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Blow up problems for a degenerate parabolic equation with nonlocal source and nonlocal nonlinear boundary condition

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Abstract

This article deals with the blow-up problems of the positive solutions to a nonlinear parabolic equation with nonlocal source and nonlocal boundary condition. The blow-up and global existence conditions are obtained. For some special case, we also give out the blow-up rate estimate.

Keywords: parabolic equation, nonlocal source, nonlocal nonlinear boundary condition, existence, blow-up

1. Introduction

In this article, we consider the positive solution of the following degenerate parabolic equation

$$\begin{array}{l} u_t = f(u)(\Delta u + a\int_\Omega u(x,t)dx), \quad x \in \Omega, \quad t > 0, \\ u(x,t) = \int_\Omega g(x,y)u^l(y,t)dy, \quad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{array} \tag{1.1}$$

where a, $l > 0$ and Ω is a bounded domain in R^N $(N \geq 1)$ with smooth boundary $\partial\Omega$.

There have been many articles dealing with properties of solutions to degenerate parabolic equations with homogeneous Dirichlet boundary condition (see [1-4] and references therein). For example, Deng et al. [5] studied the parabolic equation with nonlocal source

$$u_t = f(u)(\Delta u + a\int_\Omega u dx), \tag{1.2}$$

which is subjected to homogeneous Dirichlet boundary condition. It was proved that there exists no global positive solution if and only if $\int^\infty 1/(sf(s))ds < \infty$ and $\int_\Omega \varphi(x)dx > 1/a$, where $\phi(x)$ is the unique positive solution of the linear elliptic problem

$$-\Delta\varphi = 1, x \in \Omega; \varphi(x) = 0, x \in \partial\Omega. \tag{1.3}$$

However, there are some important phenomena formulated into parabolic equations which are coupled with nonlocal boundary conditions in mathematical modeling such as thermoelasticity theory (see [6,7]). Friedman [8] studied the problem of nonlocal boundary conditions for linear parabolic equations of the type

$$\begin{cases} u_t - Au = c(x)u, & x \in \Omega, t > 0, \\ u(x,t) = \int_\Omega K(x,y)u(y,t)dy, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

with uniformly elliptic operator $A = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)$ and $c(x) \le 0$. It was proved that the unique solution of (1.4) tends to 0 monotonically and exponentially as $t \to +\infty$ provided that $\int_\Omega |\varphi(x,y)|\,dy \le \rho < 1, x \in \partial\Omega$.

Parabolic equations with both nonlocal sources and nonlocal boundary conditions have been studied as well (see [9-12]). Lin and Liu [13] considered the problem of the form

$$\begin{cases} u_t = \Delta u + \int_\Omega g(u)dx, & x \in \Omega, t > 0, \\ u(x,t) = \int_\Omega K(x,y)u(y,t)dy, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.5}$$

They established local existence, global existence, and nonexistence of solutions, and discussed the blow-up properties of solutions.

Chen and Liu [14] considered the following nonlinear parabolic equation with a localized reaction source and a weighted nonlocal boundary condition

$$\begin{cases} u_t = f(u)(\Delta u + au(x_0,t)), & x \in \Omega, t > 0, \\ u(x,t) = \int_\Omega g(x,y)u(y,t)dy, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.6}$$

Under certain conditions, they obtained blow-up criteria. Furthermore, they derived the uniform blow-up estimate for some special $f(u)$.

In recent few years, reaction-diffusion problems coupled with nonlocal nonlinear boundary conditions have also been studied. Gladkov and Kim [15] considered the following problem for a single semilinear heat equation

$$\begin{cases} u_t = \Delta u + c(x,t)u^p, & x \in \Omega, t > 0, \\ u(x,t) = \int_\Omega \varphi(x,y,t)u^l(y,t)dy, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.7}$$

where p, $l > 0$. They obtained some criteria for the existence of global solution as well as for the solution to blow-up in finite time.

For other works on parabolic equations and systems with nonlocal nonlinear boundary conditions, we refer readers to [16-20] and the references therein.

Motivated by those of works above, we will study the problem (1.1) and want to understand how the function $f(u)$ and the coefficient a, the weight function $g(x, y)$ and the nonlinear term $u^l(y, t)$ in the boundary condition play substantial roles in determining blow-up or not of solutions.

In this article, we give the following hypotheses:

(H1) $u_0(x) \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ for $\alpha \in (0,1), u_0(x) > 0$ in Ω, $u_0(x) = \int_\Omega g(x,y)u_0^l(y)dy$ on $\partial\Omega$.

(H2) $g(x, y)$⊠0 is a nonnegative and continuous function defined for $x \in \partial\Omega$, $y \in \overline{\Omega}$.

(H3) $f \in C([0,\infty)) \cap C^1(0,\infty)$, $f > 0$, $f' \geq 0$ in $(0,\infty)$.

The main results of this article are stated as follows.

Theorem 1.1. Assume that $0 < l \leq 1$ and $\int_\Omega g(x,y)dy < 1$ for all $x \in \partial\Omega$.

(1) If a is sufficiently small, then the solution of (1.1) exists globally;

(2) If a is sufficiently large, then the solution of (1.1) also exists globally provided that $\int_\delta^{+\infty} 1/(sf(s))ds = +\infty$ for some $\delta > 0$.

Theorem 1.2. Assume that $l > 1$ and $\int_\Omega g(x,y)dy < 1$ for all $x \in \partial\Omega$. Then the solution of (1.1) exists globally provided that a and $u_0(x)$ are sufficiently small. While the solution blows up in finite time if $a, u_0(x)$ are sufficiently large and $\int_\delta^{+\infty} 1/(sf(s))ds < +\infty$ for some $\delta > 0$.

Theorem 1.3. Assume that $l > 1$ and $\int_\Omega g(x,y)dy \geq 1$ for all $x \in \partial\Omega$. If $\int_\delta^{+\infty} 1/(sf(s))ds < +\infty$ for some $\delta > 0$, then the solution of (1.1) blows up in finite time provided that $u_0(x)$ is large enough.

Theorem 1.4. If $\int_\delta^{+\infty} 1/(sf(s))ds < +\infty$ for some $\delta > 0$ and $a > \left(\int_\Omega \varphi(x)dx\right)^{-1}$, where $\phi(x)$ is the solution of (1.3), then there exists no global positive solution of (1.1).

To describe conditions for blow-up of solutions, we need an additional assumption on the initial data u_0.

(H4) There exists a constant $\varepsilon > \varepsilon_1 > 0$ such that $\Delta u_0 + a\int_\Omega u_0(x)dx \geq \varepsilon u_0$, where ε_1 will be given later.

Theorem 1.5. Assume $u_0(x)$ satisfies (H1), (H2), and (H4), $\Delta u_0 \leq 0$ in Ω holds, and let $f(u) = u^p, 0 < p \leq 1$, $l = 1$, then the following limits converge uniformly on any compact subset of Ω:

(1) If $0 < p < 1$, $\lim_{t \to T^*} u(x,t)(T^* - t)^{1/p} = (ap|\Omega|)^{-1/p}$.

(2) If $p = 1$, $\lim_{t \to T^*} \left|\ln(T^* - t)\right|^{-1} \ln u(x,t) = 1$.

This article is organized as follows. In Section 2, we establish the comparison principle and the local existence. Some criteria regarding to global existence and finite time blow-up for problem (1.1) are given in Section 3. In Section 4, the global blow-up result and the blow-up rate estimate of blow-up solutions for the special case of $f(u) = u^p$, $0 < p \leq 1$ and $l = 1$ are obtained.

2. Comparison principle and local existence

First, we start with the definition of subsolution and supersolution of (1.1) and comparison principle. Let $Q_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$, and $\overline{Q}_T = \overline{\Omega} \times [0, T)$.

Definition 2.1. A function $\underline{u}(x, t)$ is called a subsolution of (1.1) on Q_T, if $\underline{u} \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ satisfies

$$\begin{array}{ll} \underline{u}_t \leq f(\underline{u})(\Delta \underline{u} + a \int_\Omega \underline{u} dx), & x \in \Omega, t > 0, \\ \underline{u}(x, t) \leq \int_\Omega g(x, y) \underline{u}^l(y, t) dy, & x \in \partial\Omega,\ t > 0, \\ \underline{u}(x, 0) \leq u_0(x), & x \in \Omega. \end{array} \tag{2.1}$$

Similarly, a supersolution $\bar{u}(x, t)$ of (1.1) is defined by the opposite inequalities.

A solution of problem (1.1) is a function which is both a subsolution and a supersolution of problem (1.1).

The following comparison principle plays a crucial role in our proofs which can be obtained by similar arguments as [10] and its proof is therefore omitted here.

Lemma 2.2. Suppose that $w(x, t) \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and satisfies

$$\begin{array}{ll} w_t - d(x, t)\Delta w \geq c_1(x, t) w + c_2(x, t) \displaystyle\int_\Omega c_3(x, t) w(x, t) dx, & (x, t) \in Q_T, \\ w(x, t) \geq c_4(x, t) \displaystyle\int_\Omega c_5(x, y) w^l(y, t) dy, & (x, t) \in S_T, \end{array} \tag{2.2}$$

where $d(x, t)$, $c_i(x, t) (i = 1,2,3,4)$ are bounded functions and $d(x, t) \geq 0$, $c_i(x, t) \geq 0$ ($i = 2,3,4$) in Q_T, $c_5(x, y) \geq 0$ for $x \in \partial\Omega$, $y \in \Omega$ and is not identically zero. Then, $w(x, 0) > 0$ for $x \in \overline{\Omega}$ implies $w(x, t) > 0$ in Q_T. Moreover, $c_5(x, y) \equiv 0$ or if $c_4(x, t) \int_\Omega c_5(x, y) dy \leq 1$ on S_T, then $w(x, 0) \geq 0$ for $x \in \overline{\Omega}$ implies $w(x, t) \geq 0$ in Q_T.

On the basis of the above lemmas, we obtain the following comparison principle of (1.1).

Lemma 2.3. Let u and v be nonnegative subsolution and supersolution of (1.1), respectively, with $u(x, 0) \leq v(x, 0)$ for $x \in \overline{\Omega}$. Then, $u \leq v$ in Q_T if $u \geq \eta$ or $v \geq \eta$ for some small positive constant η holds.

Local in time existence of positive classical solutions of (1.1) can be obtained by using fixed point theorem [21], the representation formula and the contraction mapping principle as in [13]. By the above comparison principle, we get the uniqueness of solution to the problem. The proof is more or less standard, so is omitted here.

3. Global existence and blow-up in finite time

In this section, we will use super- and subsolution techniques to derive some conditions on the existence or nonexistence of global solution.

Proof of Theorem 1.1. (1) Let $\psi(x)$ be the unique positive solution of the linear elliptic problem

$$\begin{array}{ll} -\Delta \psi = \varepsilon_0, & x \in \Omega, \\ \psi(x) = \displaystyle\int_\Omega g(x, y) dy, & x \in \partial\Omega, \end{array} \tag{3.1}$$

where ε_0 is a positive constant such that $0 < \psi(x) < 1$ (since $\int_\Omega g(x,y)dy < 1$, there exists such ε_0). Let $\overline{K} = \max_{x\in\overline{\Omega}}\psi(x)$, $\underline{K} = \min_{x\in\overline{\Omega}}\psi(x)$.

We define a function $w(x, t)$ as following:

$$w(x,t) = M\psi(x), \tag{3.2}$$

where $M \geq 1$ is a constant to be determined later. Then, we have

$$\begin{aligned} w|_{\partial\Omega} &= M\int_\Omega g(x,y)dy \geq M\int_\Omega g(x,y)\psi^l(x)dy = M^{1-l}\int_\Omega g(x,y)w^l(y,t)dy \\ &\geq \int_\Omega g(x,y)w^l(y,t)dy. \end{aligned} \tag{3.3}$$

On the other hand, we have for $x \in \Omega$, $t > 0$,,

$$w_t - f(w)(\Delta w + a\int_\Omega w(x,t)dx) \geq f(M\psi(x))M(\varepsilon_0 - a|\Omega|\overline{K}). \tag{3.4}$$

We choose $M = \max\{\underline{K}^{-1}\max_{x\in\overline{\Omega}}u_0(x), 1\}$ and set $a_0 = \varepsilon_0(|\Omega|\overline{K})^{-1}$, then it is easy to verify that $w(x, t)$ is a supersolution of (1.1) provided that $a \leq a_0$. By comparison principle, $u(x, t) \leq w(x, t)$, then $u(x, t)$ exists globally.

(2) Consider the following problem

$$\begin{aligned} &z'(t) = b_1 f(\overline{K}z(t))z(t), \quad t > 0, \\ &z(0) = z_0, \end{aligned} \tag{3.5}$$

where $z_0 > \max\{\underline{K}^{-1}\max_{x\in\overline{\Omega}}u_0(x), 1\}$, b_1 is a positive constant to be fixed later. It follows from hypothesis (H3) and the theory of ordinary differential equation (ODE) that there exists a unique solution $z(t)$ to problem (3.5) and $z(t)$ is increasing. If $\int_\delta^{+\infty} 1/(sf(s))ds = +\infty$ for some positive δ, we know that $z(t)$ exists globally and $z(t) \geq z_0$.

Let $v(x, t) = z(t)\psi(x)$, where $\psi(x)$ is given by (3.1), then for $x \in \Omega$, $t > 0$, we obtain

$$\begin{aligned} &v_t - f(v)(\Delta v + a\int_\Omega v(x,t)dx) \\ &\quad = z'(t)\psi(x) - f(z(t)\psi(x))(z(t)\Delta\psi(x) + a\int_\Omega z(t)\psi(x)dx) \\ &\quad \geq z'(t)\underline{K} - f(\overline{K}z(t))z(t)(a\overline{K}|\Omega| - \varepsilon_0) \\ &\quad = f(\overline{K}z(t))z(t)(b_1\underline{K} - (a\overline{K}|\Omega| - \varepsilon_0)). \end{aligned} \tag{3.6}$$

Set $a_1 = \varepsilon_0(\overline{K}|\Omega|)^{-1}$, if a is sufficiently large such that $a > a_1$, then we can choose $b_1 = \underline{K}^{-1}(a\overline{K}|\Omega| - \varepsilon_0) > 0$. Thus,

$$v_t - f(v)(\Delta v + a\int_\Omega v(x,t)dx) \geq 0. \tag{3.7}$$

On the other hand, for $x \in \partial\Omega$, $t > 0$, we get

$$\begin{aligned} v(x,t) &= z(t)\int_\Omega g(x,y)dy > z(t)\int_\Omega g(x,y)\psi^l(y)dy \\ &> \int_\Omega g(x,y)z^l(t)\psi^l(y)dy = \int_\Omega g(x,y)v^l(y,t)dy. \end{aligned} \tag{3.8}$$

Here, we use the conclusions $0 < \psi(x) < 1$ and $z(t) > 1$.

Also for $x \in \overline{\Omega}$, we have

$$v(x,0) = z(0)\psi(x) = z_0\psi(x) \ge z_0\underline{K} \ge u_0(x). \tag{3.9}$$

And the inequalities (3.5)-(3.9) show that $v(x, t)$ is a supersolution of (1.1). Again by using the comparison principle, we obtain the global existence of $u(x, t)$. The proof is complete.

Proof of Theorem 1.2. The proof of global existence part is similar to the first case of Theorem 1.1. For any given positive constant $M \le 1$, $w(x) = M\psi(x)$ is a supersolution of problem (1.1) provided that $u_0(x) \le \psi(x) < 1$ and $a < \varepsilon_0(|\Omega|\overline{K})^{-1}$, so the solution of (1.1) exists globally by using the comparison principle.

To prove the bow-up result, we introduce the elliptic problem

$$-\Delta\varphi(x) = 1, \quad x \in \Omega;\ \varphi(x) = \int_\Omega g(x,y)\varphi(y)dy, \quad x \in \partial\Omega.$$

Under the hypothesis (H2) and $\int_\Omega g(x,y)dy < 1$, we know that it exists a unique positive solution $\phi(x)$. Let $K_* = \min_{x\in\overline{\Omega}}\varphi(x)$, $K^* = \max_{x\in\overline{\Omega}}\varphi(x)$, and $z(t)$ be the solution of the following ODE

$$\begin{aligned} &z'(t) = f(K_*z(t))z(t), \quad t > 0, \\ &z(0) = z_1 > 1. \end{aligned} \tag{3.10}$$

Then, $z(t)$ is increasing and $z(t) \ge z_1$. Due to the condition $\int_\delta^{+\infty} 1/(sf(s))ds < +\infty$ for some positive constant δ, we know that $z(t)$ of problem (3.10) blows up in finite time.

If a and $u_0(x)$ are so large that $a \ge (K^*)^{l-1}(K^* + l)|\Omega|^{-1}K_*^{-l}, u_0(x) \ge z_1(K^*)^l$, then we set $v_1(x, t) = z(t)\phi^l(x)$. For $x \in \Omega$, $t > 0$, we obtain

$$\begin{aligned} &v_{1t} - f(v_1)(\Delta v_1 + a\int_\Omega v_1(x,t)dx) \\ &= z'(t)\varphi^l(x) - f(z(t)\varphi^l(x))z(t)(l(l-1)\varphi^{l-2}(x)|\nabla\varphi|^2 + l\varphi^{l-1}(x)\Delta\varphi(x) + a\int_\Omega \varphi^l(x)dx) \\ &\le z'(t)(K^*)^l - f(K_*{}^l z(t))z(t)(aK_*{}^l|\Omega| - l(K^*)^{l-1}) \le 0. \end{aligned} \tag{3.11}$$

For $x \in \partial\Omega$, $t > 0$, by Jensen's inequality, we get

$$\begin{aligned} v_1(x,t) &= z(t)\varphi^l(x) = z(t)(\int_\Omega g(x,y)\varphi(y)dy)^l \\ &\le z(t)(\int_\Omega g(x,y)dy)^{l-1}(\int_\Omega g(x,y)\varphi^l(y)dy) \\ &\le \int_\Omega g(x,y)z^l(t)\varphi^l(y)dy = \int_\Omega g(x,y)v_1^l(y)dy. \end{aligned} \tag{3.12}$$

Also for $x \in \overline{\Omega}$, we have

$$v_1(x,0) = z(0)\varphi^l(x) = z_1\varphi^l(x) \le z_1(K^*)^l \le u_0(x). \tag{3.13}$$

The inequalities (3.10)-(3.13) show that $v_1(x, t)$ is a subsolution of problem (1.1). Since $v_1(x, t)$ blows up in finite time, $u(x, t)$ also blows up in finite time by comparison principle.

Proof of Theorem 1.3. Let $z(t)$ be the solution of the following ODE

$$\begin{aligned} &z'(t) = b_2 f(z(t))z(t), \quad t > 0, \\ &z(0) = z_2. \end{aligned} \tag{3.14}$$

where $0 < b_2 < a\ |\Omega|$. If $u_0(x)$ is large enough, we can set $1 < z_2 < \min_{x\in\overline{\Omega}} u_0(x)$. Then, $z\ (t)$ is increasing and satisfies $z\ (t) \ge z_2 > 1$. Moreover, $z\ (t)$ of problem (3.14) blows up in finite time.

Set $s\ (x,\ t) = z\ (t)$, then we have for $x \in\ \Omega$, $t > 0$,

$$\begin{aligned} &s_t - f(s)(\Delta s + a\int_\Omega s(x,t)dx) \\ &\quad = z'(t) - af(z(t))\,|\Omega|\,z(t) \ = \ (b_2 - a\,|\Omega|)\ f(z(t))z(t) \ < \ 0. \end{aligned} \tag{3.15}$$

For $x \in\ \partial\Omega$, $t > 0$,

$$s(x,t) = z(t) \le \int_\Omega g(x,y)z(t)dy < \int_\Omega g(x,y)z^l(t)dy = \int_\Omega g(x,y)s^l(y,t)dy. \tag{3.16}$$

$$s(x,0) = z(0) = z_2 < u_0(x), \quad x \in \overline{\Omega}. \tag{3.17}$$

From (3.14)-(3.17), we see that $s\ (x,\ t)$ is a subsolution of (1.1). Hence, $u\ (x,\ t) \ge s\ (x,\ t)$ by comparison principle, which implies $u\ (x,\ t)$ blows up in finite time. This completes the proof.

Proof of Theorem 1.4. Consider the following equation

$$\begin{aligned} &v_t = f(v)(\Delta v + a\int_\Omega v dx)\ , \quad x \in \Omega, t > 0, \\ &v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\ &v(x,0) = v_0(x), \quad x \in \Omega, \end{aligned} \tag{3.18}$$

and let $v\ (x,\ t)$ be the solution to problem (3.18). It is obvious that $v\ (x,\ t)$ is a subsolution of (1.1). By Theorem 1 in [5], we can obtain the result immediately.

4. Blow-up rate estimate

Now, we consider problem (1.1) with $f\ (u) = u^p$, $0 < p \le 1$ and $l = 1$, i.e.,

$$\begin{aligned} &u_t = u^p(\Delta u + a\int_\Omega u(x,t)dx)\ , \quad x \in \Omega, t > 0, \\ &u(x,t) = \int_\Omega g(x,y)u(y,t)dy\ , \quad x \in \partial\Omega,\ t > 0, \\ &u(x,0) = u_0(x), \quad x \in \Omega, \end{aligned} \tag{4.1}$$

where $\int_\Omega g(x,y)dy \le 1$ for all $x \in \partial\Omega$, and suppose that the solution of (4.1) blows up in finite time T^*.

Set $U(t) = \max_{x\in\overline{\Omega}} u(x,t)$, then $U(t)$ is Lipschitz continuous.

Lemma 4.1. Suppose that u_0 satisfies (H1), (H2), and (H4), then there exists a positive constant c_0 such that

$$U(t) \ge c_0(T^* - t)^{-1/p}. \tag{4.2}$$

Proof. By the first equation in (4.1), we have (see [22])

$$U'(t) \le a\,|\Omega|\,U^{1+p}(t), \qquad a.e.\ t \in (0, T^*), \tag{4.3}$$

Hence,

$$-(U^{-p}(t))' \le ap\,|\Omega|\,. \tag{4.4}$$

Integrating (4.3) over (t, T^*), we can get

$$U(t) \ge (ap\,|\Omega|)^{-1/p}(T^* - t)^{-1/p}. \tag{4.5}$$

Setting $c_0 = (ap\ |\Omega|\ p)^{-1/p}$, then we draw the conclusion.

Lemma 4.2. Under the conditions of Lemma 4.1, there exists a constant ε_1, which will be given below, such that

$$u_t - \varepsilon_1 u^{p+1} \ge 0, \qquad (x,t) \in \Omega \times (0, T^*). \tag{4.6}$$

Proof. Let $J(x, t) = u_t\text{-}\varepsilon_1 u^{p+1}$ for $(x, t) \in \Omega \times (0, T^*)$, a series of computations yields

$$\begin{aligned} & J_t - u^p\Delta J - 2p\varepsilon_1 u^p J - au^p\int_\Omega Jdx \\ & = pu^{-1}J^2 + \varepsilon_1(p+1)pu^{2p-1}|\nabla u|^2 + p{\varepsilon_1}^2u^{2p+1} + a\varepsilon_1 u^p\int_\Omega u^{p+1}dx - a\varepsilon_1(p+1)u^{2p}\int_\Omega udx \\ & \ge p{\varepsilon_1}^2u^{2p+1} + a\varepsilon_1 u^p\int_\Omega u^{p+1}dx - a\varepsilon_1(p+1)u^{2p}\int_\Omega udx. \end{aligned} \tag{4.7}$$

By virtue of Hölder inequality, we have

$$\int_\Omega udx \le |\Omega|^{p/(p+1)}\Big(\int_\Omega u^{p+1}dx\Big)^{1/(p+1)}.$$

Furthermore, by Young's inequality, for any $\theta > 0$, the following inequality holds

$$\begin{aligned} u^{2p}\int_\Omega udx &\le |\Omega|^{p/(p+1)}u^p \cdot u^p\Big(\int_\Omega u^{p+1}dx\Big)^{1/(p+1)} \\ &\le |\Omega|^{p/(p+1)}u^p\Big(p/(p+1)(\theta u^p)^{(p+1)/p} + 1/(p+1)\theta^{-(p+1)}\int_\Omega u^{p+1}dx\Big) \\ &= (1/(p+1))|\Omega|^{p/(p+1)}\Big(p\theta^{(p+1)/p}u^{2p+1} + \theta^{-(p+1)}u^p\int_\Omega u^{p+1}dx\Big). \end{aligned} \tag{4.8}$$

Using (4.8) and taking $\theta = |\Omega|^{p/(p+1)^2}$, $\varepsilon_1 = a|\Omega|$, then (4.7) becomes

$$J_t - u^p\Delta J - 2p\varepsilon_1 u^p J - au^p\textstyle\int_\Omega Jdx \ge p\varepsilon_1(\varepsilon_1 - a\,|\Omega|)u^{2p+1} = 0, \tag{4.9}$$

Fix $(x, t) \in \partial\Omega \times (0, T^*)$, then we have

$$J(x,t) = u_t - \varepsilon_1 u^{p+1} = \int_\Omega g(x,y)u_t(y,t)dy - \varepsilon_1\left(\int_\Omega g(x,y)u(y,t)dy\right)^{p+1}.$$

Since $u_t(y, t) = J(y, t) + \varepsilon_1 u^{p+1}(y, t)$, we have

$$\begin{aligned}&\int_\Omega g(x,y)u_t(y,t)dy - \varepsilon_1\left(\int_\Omega g(x,y)u(y,t)dy\right)^{p+1}\\ &= \int_\Omega g(x,y)J(y,t)dy + \varepsilon_1\left(\int_\Omega g(x,y)u^{p+1}(y,t)dy - \left(\int_\Omega g(x,y)u(y,t)dy\right)^{p+1}\right).\end{aligned}$$

Noticing that $p > 0,\ 0 < F(x) = \int_\Omega g(x,y)dy \le 1\ \ , x \in \partial\Omega$, we can apply Jensen's inequality to the last integral in the above inequality,

$$\begin{aligned}&\int_\Omega g(x,y)u^{p+1}(y,t)dy - \left(\int_\Omega g(x,y)u(y,t)dy\right)^{p+1}\\ &\ge F(x)\left(\int_\Omega g(x,y)u(y,t)dy \Big/ F(x)\right)^{p+1} - \left(\int_\Omega g(x,y)u(y,t)dy\right)^{p+1} \ge 0.\end{aligned}$$

Hence, for $(x, t) \in \partial\Omega \times (0, T^*)$, we have

$$J(x,t) \ge \int_\Omega g(x,y)J(y,t)dy\ . \tag{4.10}$$

On the other hand, (H4) implies that

$$J(x,0) > 0,\ \ x \in \Omega. \tag{4.11}$$

Owing to $u(x, t)$ is a positive continuous function for $(x,t) \in \overline{\Omega} \times [0, T^*)$, it follows from (4.9)-(4.11) and Lemma 2.2 that $J(x, t) \ge 0$ for $(x,t) \in \overline{\Omega} \times [0,T^*)$, i.e., $u_t \ge \varepsilon_1 u^{p+1}$. This completes the proof.

Integrating (4.6) from t to T^*, we conclude that

$$u(x,t) \le c_2(T^* - t)^{-1/p}, \tag{4.12}$$

where $c_2 = (\varepsilon_1 p)^{-1/p}$ is a positive constant independent of t. Combining (4.2) with (4.12), we obtain the following result.

Theorem 4.3. Under the conditions of Lemma 4.1, if $u(x, t)$ is the solution of (4.1) and blows up in finite time T^*, then there exist positive constants c_1, c_2, such that

$$c_1(T^* - t)^{-1/p} \le \max_{x\in\overline{\Omega}} u(x,t) \le c_2(T^* - t)^{-1/p}.$$

Lemma 4.4. Assume that $u_0(x)$ satisfies (H1), (H2), and (H4), $\Delta u_0 \le 0$ in Ω. $u(x, t)$ is the solution of problem (4.1). Then, $\Delta u \le 0$ in any compact subsets of $\Omega \times (0, T^*)$.

The proof is similar to that of Lemma 1.1 in [14].

Denote

$$g(t) = a\int_\Omega u dx, \quad G(t) = \int_0^t g(s)ds.$$

Lemma 4.5. Under the conditions of Lemma 4.4, it holds that

$$\lim_{t\to T^*} g(t) = \infty, \qquad \lim_{t\to T^*} G(t) = \infty\ . \tag{4.13}$$

Proof. From Lemma 4.3, we have

$$u_t \le u^p g(t), \qquad a.e.\ t \in [0, T^*). \tag{4.14}$$

Integrating (4.14) over (0, *t*), we obtain

$$\frac{1}{1-p} u^{1-p}(x,t) \le \int_0^t g(s)ds + \frac{1}{1-p} {u_0}^{1-p}(x), \quad 0 < p < 1, \tag{4.15}$$

$$\ln u(x,t) \le \int_0^t g(s)ds + \ln u_0(x), \quad p = 1. \tag{4.16}$$

In view of $\lim_{t\to T^*} u(x,t) = \infty$, $\lim_{t\to T^*} G(t) = \infty$. Noting that $u_t \ge 0$ by the assumption of the initial function, then we see that $g(t)$ is monotone nondecreasing. Therefore, $\lim_{t\to T^*} g(t) = \infty$.

Lemma 4. 6. Under the conditions of Lemma 4.4, then we have

$$(1) \quad \lim_{t\to T^*} \frac{u^{1-p}(x,t)}{(1-p)G(t)} = \lim_{t\to T^*} \frac{\|u(\cdot,t)\|_\infty^{1-p}}{(1-p)G(t)} = 1, \quad 0 < p < 1, \tag{4.17}$$

$$(2) \quad \lim_{t\to T^*} \frac{\ln u(x,t)}{G(t)} = \lim_{t\to T^*} \frac{\|\ln u(\cdot,t)\|_\infty}{G(t)} = 1, \quad p = 1, \tag{4.18}$$

uniformly on any compact subsets of Ω.

Proof. Let $\lambda > 0$ be the principal eigenvalue of $-\Delta$ in Ω with the null Dirichlet boundary condition, and $\varphi(x)$ be the corresponding eigenfunction satisfying $\varphi(x) > 0$, $\int_\Omega \phi(x)dx = 1$.

In case of (1). Define $z_1(x, t) = G(t) - u^{1-p}/(1-p)$, $\gamma_1(t) = \int_\Omega z_1(y,t)\phi(y)dy$. A direct computation shows

$$\begin{aligned}
\gamma_1'(t) &= \int_\Omega (g(t) - u^{-p}(y,t)u_t(y,t))\phi(y)dy = -\int_\Omega \Delta u(y,t)\phi(y)dy \\
&= \lambda \int_\Omega \phi(y)u(y,t)dy + \int_{\partial\Omega} u(\partial\phi/\partial n)dS \\
&\le \lambda \int_\Omega (G(t) - z_1(y,t))^{1/(1-p)}\phi(y)dy \\
&\le C_1(G^{1/(1-p)}(t) + \int_\Omega (z_1^-(y,t))^{1/(1-p)}\phi(y)dy),
\end{aligned}$$

where $z_1^- = \max\{-z_1, 0\}$ and using the equality $\int_{\partial\Omega} (\partial\phi/\partial n)dS = -\lambda < 0$. From (4.15), we know that

$$\inf_\Omega z_1(y,t) \ge -C', \tag{4.19}$$

which means $z_1^- \le C'$. Then,

$$\gamma_1'(t) \le C_2 G^{1/(1-p)}(t) + C_3, \tag{4.20}$$

Integrate (4.20) from 0 to t,

$$\gamma_1(t) \le C_4\left(1 + \int_0^t G^{1/(1-p)}(s)ds\right), \tag{4.21}$$

Thus, (4.19) and (4.21) imply

$$\int_\Omega \left|z_1(y,t)\right| \phi(y)dy \le C_5\left(1 + \int_0^t G^{1/(1-p)}(s)ds\right).$$

Define $K_\rho = \{y \in \Omega: \mathrm{dist}(y, \partial\Omega) \ge \rho\}$. Since $-\Delta z_1 \le 0$ in $\Omega \times (0, T^*)$. Using Lemma 4.5 in [1], we obtain

$$\sup_{K_\rho} z_1(x,t) \le \frac{C_6}{\rho^{N+1}}\left(1 + \int_0^t G^{1/(1-p)}(s)ds\right). \tag{4.22}$$

It follows from (4.22) and (4.15) that

$$-\frac{k_1}{G(t)} \le 1 - \frac{u^{1-p}(x,t)}{(1-p)G(t)} \le \frac{K_1}{\rho^{N+1}} \frac{1 + \int_0^t G^{1/(1-p)}(s)ds}{G(t)}, \tag{4.23}$$

for any $x \in K_\rho$ and $t \in (0,T^*)$, where k_1 and K_1 are positive constants.
We know from Theorem 4.3 that

$$\int_0^t G^{1/(1-p)}(s)ds \le C_7 \int_0^t (T^* - s)^{-1/p}ds. \tag{4.24}$$

In view of (4.15) and Theorem 4.3, it follows that

$$G(t) \ge C_8 u^{1-p} \ge C_9 (T^* - t)^{-(1-p)/p}. \tag{4.25}$$

From (4.23)-(4.25), we get

$$-\frac{k_1}{G(t)} \le 1 - \frac{u^{1-p}(x,t)}{(1-p)G(t)} \le \frac{C_{10}}{\rho^{N+1}} \frac{1 + \int_0^t (T^* - s)^{-1/p}ds}{(T^* - t)^{-(1-p)/p}}. \tag{4.26}$$

It is obvious that

$$\lim_{t \to T^*} \frac{\int_0^t (T^* - s)^{-1/p}ds}{(T^* - t)^{-(1-p)/p}} = 0.$$

Thus,

$$\lim_{t \to T^*} \frac{u^{1-p}(x,t)}{(1-p)G(t)} = \lim_{t \to T^*} \frac{\left\|u(\cdot,t)\right\|_\infty^{1-p}}{(1-p)G(t)} = 1.$$

In case of (2). We define $z_2(x,t) = G(t) - \ln u(x,t)$, $\gamma_2(t) = \int_\Omega z_2(y,t)\phi(y)dy$. Then,

$$\begin{aligned}
\gamma_2'(t) &= \int_\Omega (g(t) - u^{-1}(y,t)u_t(y,t))\phi(y)dy = -\int_\Omega \Delta u(y,t)\phi(y)dy \\
&= \lambda \int_\Omega \phi(y)u(y,t)dy + \int_{\partial\Omega} u(\partial\phi/\partial n)dS \\
&\le \lambda \int_\Omega \exp\{G(t) - z_2(y,t)\}\phi(y)dy.
\end{aligned}$$

From (4.16), we know that

$$\inf_{\Omega} z_2(y,t) \geq -C'', \tag{4.27}$$

Then,

$$y_2'(t) \leq C_{11} \exp\{G(t)\}, \tag{4.28}$$

Integrate (4.28) from 0 to t yields

$$y_2(t) \leq C_{12}\left(1 + \int_0^t \exp\{G(s)\}ds\right), \tag{4.29}$$

Thus, (4.29) and (4.27) imply

$$\int_\Omega |z_2(y,t)|\,\phi(y)dy \leq C_{13}\left(1 + \int_0^t \exp\{G(s)\}ds\right).$$

Define $K_\zeta = \{y \in \Omega: dist(y, \partial\Omega) \geq \zeta\}$. Since $-\Delta z_2 \leq 0$ in $\Omega \times (0, T^*)$, we obtain

$$\sup_{K_\zeta} z_2(x,t) \leq \frac{C_{14}}{\zeta^{N+1}}\left(1 + \int_0^t \exp\{G(s)\}ds\right). \tag{4.30}$$

It follows from (4.30) and (4.27) that

$$-\frac{k_2}{G(t)} \leq 1 - \frac{\ln u(x,t)}{G(t)} \leq \frac{K_2}{\zeta^{N+1}} \frac{1 + \int_0^t \exp\{G(s)\}ds}{G(t)}, \tag{4.31}$$

for any $x \in K_\zeta$ and $t \in (0, T^*)$.
By Theorem 4.3, we have

$$\begin{aligned} G(t) &= \int_0^t g(s)ds \leq a \int_0^t \left(\int_\Omega |U(s)|dx\right)ds \\ &\leq a\,|\Omega|\,(\varepsilon_1)^{-1} \int_0^t (T^* - s)^{-1} ds \leq \ln (T^* - t)^{-1} + \ln T^*. \end{aligned} \tag{4.32}$$

On the other hand, we know form (4.16) and Theorem 4.3 that

$$G(t) \geq C_{16}\left|\ln(T^* - t)\right|. \tag{4.33}$$

From (4.31)-(4.33), we get

$$-\frac{k_2}{G(t)} \leq 1 - \frac{\ln u(x,t)}{G(t)} \leq \frac{C_{17}}{\zeta^{N+1}} \frac{1 + T^* \int_0^t (T^* - s)^{-1} ds}{\left|\ln(T^* - t)\right|}.$$

It is easy to derive

$$\lim_{t \to T^*} \frac{1 + T^* \int_0^t (T^* - s)^{-1} ds}{\left|\ln(T^* - t)\right|} = 0.$$

Thus,

$$\lim_{t \to T^*} \frac{\ln u(x,t)}{G(t)} = \lim_{t \to T^*} \frac{\left\|\ln u(\cdot,t)\right\|_\infty}{G(t)} = 1.$$

This completes the proof.

Proof of Theorem 1.5. Case 1: $0 < p < 1$. Form (4.17), we have

$$u(x,t) \sim ((1-p)G(t))^{1/(1-p)} \text{ as } t \to T^*,$$

where the notation $u \sim v$ means $\lim_{t\to T^*} u(t)/v(t) = 1$.

Furthermore,

$$G'(t) = g(t) = a\int_\Omega u dx \sim a|\Omega|((1-p)G(t))^{1/(1-p)} \text{ as } t \to T^*. \tag{4.34}$$

Integrating (4.34) over (t, T^*) yields

$$G(t) \sim (1-p)^{-1}(ap|\Omega|(T^*-t))^{-(1-p)/p} \text{ as } t \to T^*. \tag{4.35}$$

So, we can get our conclusion by using (4.17) and (4.35).

Case 2: $p = 1$. In this case, for any given σ: $0 < \sigma \ll 1$. By (4.18), there exists $0 < t_0 < T^*$ such that

$$(1-\sigma)G(t) \le \ln u(x,t) \le (1+\sigma)G(t), \quad x \in \Omega, t \in [t_0, T^*).$$

Therefore,

$$a|\Omega|\exp\{(1-\sigma)G(t)\} \le G'(t) = a\int_\Omega u dx \le a|\Omega|\exp\{(1+\sigma)G(t)\}, \quad t_0 \le t \le T^*. \tag{4.36}$$

In view of the right-hand side of the (4.36), we have

$$\exp\{-(1+\sigma)G(t)\}dG(t) \le a|\Omega|dt, \quad t_0 \le t \le T^*.$$

Integrating the above inequality from t to T^* yields that

$$\exp\{-(1+\sigma)G(t)\} \le a(1+\sigma)|\Omega|(T^*-t), \quad t_0 \le t \le T^*.$$

Namely,

$$G(t) \ge (-1/(1+\sigma))\ln[a(1+\sigma)|\Omega|(T^*-t)], \quad t_0 \le t \le T^*. \tag{4.37}$$

Similar arguments to the left-hand side of (3.36) yield that

$$G(t) \le (-1/(1-\sigma))\ln[a(1-\sigma)|\Omega|(T^*-t)], \quad t_0 \le t \le T^*. \tag{4.38}$$

Consequently, (4.37) and (4.38) guarantee that for $t_0 \le t \le T^*$,

$$(-1/(1+\sigma))\ln[a(1+\sigma)|\Omega|(T^*-t)] \le G(t) \le (-1/(1-\sigma))\ln[a(1-\sigma)|\Omega|(T^*-t)]. \tag{4.39}$$

Letting $\sigma \to 0$, we have

$$\lim_{t\to T^*} G(t)\left|\ln(T^*-t)\right|^{-1} = 1, \tag{4.40}$$

because of $\lim_{t\to T^*} G(t) = \infty$. Due to $\ln u(x, t) \sim G(t)$ uniformly on any compact subset of Ω, the proof is complete.

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Authors' contributions
The main results in this article were derived by GZ and LT. All authors read and approved the final manuscript.

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