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Carleman estimates and unique continuation property for abstract elliptic equations

Veli B Shakhmurov

Correspondence: veli.sahmurov@okan.edu.tr
Department of Electronics Engineering and Communication, Okan University, Akfirat Beldesi, Tuzla, 34959, Istanbul, Turkey

Abstract

The unique continuation theorems for elliptic differential-operator equations with variable coefficients in vector-valued L_p-space are investigated. The operator-valued multiplier theorems, maximal regularity properties and the Carleman estimates for the equations are employed to obtain these results. In applications the unique continuation theorems for quasielliptic partial differential equations and finite or infinite systems of elliptic equations are studied.

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1 Introduction

The aim of this article, is to present a unique continuation result for solutions of a differential inequalities of the form:

$$\left\| P(x,\ D)\, u\,(x) \right\|_E \leq \ \left\| V(x)\, u\,(x) \right\|_E, \tag{1}$$

where

$$P(x; D)u = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + Au + \sum_{k=1}^{n} A_k \frac{\partial u}{\partial x_k},$$

here a_{ij} are real numbers, $A = A\ (x)$, $A_k = A_k\ (x)$ and $V\ (x)$ are the possible linear operators in a Banach space E.

Jerison and Kenig started the theory of L_p Carleman estimates for Laplace operator with potential and proved unique continuation results for elliptic constant coefficient operators in [1]. This result shows that the condition $V \in L_{n/2,loc}$ is in the best possible nature. The uniform Sobolev inequalities and unique continuation results for second-order elliptic equations with constant coefficients studied in [2]. This was latter generalized to elliptic variable coefficient operators by Sogge in [3]. There were further improvement by Wolff [4] for elliptic operators with less regular coefficients and by Koch and Tataru [5] who considered the problem with gradients terms. A comprehensive introductions and historical references to Carleman estimates and unique continuation properties may be found, e.g., in [5]. Moreover, boundary value problems for differential-operator equations (DOEs) have been studied extensively by many researchers (see [6-18] and the references therein).

In this article, the unique continuation theorems for elliptic equations with variable operator coefficients in E-valued L_p spaces are studied. We will prove that if $\frac{1}{p}+\frac{1}{p^{|}}=1\frac{1}{\mu}=\frac{1}{p}-\frac{1}{p'}, \frac{1}{p}+\frac{1}{p^{|}}=1$, $V \in L_\mu(R^n; L(E))$, $p, \mu \in (1, \infty)$ and $u \in W_p^2(R^n; E(A), E)$ satisfies (1), then u is identically zero if it vanishes in a nonempty open subset, where $W_p^2(R^n; E(A), E)$ is an E-valued Sobolev-Lions type space. We prove the Carleman estimates to obtain unique continuation. Specifically, we shall see that it suffices to show that if $w(x) = x_1 + \frac{x_1^2}{2}$, then

$$\left\|e^{tw}u\right\|_{L_{p^{|}}(R^n;E)} \leq C\left\|e^{tw}L(\varepsilon x, D)u\right\|_{L_p(R^n;E)}, \ \frac{1}{p}+\frac{1}{p^{|}}=1,$$

$$\sum_{|\alpha|\leq 1} t^{\left(1+\frac{1}{n}-|\alpha|\right)}\left\|e^{tw}D^\alpha u\right\|_{L_p(R^n;E)}+\left\|e^{tw}Au\right\|_{L_p(R^n;E)} \leq C\left\|e^{tw}L(\varepsilon x, D)u\right\|_{L_p(R^n;E)}.$$

In the Hilbert space $L_2(R^n; H)$, we derive the following Carleman estimate

$$\sum_{|\alpha|\leq 2} t^{\frac{3}{2}-|\alpha|}\left\|e^{tw}D^\alpha u\right\|_{L_2(R^n;H)}+\left\|e^{tw}Au\right\|_{L_2(R^n;H)} \leq C\left\|e^{tw}L_0 u\right\|_{L_2(R^n;H)}.$$

Any of these inequalities would follow from showing that the adjoint operator $L_t(x; D) = e^{tw}L(x; D)e^{-tw}$ satisfies the following relevant local Sobolev inequalities

$$\|u\|_{L_{p^{|}}(R^n;E)} \leq C\left\|L_t u\right\|_{L_p(R^n;E)}, \ \frac{1}{p}+\frac{1}{p^{|}}=1,$$

$$\sum_{|\alpha|\leq 1} t^{\left(1+\frac{1}{n}-|\alpha|\right)}\left\|D^\alpha u\right\|_{L_p(R^n;E)}+\|Au\|_{L_p(R^n;E)} \leq C\|L_t u\|_{L_p(R^n;E)},$$

uniformly to t, where $L_{0t} = e^{tw}L_0e^{-tw}$. In application, putting concrete Banach spaces instead of E and concrete operators instead of A, we obtain different results concerning to Carleman estimates and unique continuation.

2 Notations, definitions, and background

Let **R** and **C** denote the sets of real and complex numbers, respectively. Let

$$S_\varphi = \{\xi \in C, \ |\arg\xi| \leq \varphi\} \cup \{0\}, \ \varphi \in [0, \pi).$$

Let E and E_1 be two Banach spaces, and $L(E, E_1)$ denotes the spaces of all bounded linear operators from E to E_1. For $E_1 = E$ we denote $L(E, E_1)$ by $L(E)$. A linear operator A is said to be a ϕ-positive in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and

$$\left\|(A+\xi I)^{-1}\right\|_{L(E)} \leq M(1+|\xi|)^{-1}$$

with $\lambda \in S_\phi$, $\phi \in (0, \pi]$, I is identity operator in E. We will sometimes use $A + \xi$ or A_ξ instead of $A + \xi I$ for a scalar ξ and $(A + \xi I)^{-1}$ denotes the inverse of the operator $A + \xi I$ or the resolvent of operator A. It is known [19, §1.15.1] that there exist fractional powers A^θ of a positive operator A and

$$E(A^{\theta}) = \{u \in D(A^{\theta}),\ \|u\|_{E(A^{\theta})} = \left\|A^{\theta}u\right\|_{E} + \|u\| < \infty,\ -\infty < \theta < \infty\}.$$

We denote by L_p (Ω; E) the space of all strongly measurable E-valued functions on Ω with the norm

$$\|u\|_{L_p} = \|u\|_{L_p(\Omega;E)} = \left(\int_{\Omega}\|u(x)\|_E^p dx\right)^{1/p},\ 1 \le p < \infty.$$

By $L_{p,q}$ (Ω) and $W^l_{p,q}(\Omega)$ let us denoted, respectively, the (p, q)-integrable function space and Sobolev space with mixed norms, where $1 \le p, q < \infty$, see [20].

Let E_0 and E be two Banach spaces and E_0 is continuously and densely embedded E.

Let l be a positive integer.

We introduce an E-valued function space $W^l_p(\Omega; E_0, E)$ (sometimes we called it Sobolev-Lions type space) that consist of all functions $u \in L_p$ (Ω; E_0) such that the generalized derivatives $D^l_k u = \frac{\partial^l u}{\partial x^l_k} \in L_p(\Omega; E)$ are endowed with the

$$\|u\|_{W^l_p(\Omega;E_0,E)} = \|u\|_{L_p(\Omega;E_0)} + \sum_{k=1}^{n}\left\|D^l_k u\right\|_{L_p(\Omega;E)} < \infty,\ 1 \le p < \infty.$$

The Banach space E is called an *UMD*-space if the Hilbert operator $(Hf)(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}dy$ is bounded in L_p (R, E), $p \in (1, \infty)$ (see e.g., [21,22]). *UMD* spaces include, e.g., L_p, l_p spaces and Lorentz spaces L_{pq}, $p, q \in (1, \infty)$.

Let E_1 and E_2 be two Banach spaces. Let S (R^n; E) denotes a Schwartz class, i.e., the space of all E-valued rapidly decreasing smooth functions on R^n. Let F and F^{-1}denote Fourier and inverse Fourier transformations, respectively. A function $\Psi \in C^m$ (R^n; L (E_1, E_2)) is called a multiplier from L_p (R^n; E_1) to L_q (R^n; E_2) for $p, q \in (1, \infty)$ if the map $u \to Ku = F^{-1}\ \Psi(\xi)\ Fu$, $u \in S$ (R^n; E_1) is well defined and extends to a bounded linear operator

$$K\ :\ L_p(R^n; E_1) \to L_q(R^n; E_2).$$

We denote the set of all multipliers from L_p (R^n; E_1) to L_q (R^n; E_2) by $M^q_p(E_1,\ E_2)$. For $E_1 = E_2 = E$ and $q = p$ we denote $M^q_p(E_1,\ E_2)$ by M_p (E). The L_p-multipliers of the Fourier transformation, and some related references, can be found in [19, § 2.2.1-§ 2.2.4]. On the other hand, Fourier multipliers in vector-valued function spaces, have been studied, e.g., in [23-28].

A set $K \subset L$ (E_1, E_2) is called R-bounded [22,23] if there is a constant C such that for all $T_1, T_2, \ldots, T_m \in K$ and $u_1, u_2, \ldots, u_m \in E_1$, $m \in N$

$$\int_0^1\left\|\sum_{j=1}^{m} r_j(y)T_j u_j\right\|_{E_2} dy \le C\int_0^1\left\|\sum_{j=1}^{m} r_j(y)u_j\right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric {-1, 1}-valued random variables on [0,1]. The smallest C for which the above estimate holds is called a R-bound of the collection K and denoted by R (K).

Let

$$U_n = \{\beta = (\beta_1,\ \beta_2,\ \ldots,\ \beta_n),\ \beta_i \in \{0,1\},\ i = 1,2,\ \ldots,\ n\},$$
$$\xi^\beta = \xi_1^{\beta_1}\xi_2^{\beta_2}\ldots\xi_n^{\beta_n},\ |\xi^\beta| = |\xi_1|^{\beta_1}|\xi_2|^{\beta_2}\ldots|\xi_n|^{\beta_n}.$$

For any $r = (r_1, r_2, \ldots, r_n)$, $r_i \in [0, \infty)$ the function $(i\xi)^r$, $\xi \in R^n$ will be defined such that

$$(i\xi)^r = \begin{cases} (i\xi_1)^{r_1}\ldots(i\xi_n)^{r_n}, & \xi_1, \xi_2,\ \ldots,\ \xi_n \neq 0, \\ 0, & \xi_1,\ \xi_2,\ \ldots,\ \xi_n = 0, \end{cases}$$

where

$$(it)^\nu = |t|^\nu \exp\left(\frac{i\pi}{2}\operatorname{sign} t\right),\ t \in (-\infty,\ \infty),\ \nu \in [0,\ \infty).$$

Definition 2.1. The Banach space E is said to be a space satisfying a multiplier condition with respect to p, $q \in (1, \infty)$ (with respect to p if $q = p$) when for $\Psi \in C^{(n)}(R^n; L(E_1, E_2))$ if the set

$$\left\{\xi^{|\beta|+\frac{1}{p}-\frac{1}{q}} D^\beta \Psi(\xi) : \xi \in R^n\backslash 0,\ \beta \in U_n\right\}$$

is R-bounded, then $\Psi \in M_p^q(E_1,\ E_2)$.

Definition 2.2. The ϕ-positive operator A is said to be a R-positive in a Banach space E if there exists $\phi \in [0, \pi)$ such that the set

$$L_A = \{\xi(A + \xi I)^{-1} : \xi \in S_\varphi\}$$

is R-bounded.

Remark 2.1. By virtue of [29] or [30] UMD spaces satisfy the multiplier condition with respect to $p \in (1, \infty)$.

Note that, in Hilbert spaces every norm bounded set is R-bounded. Therefore, in Hilbert spaces all positive operators are R-positive. If A is a generator of a contraction semigroup on L_q, $1 \leq q \leq \infty$ [31], A has the bounded imaginary powers with $\nu < \frac{\pi}{2}$, $\nu < \frac{\pi}{2}$ or if A is a generator of a semigroup with Gaussian bound in $E \in$ UMD then those operators are R-positive (see e.g., [24]).

It is well known (see e.g., [32]) that any Hilbert space satisfies the multiplier condition with respect to $p \in (1, \infty)$. By virtue of [33] Mikhlin conditions are not sufficient for operator-valued multiplier theorem. There are however, Banach spaces which are not Hilbert spaces but satisfy the multiplier condition (see Remark 2.1).

Let $H_k = \{\Psi_h \in M_p^q(E_1,\ E_2),\ h = (h_1,\ h_2,\ \ldots,\ h_n) \in K\}$ be a collection of multipliers in $M_p^q(E_1,\ E_2)$. We say that H_k is a uniform collection of multipliers if there exists a constant $M > 0$, independent on $h \in K$, such that

$$\left\|F^{-1}\Psi_h F u\right\|_{L_q(R^n;E_2)} \leq M\|u\|_{L_p(R^n;E_1)}$$

for all $h \in K$ and $u \in S(R^n; E_1)$.

We set

$$C_b(\Omega;E) = \left\{u \in C(\Omega;E), \quad \lim_{|x|\to\infty} u(x) \text{ exists}\right\}.$$

In view of [17, Theorem A_0], we have

Theorem 2.0. Let E_1 and E_2 be two UMD spaces and let

$$\Psi \in C^{(n)}(R^n\backslash 0; L(E_1,\ E_2)) \text{ for } p, q \in (1,\ \infty).$$

If

$$R\left\{\xi^{|\beta|+\frac{1}{p}-\frac{1}{q}} D_\xi^\beta \Psi_h(\xi)\ :\ \xi \in R^n\backslash 0,\ \beta \in U_n\right\} \le K_\beta < \infty$$

uniformly with respect to $h \in K$ then $\Psi_h(\xi)$ is a uniformly collection of multipliers from $L_p(R^n; E_1)$ to $L_q(R^n; E_2)$.

Let

$$\chi = \frac{|\alpha| + n\left(\frac{1}{p} - \frac{1}{q}\right)}{l},\ \alpha = (\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n).$$

Embedding theorems in Sobolev-Lions type spaces were studied in [13-18,32,34]. In a similar way as [17, Theorem 3] we have

Theorem 2.1. Suppose the following conditions hold:

(1) E is a Banach space satisfying the multiplier condition with respect to $p, q \in (1, \infty)$ and A is a R-positive operator on E;

(2) l is a positive and α_k are nonnegative integer numbers such that $0 \le \mu \le 1 - \varkappa$, t and h are positive parameters.

Then the embedding

$$D^\alpha W_p^l(R^n; E(A),\ E) \subset L_q(R^n; E(A^{1-\chi-\mu}))$$

is continuous and there exists a positive constant C_μ such that for

$$u \in W_p^l(R^n; E(A),\ E)$$

the uniform estimate holds

$$\left\|D^\alpha u\right\|_{L_q(R^n;E(A^{1-\chi-\mu}))} \le C_\mu\left[h^\mu \|u\|_{W_p^l(R^n;E(A),E)} + h^{-(1-\mu)}\|u\|_{L_p(R^n;E)}\right].$$

Moreover, for $u \in W_p^l(R^n; E(A),\ E)$ the following uniform estimate holds

$$\left\|A^{1-\chi-\mu}u\right\|_{L_p(R^n;E)} \le C_\mu\left[h^\mu \|u\|_{W_p^l(R^n;E(A),E)} + h^{-(1-\mu)}\|u\|_{L_p(R^n;E)}\right].$$

3 Carleman estimates for DOE

Consider at first the equation with constant coefficients

$$L_0 u = \sum_{k=1}^n D_k^2 u + Au = f(x), \tag{2}$$

where $D_k = \frac{\partial}{i\partial_k}$ and A is the possible unbounded operator in a Banach space E.

Let $w(x) = x_1 + \frac{x_1^2}{2}$ and t is a positive parameter.

Remark 3.1. It is clear to see that

$$\begin{aligned} e^{tw}L_0[e^{-tw}u] = L_{0t}(x,\ D)u = e^{tw}\left(\sum_{k=1}^{n} D_k^2(e^{-tw}u) + e^{-tw}Au\right) \\ = \sum_{k=1}^{n} D_k^2 u + Au + 2tw_1\frac{\partial u}{\partial x_1} + [-t^2 w_1^2 + t]u, \end{aligned} \tag{3}$$

where $w_1 = \frac{\partial w}{\partial x_1}$. Let L_{0t} (x, ξ) is the principal operator symbol of L_{0t} (x, D) on the domain B_0, i.e.,

$$L_{0t}(x,\ \xi) = \xi_1^2 - 2i\xi_1 w_1 t + A + |\xi^{|}|^2 - t^2 w_1^2 = G_t(x,\ \xi)B_t(x,\ \xi),$$

where

$$G_t(x,\ \xi) = \xi_1 - i\left[\left(A + |\xi^{|}|^2\right)^{\frac{1}{2}} + tw_1\right],$$

$$B_t(x,\ \xi) = \xi_1 + i\left[\left(A + |\xi^{|}|^2\right)^{\frac{1}{2}} - tw_1\right],\ |\xi^{|}|^2 = \sum_{k=2}^{n} \xi_k^2.$$

Our main aim is to show the following result:

Remark 3.2. Since $Q(\xi) \in S(\phi)$ for all $\phi \in [0, \pi)$ due to positivity of A, the operator function $A + |\xi^{|}|^2$, $\xi \in R^n$ is uniformly positive in E. So there are fractional powers of $A+|\xi^{|}|^2$ and the operator function $\left(A + |\xi^{|}|^2\right)^{\frac{1}{2}}$ is positive in E (see e.g., [19, §1. 15.1]).

First, we will prove the following result.

Theorem 3.1. Suppose A is a positive operator in a Hilbert space H. Then the following uniform Sobolev type estimate holds for the solution of Equation (3)

$$\sum_{|\alpha|\le 2} t^{\frac{3}{2}-|\alpha|} \left\|e^{tw}D^\alpha u\right\|_{L_2(R^n;H)} + \left\|e^{tw}Au\right\|_{L_2(R^n;H)} \le C\left\|e^{tw}L_0 u\right\|_{L_2(R^n;H)}. \tag{4}$$

By virtue of Remark 3.1 it suffices to prove the following uniform coercive estimate

$$\sum_{|\alpha|\le 2} t^{\frac{3}{2}-|\alpha|} \left\|D^\alpha u\right\|_{L_2(R^n;H)} + \left\|Au\right\|_{L_2(R^n;H)} \le C\left\|L_{0t}u\right\|_{L_2(R^n;H)} \tag{5}$$

for $u \in W_2^2(R^n; H(A),\ E)$.

To prove the Theorem 3.1, we shall show that L_{0t} (x, D) has a right parametrix T, with the following properties.

Lemma 3.1. For $t > 0$ there are functions $K = K_t$ and $R = R_t$ so that

$$L_{0t}(x,\ D)K(x,\ y) = \delta(x - y) + R(x,\ y),\ x, y \in B_0, \tag{6}$$

where δ denotes the Dirac distribution. Moreover, if we let $T = T_t$ be the operator with kernel K, i.e.,

$$Tf(x) = \int_{B_0} K(x,\ y)f(y)dy,\ f \in C_0^\infty(B_0; E),$$

and R is the operator with kernel $R(x, y)$, then for large $t>0$, the adjoint of these operators satisfy the following estimates

$$\sum_{|\alpha|\leq 2} t^{2-|\alpha|} \left\| D^{\alpha} T^{*} f \right\|_{L_2(B_0;H)} \leq C \left\| f \right\|_{L_2(B_0;H)}, \left\| A T^{*} f \right\|_{L_2(B_0;H)} \leq C \left\| f \right\|_{L_2(B_0;H)}, \quad (7)$$

$$t^{\frac{1}{2}} \left\| R^{*} f \right\|_{L_2(B_0;H)} \leq C \left\| f \right\|_{L_2(B_0;H)}, \quad (8)$$

$$t^{-\frac{1}{2}} \left\| D^{\nu} R^{*} f \right\|_{L_2(B_0;H)} \leq C \sum_{|\alpha|\leq|\nu|-1} \left\| D^{\alpha} f \right\|_{L_2(B_0;H)}, 1 \leq |\nu| \leq 2. \quad (9)$$

Proof. By Remark 3.2 the operator function $\left(A + |\xi^{|}|^2\right)^{\frac{1}{2}}$ is positive in E for all $\xi \in R^n$. Since $tw_1 + i\xi_1 \in S(\phi)$, due to positivity of A, for $\varphi \in \left[\frac{\pi}{2}, \pi\right)$ the factor $G_t(x, \xi) = -i\left[\left(A + |\xi^{|}|^2\right)^{\frac{1}{2}} + w_1 t + i\xi_1\right]$ has a bounded inverse $G_t^{-1}(x, \xi)$for all $\xi \in R^n$, $t > 0$ and

$$\left\| G_t^{-1}(x, \xi) \right\|_{B(H)} \leq C\left(1 + |tw_1 + i\xi_1|\right)^{-1}. \quad (10)$$

Therefore, we call $G_t(x, \xi)$ the regular factor. Consider now the second factor

$$B_t(x, \xi) = i\left[\left(A + |\xi^{|}|^2\right)^{\frac{1}{2}} - (w_1 t + i\xi_1)\right].$$

By virtue of operator calculus and fractional powers of positive operators (see e.g., [19, §1.15.1] or [35]) we get that - $[tw_1 + i\xi_1] \notin S(\phi)$ for $\xi_1 = 0$ and $tw_1 = |\xi^{|}|$, i.e., the operator $B_t(x, \xi)$ does not has an inverse, in the following set

$$\Delta_t = \{(x, \xi) \in B_0 \times R^n : \xi_1 = 0, \ |\xi^{|}| = tw_1\}.$$

So we will called B_t the singular factor and the set Δ_t call singular set for the operator function B_t. The operator B_t^{-1} cannot be bounded in the set Δ_t. Nevertheless, the operator B_t^{-1}, and hence L_{0t}^{-1}, can be bounded when (x, ξ) is sufficiently far from Δ_t. For instance, if we define

$$\Gamma_t = \left\{(x, \xi) \in B_0 \times R^n : |\xi^{|}| \in \left[\frac{t}{4}, 4t\right], |\xi_1| \leq \frac{t}{4}\right\},$$

by properties of positive operators we will get the same estimate of type (10) for the singular factor B_t. Hence, using this fact and the resolvent properties of positive operators we obtain the following estimate

$$\left\| L_{0t}^{-1}(x, \xi) \right\|_{B(E)} \leq C\left(1 + |\xi|^2 + t^2\right)^{-1} \text{when } (x, \xi) \in {}^{c}\Gamma_t, \quad (11)$$

where the constant C is independent of x, ξ, t and ${}^{c}\Gamma_t$ denotes the complement of Γ_t.

Let $\beta \in C_0^{\infty}(R)$ such that, $\beta(\xi) = 0$ if $|\xi| \in \left[\frac{1}{4}, 4\right]$ and $\beta(\xi) = 0$ near the origin. We then define

$$\beta_0(\xi) = \beta_{0t}(\xi)\beta_0(\xi) = 1 - \beta(|\xi^{|}|/t)\beta(1 - \xi_1/t)$$

and notice that $\beta_0\ (\xi) = 0$ on Γ_t. Hence, if we define

$$K_0(x,\ y) = (2\pi)^{-n} \int_{R^n} \beta_0(\xi) e^{i((x-y),\xi)} L_{0t}^{-1}(y,\ \xi) d\xi \tag{12}$$

and recall (11), then by [31] it follows from standard microlocal arguments that

$$L_{0t}(x,\ D) K_0(x,\ y) = (2\pi)^{-n} \int_{R^n} \beta_0(\xi) e^{i((x-y),\xi)} d\xi + R_{0t}(x,\ y),$$

where R_{0t} belongs to a bounded subset of S^{-1} which is independent of t. Since operator R_{0t}^* also has the same property, it follows that for all $f \in C_0^\infty(B_0; H)$

$$\left\| D^\nu R_{0t}^* f \right\|_{L_2(B_0;H)} \leq C \sum_{|\alpha| \leq |\nu|-1} \left\| D^\alpha f \right\|_{L_2(B_0;H)},\ \ 1 \leq\ |\nu|\ \leq 2.$$

By reasoning as in [31] we get that tR_{0t} belongs to a bounded subset of S^0. So, we have the following estimate

$$t \left\| D^\nu R_{0t}^* f \right\|_{L_2(B_0;H)} \leq C \left\| f \right\|_{L_2(B_0;H)}.$$

Moreover, the Remark 3.2, positivity properties of A and, (11) and (12) imply that, the operator functions $\sum_{|\alpha| \leq 2} \beta_0(\xi) t^{2-|\alpha|} \xi^\alpha L_{0t}^{-1}(x,\ \xi)$ and $\beta_0(\xi) A L_{0t}^{-1}(x,\ \xi)$ are uniformly bounded. Then, if we let T_0 be the operator with kernel $K_0\ (x, y)$, by using the Minkowski integral inequality and Plancherel's theorem we obtain

$$\sum_{|\alpha| \leq 2} t^{2-|\alpha|} \left\| D^\alpha T_0 f \right\|_{L_2(B_0;H)} \leq C \left\| f \right\|_{L_2(B_0;H)},\ \ \left\| A T_0 f \right\|_{L_2(B_0;H)} \leq C \left\| f \right\|_{L_2(B_0;H)}.$$

For inverting $L_{0t}\ (x, D)$ on the set Γ_t we will require the use of Fourier integrals with complex phase. Let $\beta_1\ (\xi) = 1 - \beta_0\ (\xi)$. We will construct a Fourier integral operator T_1 with kernel

$$K_1(x,\ y) = (2\pi)^{-n} \int_{R^n} \beta_1(\xi) e^{i\Phi(x,y,\xi)} L_{0t}^{-1}(y,\ \xi)\, d\xi \tag{13}$$

so that the analogs of (16) and the estimates (7)-(9) are satisfied. Since $G_t^{-1}(x,\ \xi)$ is uniformly bounded on Γ_t, we should expect to construct the phase function Φ in (13) using the factor $B_t\ (x, \xi)$. Specifically, we would like Φ to satisfy the following equation

$$B_t(x,\ \Phi_x) = B_t(y,\ \xi), y \in B_0, (x,\ \xi \in \Gamma_t). \tag{14}$$

The Equation (14) leads to complex eikonal equation (i.e., a non-linear partial differential equation with complex coefficients).

$$\begin{aligned}(A + |\Phi_{x^{|}}(x,\ y,\ \xi)|^2)^{\frac{1}{2}} - [w_1(x)t + i\Phi_{x_1}(x,\ y,\ \xi)] = \\ \left(A + |\xi^{|}|^2\right)^{\frac{1}{2}} - (w_1(y)t + i\xi_1).\end{aligned} \tag{15}$$

Since $w_1\ (x) = 1 + x_1$, $w_1\ (y) = 1 + y_1$, we have

$$\Phi = (x - y,\ \xi) + \frac{(x_1 - y_1)^2 \xi_1}{2(1 + y_1)} + \frac{i(x_1 - y_1)^2 |\xi^{|}|}{2(1 + y_1)} \tag{16}$$

is a solution of (15). To use this we get

$$L_{0t}(x,\ D)\, e^{i\Phi(x,y,\xi)} = e^{i\Phi} L_{0t}(x,\ \Phi_x) + e^{i\Phi}\ \frac{\partial^2 \Phi}{\partial x_1^2}.$$

Next, if we set

$$r(x,\ y,\ \xi) = G_t(y,\ \xi) - G_t(x,\ \xi) = -i\,[w_1(y) - w_1(x)]t \tag{17}$$

then it follows from $L_{0t}\ (x, \xi) = G_t\ (x, \xi) B_t\ (x, \xi)$ and (14) that

$$L_{0t}(x,\ \Phi_x) = L_{0t}(y,\ \xi) + B_t(y,\ \xi) r(x,\ y,\ \xi). \tag{18}$$

Consequently, (16)-(18) imply that

$$\begin{aligned}(2\pi)^n L_{0t}(x,\ D) K_1(x,\ y) = \int_{R^n} \beta_1(\xi) e^{i\Phi} d\xi + \int_{R^n} \beta_1(\xi) r(x,\ y,\ \xi) G_t^{-1}(y,\ \xi) e^{i\Phi} d\xi \\ \int_{R^n} \beta_1(\xi) A L_{0t}^{-1}(y,\ \xi) e^{i\Phi} d\xi + \int_{R^n} \beta_1(\xi) \frac{\partial^2 \Phi}{\partial x_1^2} L_{0t}^{-1}(y,\ \xi) e^{i\Phi} d\xi.\end{aligned} \tag{19}$$

By reasoning as in [3] we obtain that the first and second summands in (19) belong to a bounded subset of S^0. So, we see that the equality (5) must hold. Now we let $K\ (x, y) = K_0\ (x, y) + K_1\ (x, y)$ and $R\ (x, y) = R_0\ (x, y) + R_1\ (x, y)$, where

$$R_1(x,\ y) = R_{10}(x,\ y) + R_{11}(x,\ y), R_{10}(x,\ y) = \int_{R^n} \beta_1(\xi) r(x,\ y,\ \xi) G_t^{-1}(y,\ \xi) e^{i\Phi} d\xi,$$

$$R_{11}(x,\ y) = \int_{R^n} \beta_1(\xi) \frac{\partial^2 \Phi}{\partial x_1^2} L_{0t}^{-1}(y,\ \xi) e^{i\Phi} d\xi,\ T_0 f(x) =$$

$$\int_{B_0} K_0(x,\ y) f(y) dy,\ T_1 f(x) = \int_{B_0} K_1(x,\ y) f(y) dy.$$

Due to regularity of kernels, by using of Minkowski and Hölder inequalities we get the analog estimate as (7) and (9) for the operators T_0 and R_{10}. Thus, in order to finish the proof, it suffices to show that for $f \in L_2\ (B_0;\ E)$ one has

$$\sum_{|\alpha| \le 2} t^{2-|\alpha|} \left\| D^\alpha T_1^* f \right\|_{L_2(B_0;H)} + \left\| A T_1^* f \right\|_{L_2(B_0;H)} \le C \left\| f \right\|_{L_2(B_0;H)}, \tag{20}$$

$$t^{\frac{1}{2}} \left\| R_{11}^* f \right\|_{L_2(B_0;H)} \le C \left\| f \right\|_{L_2(B_0;H)}, \tag{21}$$

$$t^{-\frac{1}{2}} \| D^\nu R_{11}^* f \|_{L_2(B_0;H)} \le C \sum_{|\alpha| \le |\nu| - 1} \left\| D^\alpha f \right\|_{L_2(B_0;H)},\ 1 \le |\nu| \le 2. \tag{22}$$

However, since $R_{1,1} \approx t T_1$, we need only to show the following

$$t^{3/2} \left\| T_1^* f \right\|_{L_2(B_0;H)} \le C \left\| f \right\|_{L_2(B_0;H)}. \tag{23}$$

By using the Minkowski inequalities we get

$$\left\| T_1^* f \right\|_{L_2(R^{n-1};E)} \le \int_{-\frac{1}{4}}^{\frac{1}{4}} \left\| \int_{B_0} K_1^*(x,\ y) f(y) dy' \right\| dy_1,$$

where $K_1^*(x, y) = \bar{K}_1(y, x)$. The estimates (13) and (16) imply that

$$K_1^*(x,y) = (2\pi)^{-n} \int_{R^{n-1}} e^{i(x'-y')} m(x_1, y_1, \xi^|) d\xi^|,$$

where

$$m(x_1, y_1, \xi^|) = \int_{-\infty}^{\infty} \beta_1(\xi) e^{i\left[(x_1-y_1)\xi_1+(x_1-y_1)^2(i|\xi^||-\xi_1)/2(1+x_1)\right]} L_{0t}^{-1}(y, \xi) d\xi_1.$$

Consequently, it follows from Plancherel's theorem that

$$\left\| \int_{R^{n-1}} K_1^*(x, y) f(y) dy^| \right\| \leq \sup_{\xi^|} |m(x_1, y_1, \xi^|)| \left(\int_{R^{n-1}} |f(y)|^2 dy^| \right)^{\frac{1}{2}}. \tag{24}$$

Note that for every N we have

$$e^{i[(x_1-y_1)^2|\xi^||/2(1+x_1)]} \leq C_N[1 + t(x_1 - y_1)^2]^{-N} \text{on supp } \beta_1.$$

Since A is a positive operator in E, we have

$$\left\|L_{0t}^{-1}(x, \xi)\right\|_{B(E)} \leq 1 + |-2i\xi_1 w_1 t + |\xi|^2 - t^2 w_1^2|^{-1}$$

when $-2i\xi_1 w_1 t + A + |\xi|^2 - t^2 w_1^2 \in S(\varphi)$. Then by using the above estimate it not easy to check that

$$\int_{-\infty}^{\infty} \beta_1(\xi) e^{i\xi_1\left[(x_1-y_1)-(x_1-y_1)^2/2(1+x_1)\right]} L_{0t}^{-1}(y, \xi) d\xi_1 = O(t^{-1}),$$

i.e.,

$$|m(x_1, y_1, \xi^|)| \leq Ct^{-1}[1 + t(x_1 - y_1)^2]^{-1}.$$

Moreover, it is clear that

$$\int_{-\infty}^{\infty} (1 + tx_1)^{-1} dx_1 = O\left(t^{-\frac{1}{2}}\right).$$

Thus from (24) by using the above relations and Young's inequality we obtain the desired estimate

$$\begin{aligned}\left\|T_1^* f\right\|_{L_2(B_0;H)} &\leq Ct^{-1} \int \left| \int \left[1 + t(x_1 - y_1)^2\right] \left\|f(y_1, \cdot)\right\|_{L_2} dy_1 \right| dx_1 \\ &\leq Ct^{-3/2} \left\|f\right\|_{L_2(R^n;H)}.\end{aligned}$$

Moreover, by using the estimate (10) and the resolvent properties of the positive operator A we have

$$\left\|AT_1^* f\right\|_{L_2(B_0;H)} \leq C\left\|f\right\|_{L_2(B_0;H)}.$$

The last two estimates then, imply the estimates (20)-(22).

Proof of Theorem 3.1: The estimates (7)-(9) imply the estimate (5), i.e., we obtain the assertion of the Theorem 3.1.

4 L_p-Carleman estimates and unique continuation for equation with variable coefficients

Consider the following DOE

$$L(x,\ D)u = \sum_{i,j=1}^{n} a_{ij}(x)D_{ij}^2 u + Au = f(x),\ x \in R^n, \tag{25}$$

where $D_k = \frac{\partial}{i\partial_k}$ and A is the possible unbounded operator in a Banach space E and a_{ij} are

real-valued smooth functions in $B_\varepsilon = \{x \in\ R^n,\ |x| < \varepsilon\}$.

Condition 4.1. There is a positive constant γ such that $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2$ for all $\xi \in\ R^n$, $x \in\ B_0 = \{x \in R^n, |x| < \frac{1}{4}\}$.

The main result of the section is the following

Theorem 4.1. Let E be a Banach space satisfies the multiplier condition and A be a R-positive operator in E. Suppose the Condition 4.1 holds, $n \geq 3$, $p = \frac{2n}{n+2}$and p' is the conjugate of p, $w\ = x_1 + \frac{x_1^2}{2}$ and $a_{ij} \in\ C^\infty\ (B_\varepsilon)$. Then for $u \in C_0^\infty(\ B_\varepsilon;\ E(A))$ and $\in> 0, \frac{1}{t} < \frac{1}{2}$ the following estimates are satisfied:

$$\left\|e^{tw}u\right\|_{L_{p^l}(R^n;E)} \leq C\left\|e^{tw}L(\varepsilon x,\ D)u\right\|_{L_p(R^n;E)},\ \frac{1}{p} + \frac{1}{p^l} = 1, \tag{26}$$

$$\sum_{|\alpha|\leq 1} t^{\left(1+\frac{1}{n}-|\alpha|\right)}\left\|e^{tw}D^\alpha u\right\|_{L_p(R^n;E)} + \left\|e^{tw}Au\right\|_{L_p(R^n;E)} \leq \tag{27}$$

$$C\left\|e^{tw}L(\varepsilon x,\ D)u\right\|_{L_p(R^n;E)}.$$

Proof. As in the proof of Theorem 3.1, it is sufficient to prove the following estimates

$$\|v\|_{L_{p^l}(R^n;E)} \leq C\|L_t\,(\varepsilon x,\ D)\ v\,\|_{L_p(R^n;E)},\ \frac{1}{p} + \frac{1}{p^l} = 1, \tag{28}$$

$$\sum_{|\alpha|\leq 1} t^{(1+\frac{1}{n}-|\alpha|)}\left\|D^\alpha v\right\|_{L_p(R^n;E)} + \|Av\|_{L_p(R^n;E)} \leq C\left\|L_t(\varepsilon x,\ D)v\right\|_{L_p(R^n;E)} \tag{29}$$

where,

$$L_t\,(\varepsilon x, D) = e^{tw}L\,(\varepsilon x,\ D)\,e^{-tw} = L\,(\varepsilon x,\ D) + 2tw_1\frac{\partial}{\partial x_1} - (tw_1)^2 - t^2,\ w_1 = \frac{\partial w}{\partial x_1}.$$

Consequently, since $w_1 \simeq 1$ on B_ε, it follows that, if we let $Q_t\ (\varepsilon x,\ D)$ be the differential operator whose adjoint equals

$$Q_t^*(\varepsilon x,\ D) = w_1^{-2}L(\varepsilon x,\ D) + 2tw_1^{-1}\frac{\partial}{\partial x_1} - t^2,$$

then it suffices to prove the following

$$\|v\|_{L_{p^|}(R^n;E)} \leq C\left\|Q_t(\varepsilon x,\ D)v\right\|_{L_p(R^n;E)},\ \frac{1}{p}+\frac{1}{p^|}=1,$$
$$\sum_{|\alpha|} t^{(1+\frac{1}{n}-|\alpha|)}||D^\alpha v||_{L_p(R^n;E)} + ||Av||_{L_p(R^n;E)} \leq C||Q_t(\varepsilon x, D)v||_{L_p}(R^n;E), \tag{30}$$
$$v\in C_0^\infty(B_\varepsilon;E(A)).$$

The desired estimates will follow if we could constrict a right operator-valued parametrix T, for $Q_t{}^*$ $(\varepsilon x, D)$ satisfying L_p estimates. these are contained in the following lemma.

Lemma 4.1. For t >0 there are functions $K = K_t$ and $R = R_t$, so that

$$Q_t^*\left(\varepsilon x,\ D\right)K\left(x,\ y\right) = \delta\left(x-y\right) + R\left(x,\ y\right), x, y\in B_\varepsilon, \tag{31}$$

where δ denotes the Dirac distribution. Moreover, if we let $T = T_t$ be the operator with kernel K (x, y) and R be the operator with kernel R (x, y), then if ε and $\frac{1}{t}$ are sufficiently small, the adjoint of these operators satisfy the following uniform estimates

$$\left\|T^*f\right\|_{L_{p^1}(R^n;E)} \leq C\left\|f\right\|_{L_p(R^n;E)},\ \frac{1}{p}+\frac{1}{p^|}=1, \tag{32}$$

$$\sum_{|\alpha|\leq 1} t^{(1+\frac{1}{n}-|\alpha|)}\left\|D^\alpha T^*f\right\|_{L_p(R^n;E)} \leq C\left\|f\right\|_{L_p(R^n;E)}, \tag{33}$$

$$\left\|AT^*f\right\|_{L_p(R^n;E)} \leq C\left\|f\right\|_{L_p(R^n;E)},$$

$$t^{\frac{1}{n}}\left\|R^*f\right\|_{L_q(R^n;\,E)} \leq C\left\|f\right\|_{L_q(R^n;\,E)}, q = p, p^|, \tag{34}$$

$$t^{-1+\frac{1}{n}}\left\|\nabla R^*f\right\|_{L_p(R^n;\,E)} \leq C\left\|f\right\|_{L_p(R^n;\,E)},\ f\in C_0^\infty\left(B_\varepsilon;E\right). \tag{35}$$

Proof. The key step in the proof is to find a factorization of the operator-valued symbol Q_t^* $(\varepsilon x, \xi)$ that will allow to microlocally invert Q_t^* $(\varepsilon x, D)$ near the set where Q_t^* $(\varepsilon x, \xi)$ vanishes. Note that, after making a suitable choice of coordinates, it is enough to show that if L (x, D) is of the form

$$L\left(x, D\right) = D_1^2 + \sum_{i,j=2}^{n} a_{ij}D_iD_j, D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$$

therefore, we can expressed Q_t^* $(\varepsilon x, \xi)$ as

$$Q_t^*\left(\varepsilon x, \xi\right) = B_t\left(\varepsilon x, \xi\right)G_t\left(\varepsilon x, \xi\right), \tag{36}$$

where

$$B_t(x,\xi) = w_1^{-1}\xi_1 + i\left[\left(A + w_1^{-1}a\left(\varepsilon x,\xi^{|}\right)\right) - t\right],$$
$$G_t(x,\xi) = w_1^{-1}\xi_1 - i\left[\left(A + w_1^{-1}a\left(\varepsilon x,\xi^{|}\right)\right) + t\right],$$

where

$$a\left(x,\xi^{|}\right) = \sum_{i,j=2}^{n} a_{ij}(x)\,\xi_i\xi_j.$$

The ellipticity of $Q(x, D)$ and the positivity of the operator A, implies that the factor $G_t(x,\xi)$ never vanishes and as in the proof of Theorem 3.1 we get that

$$\left\| G_t^{-1}(\varepsilon x,\xi)\right\|_{B(H)} \le C\left(1 + |w_1^{-1}a\left(\varepsilon x,\xi^{|}\right)|^{\frac{1}{2}} + |t + w_1^{-1}\xi_1|\right)^{-1}, \tag{37}$$

$$x \in B_\varepsilon,\ \ \xi \in R^n,$$

i.e., the operator function $G_t(\varepsilon x,\xi)$ has uniformly bounded inverse for $(x,\xi) \in B_\varepsilon \times R^n$. One can only investigate the factor $B_t(\varepsilon x,\xi)$. In fact, if we let

$$\Delta_t = \left\{(x,\xi) \in B_\varepsilon \times R^n : \xi_1 = 0,\ |\xi^{|}| = tw_1\right\},$$

then the operator function $B_t(x,\xi)$ is not invertible for $(x,\xi) \in \Delta_t$. Nonetheless, $B_t(\varepsilon x,\xi)$ and $Q_t^*(\varepsilon x,\xi)$ can be have a bounded inverse when (x,ξ) is sufficiently far away. For instance, if we define

$$\Gamma_t = \left\{(x,\xi) \in B_\varepsilon \times R^n : |\xi^{|}| \in \left[\frac{t}{4}, 4t\right],\ |\xi_1| \le \frac{t}{4}\right\},$$

by properties of positive operators we will get the same estimate of type (37) for the singular factor B_t. Hence, we using this fact and the resolvent properties of positive operators we obtain the following estimate

$$\left\|\left(Q_t^*\right)^{-1}(\varepsilon x,\xi)\right\|_{B(E)} \le C\left(1 + |\xi^{|}| + |t + w_1^{-1}\xi_1|\right)^{-1} \text{when } (x,\xi) \in {}^c\Gamma_t. \tag{38}$$

As in § 3, we can use (38) to microlocallity invert $Q_t^*(\varepsilon x, D)$ away from Γ_t. To do this, we first fix $\beta \in C_0^\infty(R)$ as in § 3. We then define

$$\beta_0 = \beta_{0t} = 1 - \beta\left(\left|\xi^{|}\right|/t\right)\beta\left(1 - \xi_1/t\right).$$

It is clear that $\beta_0(\xi) = 0$ on Γ_t. Consequently, if we define

$$K_0(x,y) = (2\pi)^{-n}\int_{R^n} \beta_0(\xi)\, e^{i((x-y),\xi)}\left(Q_t^*\right)^{-1}(\varepsilon y,\xi)\, d\xi \tag{39}$$

and recall (37), then we can conclude that standard microlocal arguments give that

$$Q_t^*(\varepsilon x, D)\,K_0(x,y) = (2\pi)^{-n}\int_{R^n}\beta_0(\xi)\, e^{i((x-y),\xi)}d\xi + R_0(x,y), \tag{40}$$

where R_0 belongs to a bounded subset of S^{-1} that independent of t. Since the adjoint operator R_0^* also is abstract pseudodifferential operator with this property, by reasoning as in [31, Theorem 6] it follows that

$$\left\|\nabla R_0^* f\right\|_{L_p(R^n;E)} \leq C\|f\|_{L_p(R^n;E)}, f \in C_0^\infty\left(B_\varepsilon; E\right), \tag{41}$$

$$t\left\|R_0^* f\right\|_{L_q(R^n;E)} \leq C\|f\|_{Lq(R^n;E)}, f \in C_0^\infty\left(B_\varepsilon; E\right), \tag{42}$$

$$q = p, p', \frac{1}{p} + \frac{1}{p'} = 1.$$

Moreover, the positivity properties of A and the estimate (38) imply that the operator functions $\sum\limits_{|\alpha| \leq 2} \beta_0\left(\xi\right) t^{2-|\alpha|} \xi^\alpha \left(Q_t^*\right)^{-1}\left(\varepsilon x, \xi\right)$ and $\beta_0\left(\xi\right) A\left(Q_t^*\right)^{-1}\left(\varepsilon x, \xi\right)$ are uniformly bounded. Next, let T_0 be the operator with kernel K_0. Then in a similar way as in [31] we obtain that

$$\sum_{|\alpha|\leq 1} t^{(2-|\alpha|)}\left\|D^\alpha T_0^* f\right\|_{L_p(R^n;E)} \leq C\|f\|_{L_p(R^n;E)}, \tag{43}$$

$$\left\|A T_0^* f\right\|_{L_p(R^n;E)} \leq C\|f\|_{L_p(R^n;E)}$$

which also the first estimate is stronger than the corresponding inequality in Lemma 4.1. Finally, since $T_0 \in S^{-2}$ and $\frac{1}{p} - \frac{1}{p^|} = \frac{2}{n}$ it follows from imbedding theorem in abstract Sobolev spaces [17] that

$$\left\|T_0^* f\right\|_{L_{p^|}(R^n;E)} \leq C\|f\|_{L_p(R^n;E)}, f \in C_0^\infty\left(B_\varepsilon; E\right). \tag{44}$$

Thus, we have shown that the microlocal inverse corresponding to ${}^c\Gamma_t$, satisfies the desired estimates.

Let $\beta_1(\xi) = 1\text{-}\beta_0(\xi)$. To invert $Q_t^*(\varepsilon x, D)$ for $(x, \xi) \in \Gamma_t$, we have to construct a Fourier integral operator T_1, with kernel

$$K_1\left(x, y\right) = (2\pi)^{-n} \int_{R^n} \beta_1\left(\xi\right) e^{i\Phi(x,y,\xi)} Q_{0t}^{*^{-1}}\left(\varepsilon y, \xi\right) d\xi, \tag{45}$$

such that the analogs of (39) and (32)-(35) are satisfied. For this step the factorization (36) of the symbol $Q_t^*\left(\varepsilon y, \xi\right)$ will be used. Since the factor $G_t(\varepsilon x, \xi)$ has a bounded inverse for $(x, \xi) \in \Gamma_t$, the previous discussions show that we should try to construct the phase function in (46) using the factor $B_t(\varepsilon x, \xi)$. We would like $\Phi(x, y, \xi)$ to solve the complex eikonal equation

$$B_t\left(\varepsilon x, \Phi_x\right) = B_t\left(\varepsilon y, \xi\right), \; x, y \in B_\varepsilon, \xi \in \operatorname{supp} \beta_1, \tag{46}$$

Since $B_t(\varepsilon x, \Phi_x)$ - $B_t(\varepsilon y, \xi)$ is a scalar function (it does not depend of operator A), by reasoning as in [3, Lemma 3.4] we get that

$$\Phi(x, y, \xi) = \phi(x', y, \xi') + \psi\left(x, y, \xi\right),$$

where φ is real and defined as

$$\phi\left(x', y, \xi'\right) = \left(x_1 - y_1\right)\xi_1 + O\left(\left|x' - y'\right|^2 \left|\xi'\right|\right),$$

while

$$\psi\left(x,y,\xi\right)=\left(x_1-y_1\right)\xi_1+O\left(\left|x_1-y_1\right|^2\left|\xi'\right|\right)$$

and

$$\operatorname{Im}\,\psi\left(x,y,\xi\right)\geq c\left(x_1-y_1\right)^2\left|\xi'\right|,\ c>0. \tag{47}$$

Then we obtain from the above that

$$Q_t^*\left(\varepsilon x,D\right)e^{i\Phi(x,y,\xi)}=e^{i\Phi}Q_t^*\left(\varepsilon x,\Phi_x\right)+e^{i\Phi}w_1^{-2}L\left(\varepsilon x,D\right)\Phi.$$

Next, if we set

$$\begin{aligned}&r\left(x,\,y,\,\xi\right)\;=\;G_t\;\left(\varepsilon y,\,\xi\right)\;-\;G_t\;\left(\varepsilon x,\,\xi\right)\;=\;w_1^{-1}\;\left(y\right)\;\left[\xi_1\;-\;ia\;\left(\varepsilon y,\,\xi'\right)\right]\\&-w_1^{-1}\left(x\right)\left[\xi_1-ia\left(\varepsilon x,\xi'\right)\right]\end{aligned} \tag{48}$$

then it follows from (36) and (48) that

$$e^{i\Phi}Q_t^*\left(\varepsilon x,\Phi_x\right)=e^{i\Phi}Q_t^*\left(\varepsilon y,\xi\right)+e^{i\Phi}B_t\left(\varepsilon y,\xi\right)r\left(x,y,\xi\right)+O\left(t^{-N}\right) \tag{49}$$

for every N when $\beta_1\ (\xi)\neq 0$. Consequently, (49), (50) imply that

$$\begin{aligned}&(2\pi)^nQ_t^*\left(\varepsilon x,D\right)K_1\left(x,y\right)=\int\beta_1\left(\xi\right)e^{i\Phi}d\xi+\int\beta_1\left(\xi\right)r\left(x,y,\xi\right)G_t^{-1}\left(\varepsilon y,\xi\right)e^{i\Phi}d\xi\\&w_1^{-2}\int\beta_1(\xi)Q_t^{*-1}(\varepsilon y,\xi)\left(L(\varepsilon x,D)\Phi\right)e^{i\Phi}d\xi+O(t^{-N}).\end{aligned} \tag{50}$$

By reasoning as in Theorem 3.1 we obtain from (51) that

$$Q_t^*(\varepsilon x,D)K_1(x,y)=(2\pi)^{-n}\int\beta_1(\xi)e^{i(x-y,\xi)}d\xi+R_{10}(x,y)+R_{11}(x,y),$$

where

$$R_{11}(x,y)=(2\pi)^{-n}w_1^{-2}\int\beta_1(\xi)Q_t^{*-1}(\varepsilon y,\xi)\left(L(\varepsilon x,D)\Phi\right)e^{i\Phi}d\xi \tag{51}$$

while R_{10} belongs to a bounded subset of S^{-1} and tR_{10} belongs to a bounded subset of S^0. In view of this formula, we see that if we let $K\ (x,\ y)=K_0\ (x,\ y)+K_1\ (x,\ y)$ and $R\ (x,\ y)=R_0\ (x,\ y)+R_1\ (x,\ y)$, where $R_1=R_{10}+R_{11}$, then we obtain (31). Moreover, since R_{10} satisfies the desired estimates, we see from Minkowski inequality that, in order to finish the proof of Lemma 4.1, it suffices to show that for $f\in C_0^\infty(B_\varepsilon;E)$

$$\left\|T_1^*f\right\|_{L_{p^|}(R^n;E)}\leq C\left\|f\right\|_{L_p(R^n;E)}, \tag{52}$$

$$\sum_{|\alpha|\leq 1}t^{\left(1+\frac{1}{n}-|\alpha|\right)}\left\|D^\alpha T_1^*f\right\|_{L_p(R^n;E)}\leq C\left\|f\right\|_{L_p(R^n;E)}, \tag{53}$$

$$t^{\frac{1}{n}}\left\|R_{11}^*f\right\|_{L_q(R^n;E)}\leq C\left\|f\right\|_{L_q(R^n;E)},\ q=p,p^|, \tag{54}$$

$$t^{-1+\frac{1}{n}}\ \left\|\nabla R_{11}^{*}f\ \right\|_{L_p(R^n;E)} \leq C\ \left\|f\ \right\|_{L_p(R^n;E)}, \tag{55}$$

where $\frac{1}{p}+\frac{1}{p^{|}}=1$.

To prove the above estimates we need the following prepositions for oscillatory integral in E-valued L_p spaces which generalize the Carleson and Sjolin result [36].

Preposition 4.1. Let E be Banach spaces and $A \in C_0^{\infty}(R^n, L(E))$. Moreover, suppose $\Phi \in C^{\infty}$ satisfies $|\nabla\Phi| \geq \gamma > 0$ on supp A. Then for all $\lambda > 1$ the following holds

$$\left\|\int e^{i\lambda\Phi(x)}A(x)dx\right\|_{L(E)} \leq C_N\lambda^{-N},\ N=1,2,\ldots$$

where C_N-depends only on γ if Φ and A (x) belong to a bounded subset of C^{∞} and C^{∞} $(R^n$, L $(E))$ and A is supported in a fixed compact set.

Proof. Given $x_0 \in$ supp A. There is a direction $v \in S^{n-1}$ such that $|(v, \nabla\Phi)| \geq \frac{\gamma}{2}$ on some ball centered at x_0. Thus, by compactness, we can choose a partition of unity $\varphi_j \in C_0^{\infty}$ consisting of a finite number of terms and corresponding unit vectors v_j such that $\sum\limits_{j=1}^{m}\varphi_j(x)=1$ on supp A and $|(v_j, \nabla\Phi)| \geq \frac{\gamma}{2}$ on supp ϕ_j. For $A_j = \phi_j A$ it suffices to prove that for each j

$$\left\|\int e^{i\lambda\Phi(x)}A_j(x)dx\right\|_{L(E)} \leq C_N\lambda^{-N}, N=1,2,\ldots.$$

After possible changing coordinates we may assume that $v_j = (1, 0, \ldots, 0)$ which means that $\left|\frac{\partial\Phi}{\partial x_1}\right| \geq \frac{\gamma}{2}$ on supp ϕ_j. If let $L(x;D)=\left(\frac{\partial\Phi}{\partial x_1}\right)^{-1}\frac{1}{i\lambda}\frac{\partial}{\partial x_1}$, then $L(x;D)e^{i\lambda\Phi(x)}=e^{i\lambda\Phi(x)}$. Consequently, if $L^{*}=\frac{\partial}{\partial x_1}\left(\frac{1}{i\lambda}\left(\frac{\partial\Phi}{\partial x_1}\right)^{-1}\right)$ is a adjoint, then

$$\int e^{i\lambda\Phi(x)}A(x)dx=\int e^{i\lambda\Phi(x)}(L^{*})^{N}A_j(x)dx.$$

Since our assumption imply that $(L^{*})^{N}$ A_j $(x) = O$ (λ^{-N}), the result follows.

Preposition 4.2. Suppose $\Phi \in C^{\infty}$ is a phase function satisfying the non-degeneracy condition $\det\left[\dfrac{\partial^2\Phi}{\partial x_i\partial x_j}\right] \neq 0$on the support of

$$A(x,y)\in C_0^{\infty}(R^n\times R^n, L(E)).$$

Then for $T_\lambda f=\int\limits_{R^n} e^{i\lambda\Phi(x,y)}A(x,y)f(y)dx,\ \lambda>0$ the following estimates hold

$$\left\|T_\lambda f\right\|_{L_p(R^n;E)} \leq C\lambda^{-\frac{n-1}{p'}}\left\|f\right\|_{L_p(R^n;E)},\ \ 1\leq p\leq 2,$$
$$\left\|T_\lambda f\right\|_{L_p(R^n;E)} \leq C\lambda^{-\frac{n}{p'}}\left\|f\right\|_{L_p(R^n;E)},\ \frac{1}{p}+\frac{1}{p'}=1.$$

Proof. In view of [3, Remark 2.1] we have

$$|\nabla_x[\Phi(x,y)-\Phi(x,z)]| \simeq |y-z| \tag{56}$$

where $|y$ - $z|$ is small. By using a smooth partition of unity we can decompose A $(x$, $y)$ into a finite number of pieces each of which has the property that (57) holds on its support. So, by (57) we can assume

$$|\nabla_x[\Phi(x,y)-\Phi(x,z)]| \geq C|y-z| \tag{57}$$

on supp A for same $C>0$. To use this we notice that

$$\left\|T_\lambda f\right\|_2^2 = \int\int K_\lambda(y,z)f(y)\bar{f}(z)dydz,$$

where

$$K_\lambda(y,z) = \int_{R^n} e^{i\lambda[\Phi(x,y)-\Phi(x,z)]}A(x,y)\bar{A}(x,z)dx. \tag{58}$$

Hence, by virtue of Preposition 4.1 and by (58) we obtain that

$$\left\|K_\lambda(y,z)\right\|_{L(E)} \leq C_N\left(1+|\lambda|\ |y-z|^{-N}\right),\ \text{for all } N.$$

Consequently, by Young's inequality, the operator with kernel K_λ acts

$$L_p(R^n;E) \text{ to } L_p(R^n;E).$$

By (59) we get that

$$\left\|T_\lambda f\right\|_{L_2(R^n;E)} \leq C\lambda^{-n}\left\|f\right\|_{L_2(R^n;E)}.$$

Moreover, it is clear to see that

$$\left\|T_\lambda f\right\|_{L_\infty(R^n;E)} \leq C\lambda^{-n}\left\|f\right\|_{L_1(R^n;E)}.$$

Therefore, by applying Riesz interpolation theorem for vector-valued L_p spaces (see e. g., [19, § 1.18]) we get the assertion.

In a similar way as in [3, Preposition 3.6] we have.

Preposition 4.3. The kernel K_1 (x, y) can be written as

$$K_1(x,y) = \sum_{j=0,1} A_j(x,y)\frac{t^{n-2}e^{it\varphi_j(x',y)}}{|t(x'-y')|^{(n-2)/2}|t(x-y)|},$$

where, for every fixed N, the operator functions A_j satisfy

$$\left\|D^\alpha A_j(x,y)\ \right\| \leq C_\alpha\left(1+t(x_1-y_1)^2\right)^{-N}|x'-y'|^{-|\alpha|},$$

and moreover, the phase functions ϕ_j are real and the property that when ε is small enough, $0<\delta\leq\varepsilon$ and $y_1\in[-\varepsilon,\varepsilon]$ is fixed, the dilated functions

$$(x',y') \to (-1)^j\delta^{-1}\varphi_j(\delta x', y_1, \delta y')$$

in the some fixed neighborhood of the function $\varphi_0(x',y') = |x'-y'|$ in the C^∞ topology. Then, the following estimates holds

$$|K_1(x,y)| \leq Ct^{n-2}(1+t|x_1-y_1|)^{-1}. \tag{59}$$

Proof. By representation of K_1 (x, y) and Φ (x, y, ξ) we have

$$K_1(x,y) \simeq t^{n-2}\int_{R^n}\beta_1(t\xi)e^{it\Phi(x,y,\xi)}Q_{0t}^{*-1}(\varepsilon y,\xi)\,d\xi.$$

Then, by using (36) in view of positivity of operator A, by reasoning as in [3, Preposition 3.6] we obtain the assertion.

Let us now show the end of proof of Lemma 4.1. Let $\eta \in C_0^\infty(R)$ be supported in $\left[\frac{1}{4}, 4\right]$ such that $\sum_{\nu=-\infty}^{\infty} \eta(2^\nu s) = 1, s > 0$ and set $\eta_0(s) = 1 - \sum_{\nu=-\infty}^{0} \eta(2^\nu s)$. Then we define kernels $K_{1,\nu}$, $\nu = 0, 1, 2, \ldots$, as follows

$$K_{1,\nu} = \begin{cases} \eta(t2^{-\nu}|x' - y'|)K_1(x, y), \nu > 0 \\ \eta_0(t|x' - y'|)K_1(x, y), \nu = 0. \end{cases}$$

Let $T_{1,\nu}$ denotes the operators associated to these kernels. Then, by positivity properties of the operator A and by Prepositions 4.2, 4.3 we obtain for $f \in C_0^\infty(B_\varepsilon; E)$ the following estimates

$$\left\| T_{1,\nu}^* f \right\|_{L_{p'}(R^n;E)} \leq C 2^{-2\nu/n} \, ||f||_{L_p(R^n;E)}, \frac{1}{p} + \frac{1}{p^1} = 1, \tag{60}$$

$$||T_{1,\nu}^* f \Big\|_{L_p(R^n;E)} \leq C\left(t 2^{-\nu}\right)^{-1/p'} t^{-\left(1+\frac{1}{n}\right)} \, \|f\,\|_{L_p(R^n;E)}. \tag{61}$$

By summing a geometric series one sees that these estimates imply (52) and (53) for case of $\alpha = 0$.

Let us first to show (60). One can check that the estimate (59) implies that the L_r norm of K_{10}^* is O ($t^{n-2} t^{-n/r}$). But, if we let $r = n/n - 2$, it is follows from Young inequality and the fact that $\frac{1}{p} - \frac{1}{p'} = \frac{2}{n}$ that

$$\left\| T_{1,0}^* f \right\|_{L_{p^|}(R^n;E)} \leq C t^{n-2} t^{-n/r} \, \|f\,\|_{L_p(R^n;E)} = C \, \|f\,\|_{L_p(R^n;E)}$$

as desired. To prove the result for $\nu > 0$, set $B'_\varepsilon = \{x' \in R^{n-1}, |x'| < \varepsilon\}$ and let $K_{1\nu}^*$ be the kernel of the operator $T_{1,\nu}^*$. Then, if we fix x_1 and y_1, it follows that the $L_p(B'_\varepsilon; E) \to L_p(B'_\varepsilon; E)$ norm of the operator

$$T_{1,\nu}^{'*} g(x') = \int_{B'_\varepsilon} K_{1\nu}^*(x, y) g(y') dy'$$

equal $(2^\nu t^{-1})^{(n-1)\left(1-\frac{1}{p}+\frac{1}{p'}\right)}$ times the norm of the dilated operator

$$\tilde{T}_{1,\nu}^* g(x') = \int_{B'_\varepsilon} K_{1\nu}^*(x_1, \delta x', y_1, \delta y') g(y') dy',$$

where $\delta = 2^\nu t^{-1}$. By Preposition 4.3, the kernel in last integral equals the complex conjugate of

$$t^{n-2} \eta(t2^{--\nu}|x' - y'|) \sum_{j=0,1} A_j(y_1, \delta y', x_1, \delta x') \frac{e^{i(t\delta)\delta^{-1}\varphi_j(\delta y', x_1, \delta x')}}{|t(x' - y')|^{(n-2)/2} |t(x_1, \delta x', y_1, \delta y')|},$$

and, consequently by using the Proposition 4.2, for $0 < \delta \leq \varepsilon$ and for supp $g \subset B'_\varepsilon$ we obtain that

$$\left\| \tilde{T}^*_{1,\nu} g(x') \right\|_{L_{p|}(R^n;E)} \leq$$
$$C(t\delta)^{-(n-2)/p'} t^{n-2} (t\delta)^{-(n-2)/2} t^{-1} [(x_1 - y_1)^2 + \delta^2]^{-1/2} \| g \|_{L_p(R^n;E)}.$$

This estimate implies

$$\left\| \int_{B'_\varepsilon} K^*_{1\nu}(x, y)\, g\,(y')\, dy' \right\|_{L_p(B'_\varepsilon;E)} \leq Ct^{-\frac{2}{n}} [(x_1 - y_1)^2 + (2^\nu/t)^2]^{-1/2} \| g \|_{L_p(B'_\varepsilon;E)}.$$

For $r = \frac{n}{n-2}$ we set

$$\left(\int_{-\infty}^{\infty} \left[(x_1 - y_1)^2 + (2^\nu/t)^2 \right]^{-r/2} dx_1 \right)^{1/r} = C(t/2^\nu)^{2/n}.$$

Then, the desired estimate (60) follows from the above estimate and Young's inequality. The other inequality (61), follows from a similar argument.

Preposition 4.4. The estimates (32)-(34) imply (30).

Proof. Indeed, (31) implies that

$$v(x) = T^*(Q_t(\varepsilon x, D)v) - R^* v(x),$$

and so Minkowski's inequality, (32) and (34) give that

$$\| v \|_{p',E} \leq \left\| T^*(Q_t(\varepsilon x, D)v) \right\|_{p',E} + \left\| R^* v \right\|_{p',E} \leq$$
$$\left\| Q_t(\varepsilon x, D)v \right\|_{p,E} + Ct^{-\frac{1}{n}} \| v \|_{p',E}$$

which implies that the first inequality in (30) for sufficiently large t. Moreover, in a similar way, using (32) and (33) we get (30) for $\alpha = 0$. To prove (30) for $|\alpha| = 1$, we use (33), (34) and obtain

$$\| \nabla v \|_{p,E} \leq \left\| \nabla T^*(Q_t(\varepsilon x, D)v) \right\|_{p,E} + \left\| \nabla R^* v \right\|_{p,E} \leq$$
$$Ct^{-\frac{1}{n}} \left\| Q_t(\varepsilon x, D)v \right\|_{p,E} + Ct^{1-\frac{1}{n}} \| v \|_{p,E}.$$

Hence, the result follows.

Now we can show the end of the proof of Theorem 4.1. Really, we obtain the estimate (30), which implies the estimates (26) and (27). That is the assertion of Theorem 4.1 is hold.

Theorem 4.2. Assume all conditions of Theorem 4.1 are satisfied, then for $u \in W^2_{p,1oc}(B_0; E(A), E)$ if $\left\| L(x, D)u \right\|_E \leq \| Vu \|_E$ and $V \in L_{\frac{n}{2},1oc}(B_0; E)$ then u is identically 0 if it vanishes in a nonempty open subset.

Proof. Suppose

$$\left\| L(x, D)\, u \right\|_E \leq \| Vu \|_E + \left\| V'.\nabla u \right\|_E \tag{62}$$

in a connected open set G, where $V \in L_{\frac{n}{2},1oc}(G; E), V' \in L_{\infty,1oc}(G; E)$ and $u \in W^2_{p,1oc}(G; E(A), E)$. Then, after the possibly change of variables, one sees that Theorem 4.2 would follow if we could show that if

$$\text{supp } u \cap \{x \in B_\varepsilon, x_1 \geq 0\} \subset \{0\} \tag{63}$$

then $0 \notin \text{supp } u$. Moreover, by making a proper choice of geodesic coordinate system, we may assume $L\,(x, D)$ as

$$L\,(x, D) = D_1^2 + \sum_{i,j=2}^{n} a_{ij} D_i D_j,\ D_j = \frac{1}{i}\frac{\partial}{\partial x_j}.$$

Then argue as in [29], first set $u_\varepsilon\,(x) = u\,(\varepsilon x)$ where ε is chosen small enough so that (26) and (27) hold for B_ε. Let $\eta \in C_0^\infty(B_\varepsilon)$ be equal to one when $|x| < \frac{\varepsilon}{2}$ and set $U_\varepsilon = \eta u_\varepsilon$. Then if $V_\varepsilon\,(x) = V\,(\varepsilon x)$ and

$$L(\varepsilon x, D)U_\varepsilon = \varepsilon^2 \eta(Lu)(\varepsilon x) + \sum_{0<|\alpha|\leq 2} \frac{1}{\alpha!} D^\alpha \eta(L^{(\alpha)}(\varepsilon x, D))u_\varepsilon$$

which implies that

$$\left\| L(\varepsilon x, D)U_\varepsilon \right\|_E \leq C_0(1 + \|V_\varepsilon\|_E)\,\|U_\varepsilon\|_E + C_0\,\|\nabla U_\varepsilon\|_E\,,\ x \in B_{\varepsilon/2}. \tag{64}$$

Let

$$S_\delta = \{x \in B_\varepsilon : -\delta \leq x_1 \leq 0, \delta > 0\}.$$

If the condition (63) holds, then we can always choose δ to be small enough that

$$S_\delta \cap \text{supp } u \subset B_{\varepsilon/2},$$

and so that if C is as in (26), (27) and C_0 is as in (64) then

$$CC_0 \left(\int_{S_\delta 0} (1 + \|V_\varepsilon\|_E)^{n/2} dx \right)^{2/n} < \frac{1}{2}.$$

Next, (26), (27) imply

$$\begin{aligned}
&\left\| e^{tw} U_\varepsilon \right\|_{L_{p'}(S_\delta;E)} + t^{1/n} \left\| e^{tw} \nabla U_\varepsilon \right\|_{L_p(S_\delta;E)} \\
&\quad \leq \left\| e^{tw} L(\varepsilon x, D)U_\varepsilon \right\|_{L_p(B_\varepsilon;E)} \\
&\leq C \left\| e^{tw} L(\varepsilon x, D)U_\varepsilon \right\|_{L_p(S_\delta;E)} + C \left\| e^{tw} L(\varepsilon x, D)U_\varepsilon \right\|_{L_p({}^c S_\delta;E)}.
\end{aligned}$$

If we recall that $\frac{1}{p} - \frac{1}{p'} = \frac{n}{2}$, then we see that (64) and Hölder's inequality imply

$$\begin{aligned}
&C \left\| e^{tw} L(\varepsilon x, D)U_\varepsilon \right\|_{L_p(S_\delta;E)} \leq CC_0 \left\|{}_0(1 + \|V_\varepsilon\|_E) e^{tw} U_\varepsilon \right\|_{L_p(S_\delta;E)} + CC_0 \left\| e^{tw} \nabla U_\varepsilon \right\|_{L_p(S_\delta;E)} \\
&\quad \leq \frac{1}{2} \left\| e^{tw} U_\varepsilon \right\|_{L_{p'}(S_\delta;E)} + CC_0 \left\| e^{tw} \nabla U_\varepsilon \right\|_{L_p(S_\delta;E)}.
\end{aligned}$$

Thus, by (63) for sufficiently large $t > 0$ and $\tilde{B}_\delta = \{x \in B_\varepsilon : x_1 < -\delta\}$ we can conclude that

$$\left\| e^{tw} U_\varepsilon \right\|_{L_{p'}(S_\delta;E)} + \left\| e^{tw} \nabla U_\varepsilon \right\|_{L_p(S_\delta;E)} \leq 2C \left\| e^{tw} L(\varepsilon x, D)U_\varepsilon \right\|_{L_p(S_\delta;E)}.$$

finally, since $w'\,(x) = 1 + x_1 > 0$ on B_ε, this forces $U_\varepsilon\,(x) = 0$ for $x \in S_\delta$ and so $0 \notin$ supp u which completes the proof.

Consider the differential operator

$$P(x,D)u=\sum_{i,j=1}^{n}a_{ij}D_iD_ju+Au+\sum_{k=1}^{n}A_kD_ku,$$

where a_{ij} are real-valued functions numbers, $A = A(x)$, $A_k = A_k(x)$, $V(x)$ are the possible linear operators in a Banach space E.

By using Theorem 4.2 and perturbation theory of linear operators we obtain the following result

Theorem 4.3. Assume:

(1) all conditions of Theorem 4.1 are satisfied;

(2) $A_kA^{\left(\frac{1}{2}-\mu_k\right)}\in L_\infty(B_0;L(E))$ for $0<\mu_k<\frac{1}{2}$.

Then, for $D^\alpha u\in L_{p,\mathrm{loc}}(B_0;E)$ if $\|P(x,D)u\|_E\leq\|Vu\|_E$ and $V\in L_{\frac{n}{2},\mathrm{loc}}(B_0;E)$, then u is identically 0 if it vanishes in a nonempty open subset.

Proof. By condition (2) and by Theorem 2.1, for all $\varepsilon>0$ there is a $C(\varepsilon)$ such that

$$\sum_{k=1}^{n}\left\|A_k\frac{\partial u}{\partial x_k}\right\|_{L_p(B_0;E)}\leq\varepsilon\|u\|_{W_p^2(B_0;E(A),E)}+C(\varepsilon)\|u\|_{L_p(B_0;E)}.$$

Then, by using (29) and the above estimate we obtain the assertion.

5 Carleman estimates and unique continuation property for quasielliptic PDE

Let $\Omega\subset R^l$ be an open connected set with compact C^{2m}-boundary $\partial\Omega$. Let us consider the BVP for the following elliptic equation

$$\begin{aligned}Lu=&\sum_{i,j=1}^{n}a_{ij}(x)D_iD_ju+\sum_{k=1}^{n}d_k(x,y)D_ku\\&+\sum_{|\alpha|\leq 2m}a_\alpha(y)D_y^\alpha u=f(x,y),\ x\in R^n,y\in\Omega\subset R^l,\end{aligned}\tag{65}$$

$$B_ju=\sum_{|\beta|\leq m_j}b_{j\beta}(y)D_y^\beta u(x,y)=0,x\in R^n,y\in\partial\Omega,\ j=1,2,\ldots,m,\tag{66}$$

where $u = (x, y)$, $D_j=-i\frac{\partial}{\partial\tau_j}$, $\mathcal{T}=(\mathcal{T}_1,\ldots,\mathcal{T}_{n+l})$. Let $\tilde{\Omega}=R^n\times\Omega$.

Let Q denotes the operator generated by the problem (64), (65).

Theorem 5.1. Let the following conditions be satisfied;

(1) $a_\alpha\in C(\bar{\Omega})$ for each $|\alpha|=2m$ and $a_\alpha\in[L_\infty+L_{r_k}](\Omega)$ for each $|\alpha|=k<2m$ with $r_k\geq q$ and $2m-k>\frac{1}{r_k}$;

(2) $b_{j\beta}\in C^{2m-mj}(\partial\Omega)$ for each j, β and $m_j<2m$, $\sum_{j=1}^{m}b_{j\beta}(y^l)\sigma_j\neq 0$, for $|\beta|=m_j$, $y^l\in\partial G$, where $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_n)\in R^m$ is a normal to ∂G ;

(3) for $y\in\bar{\Omega},\xi\in R^l,\lambda\in S(\varphi),\varphi\in(0,\frac{\pi}{2})$, $|\xi|+|\lambda|\neq 0$ let $\lambda+\sum_{|\alpha|=2m}a_\alpha(y)\xi^\alpha\neq 0$;

(4) for each $y_0 \in \partial\Omega$ local BVP in local coordinates corresponding to y_0

$$\lambda + \sum_{|\alpha|=2m} a_\alpha(y_0) D^\alpha \vartheta(y) = 0,$$

$$B_{j0}\vartheta = \sum_{|\beta|=m_j} b_{j\beta}(y_0) D^\beta u(y) = h_j, \ j = 1, 2, \ldots, m$$

has a unique solution $\vartheta \in C_0(R_+)$ for all $h = (h_1, h_2, \ldots, h_m) \in R^m$, and for $\xi^1 \in R^{l-1}$ with

$$|\xi^l| + |\lambda| \neq 0;$$

(5) Condition 4.1 holds, $a_{ij} \in C^\infty(B_\varepsilon)$, $n \geq 3$, $p = \frac{2n}{n+2}$ and p' is the conjugate of p and $w = x_1 + \frac{x_1^2}{2}$;
(6) $d_k \in L_\infty(R^n \times \Omega)$.

Then:

(a) for sufficiently large $b > 0$, $t \geq t_0$ and for $n\left(\frac{1}{p} - \frac{1}{p'}\right) \leq 2, p \in (1, \infty)$ the Carleman type estimate

$$\left\| e^{-tw} u \right\|_{L_{p_1 q}(\tilde{\Omega})} \leq C \left\| e^{-tw}(Q + b)u \right\|_{L_{p_2 q}(\tilde{\Omega})}$$

holds for $u \in W^2_{p_1 q}(\tilde{\Omega})$.

(b) for $V \in L_\mu(\tilde{\Omega})$ and $\frac{1}{\mu} = \frac{1}{p} - \frac{1}{p'}$ the differential inequality

$$\left\| (Q + b)u(x, \ .) \right\|_{L_q(\Omega)} \leq \left\| V(x)u(x, \ .) \right\|_{L_q(\Omega)}$$

has a unique continuation property.
Proof. Let $E = L_q(\Omega)$. Consider the following operator A which is defined by

$$D(A) = W_q^{2m}(\Omega; B_j u = 0), \ Au = \sum_{|\alpha| \leq 2m} a_\alpha(y) D^\alpha u(y).$$

For $x \in R^n$ also consider operators

$$A_k(x)u = d_k(x, y)u(y), \quad k = 1, 2, \ldots, n.$$

The problem (5.1), (5.2) can be rewritten in the form (4.1), where $u(x) = u(x, .)$, $f(x) = f(x, .)$ are functions with values in $E = L_q(\Omega)$. Then by virtue of [24, Theorems 3.6 and 8.2] the (1) condition of Theorem 4.1 is satisfied. Moreover, by using the embedding $W_q^{2m}(\Omega) \subset L_q(\Omega)$ and interpolation properties of Sobolev spaces (see e.g., [19, §4]) we get that there is $\varepsilon > 0$ and a continuous function $C(\varepsilon)$ such that

$$\left\| d_k \frac{\partial u}{\partial x_k} \right\|_{L_q} \leq \varepsilon \left\| u \right\|_{W_q^{2m}} + C(\varepsilon) \left\| u \right\|_{L_q}.$$

Due to positive of the operator A, then we obtain that

$$\left\| d_k \frac{\partial u}{\partial x_k} \right\|_{L_q} \leq \varepsilon \, \|Au\|_{L_q} + C(\varepsilon) \, \|u\|_{L_q}.$$

Then it is easy to get from the above estimate that (2) condition of the Theorem 4.3 is satisfied. By virtue of (5) condition, (2) condition of the Theorem 4.1 is fulfilled too. Hence, by virtue of Theorems 4.1 and 4.3 we obtain the assertions.

6 Carleman estimates and unique continuation property for infinite systems of elliptic equations

Consider the following infinity systems of PDE

$$\begin{aligned} &\sum_{k=1}^{n} a_k(x) D_k^2 u_m(x) + (d_m(x) + \lambda) u_m(x) \\ &+ \sum_{k=1}^{n} \sum_{j=1}^{\infty} d_{kjm}(x) D_k u_j(x) = f_m(x), x \in R^n, m = 1, 2, \ldots. \end{aligned} \tag{67}$$

Let

$$D(x) = \{d_m(x)\}, d_m > 0, \ u = \{u_m\}, \ Du = \{d_m u_m\}, \ m = 1, 2, \ldots,$$

$$l_q(D) = \left\{ u : u \in l_q, \|u\|_{l_q(D)} = \|Du\|_{l_q} = \left(\sum_{m=1}^{\infty} |d_m u_m|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$x \in R^n, \ 1 < q < \infty.$$

Let O denotes the operator generated by the problem (66).

Theorem 6.1. Let the following conditions are satisfied:

(1) $a_k \in C_b(R^n)$, $a_k(x) \neq 0$, $x \in R^n$, $k = 1, 2, \ldots, n$ and the Condition 4.1 holds;

(2) there are $0 < \nu < \frac{1}{2}$ such that

$$\sup_m \sum_{j=1}^{N} b_{mj}(x) d_{kjm}^{-(\frac{1}{2}-\nu)}(x) < M,$$

a.e. for $x \in R^n$.

Then:

(a) for sufficiently large $b > 0$, $t \geq t_0$ and for $n(\frac{1}{p} - \frac{1}{p^|}) \leq 2, \ 1 < p \leq p^| < \infty$ the Carleman type estimate

$$\|e^{-tw} u\|_{L_{p^|}(R^n; l_q)} \leq C \, \|e^{-tw}(O + b)u\|_{L_p(R^n; l_q)}$$

holds for $u \in W_p^2(R^n; l_q(D), \ l_q)$.

(b) for $V \in L_\mu\left(\tilde{\Omega}; L(E)\right)$and $\frac{1}{\mu} = \frac{1}{p} - \frac{1}{p^{|}}$ the differential inequality

$$\left\|(O + b)u(x)\right\|_{l_q} \leq \left\|V(x)u(x)\right\|_{l_q}$$

has a unique continuation property.

Proof. Let $E = l_q$ and A, $A_k\ (x)$ be infinite matrices, such that

$$A = [d_m(x)\delta_{jm}],\ A_k(x) = [d_{kjm}(x)],\ m,\ j = 1, 2, \ldots, \infty.$$

It is clear to see that this operator A is R-positive in l_q and all other conditions of Theorems 4.1 and 4.3 are hold. Therefore, by virtue of Theorems 4.1 and 4.3 we obtain the assertions.

Competing interests

The author declares that they have no competing interests.

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