, we have $v_3(x) \geq z_1(x)$ on Ω_ρ. Thus

$$u(x) \le C_4 \left(\lambda^{p^+} (L_2 + 1)\, \mu\right)^{1/(p^- - 1)} (d(x))^{\theta}, \quad as\, d(x) \to 0.$$

Let $\alpha = \frac{1}{\rho} C_2 \left\{ \lambda^{p^+} (L_2 + 1)\, h \left[C_2 \left(\lambda^{p^+} (L_2 + 1)\, \mu\right)^{\frac{1}{p^- - 1}} \right] \right\}^{\frac{1}{p^- - 1}}$, when $\rho > 0$ is small enough, it is easy to see that

$$(\alpha)^{p(x)-1} \ge \lambda^{p^+} (L_2 + 1)\, h \left[C_2 \left(\lambda^{p^+} (L_2 + 1)\, \mu\right)^{\frac{1}{p^- - 1}} \right].$$

Similarly, when $\rho > 0$ is small enough, we have

$$v(x) \le C_4 \left\{ \lambda^{p^+} (L_2 + 1)\, h \left[C_2 \left(\lambda^{p^+} (L_2 + 1)\, \mu\right)^{\frac{1}{p^- - 1}} \right] \right\}^{\frac{1}{p^- - 1}} (d(x))^{\theta}\, as\, d(x) \to 0$$

Obviously, when $d(x) < \sigma$, we have

$$u(x),\, v(x) \ge \phi(x) = e^{kd(x)} - 1 \ge C_3 \lambda d(x).$$

Thus (18) and (19) are valid. This completes the proof. □

Acknowledgements
The authors would like to appreciate the referees for their helpful comments and suggestions. The third author partly supported by the National Science Foundation of China (10701066 & 10971087).

Author details
[1]Department of Mathematics, Sciences and Research, Islamic Azad University (IAU) Tehran, Iran [2]Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran [3]Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, Henan 450002, China [4]Department of Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran

Authors' contributions
All authors typed, read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 30 October 2011 Accepted: 3 April 2012 Published: 3 April 2012

References

1. Chen, Y, Levine, S, Rao, M: Variable exponent, linear growth functionals in image restoration. SIAM J Appl Math. **66**(4):1383–1406 (2006)
2. Ruzicka, M: Electrorheological fluids: Modeling and mathematical theory. In Lecture Notes in Math, vol. 1784,Springer-Verlag, Berlin (2000)
3. Zhikov, VV: Averaging of functionals of the calculus of variations and elasticity theory. Math USSR Izv. **29**, 33–36 (1987)
4. Acerbi, E, Mingione, G: Regularity results for a class of functionals with nonstandard growth. Arch Rat Mech Anal. **156**, 121–140 (2001)
5. Chen, M: On positive weak solutions for a class of quasilinear elliptic systems. Nonlinear Anal. **62**, 751–756 (2005)
6. Fan, XL, Zhao, D: On the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. J Math Anal Appl. **263**, 424–446 (2001)
7. Fan, XL: Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form. J Di Equ. **235**, 397–417 (2007)
8. El Hamidi, A: Existence results to elliptic systems with nonstandard growth conditions. J Math Anal Appl. **300**, 30–42 (2004)
9. Zhang, QH: Existence of positive solutions for a class of $p(x)$-Laplacian systems. J Math Anal Appl. **333**, 591–603 (2007)
10. Hai, DD, Shivaji, R: An existence result on positive solutions of p-Laplacian systems. Nonlinear Anal. **56**, 1007–1010 (2004)
11. Fan, XL, Zhang, QH, Zhao, D: Eigenvalues of $p(x)$-Laplacian Dirichlet problem. J Math Anal Appl. **302**, 306–317 (2005)
12. Zhang, QH: Existence and asymptotic behavior of positive solutions for variable exponent elliptic systems. Nonlinear Anal. **70**, 305–316 (2009)
13. Samko, SG: Densness of $C_0^{\infty}\left(R^N\right)$ in the generalized Sobolev spaces $W^{m,p(x)}$ (R^N). Dokl Ross Akad Nauk. **369**(4):451–454 (1999)
14. Fan, XL, Zhang, QH: Existence of solutions for $p(x)$-Laplacian Dirichlet problem. Nonlinear Anal. **52**, 1843–1852 (2003)

15. Zhang, QH: A strong maximum principle for differential equations with nonstandard *p*(*x*)-growth con-ditions. J Math Anal Appl. **312**(1):24–32 (2005)
16. Fan, XL: On the sub-supersolution method for *p*(*x*)-Laplacian equations. J Math Anal Appl. **330**, 665–682 (2007)

doi:10.1186/1687-2770-2012-37
Cite this article as: Ala *et al.*: Existence of positive solutions for variable exponent elliptic systems. *Boundary Value Problems* 2012 **2012**:37.