Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence results for higher order fractional differential equation with multi-point boundary condition

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Abstract

The fixed point theorems on cones are used to investigate the existence of positive solution for higher order fractional differential equation with multi-point boundary condition.

MSC: 26A33; 34B15.

Keywords: fractional differential equation, fixed point, positive solution, cone

1 Introduction

Recently, much attention has been paid to the fractional differential equations due to its wide application in physics, engineering, economics, aerodynamics, and polymer rheology etc. For the basic theory and development of the subject, we refer some contributions on fractional calculus, fractional differential equations, see Delbosco [1], Miller [2], and Lakshmikantham et al. [3-7]. Especially, there have been some articles dealing with the existence of solutions or positive solutions of boundary-value problems for nonlinear fractional differential equations (see [8-20] and references along this line). For examples, Jiang [16] obtained the existence of positive solution for boundary value problem of fractional differential equation

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0,\ u(0)=0,\ u(1)=0,\ 1<\alpha\leq 2,$$

where $D_{0+}^{\alpha}u(t)$ denotes the standard Riemann-Liouville fractional order derivative.

Agarwal et al. [17] investigated the existence of positive solution of singular problem

$$D_{0+}^{\alpha}u(t)=f(t,\ u(t),\ \ D^{\mu}u(t)),\ u(0)=u(1)=0,$$

where $1<\alpha<2$, $0\leq\mu\leq\alpha-1$ and f satisfies the Caratheodory conditions on $[0,1]\times[0,\infty)\times R$ and $f(t, x, y)$ is singular at $x = 0$. The existence results of positive solutions are established by using regularization and sequential techniques.

As to the nonlocal problem, Bai [18] established the existence of positive solution for three-point boundary value problem of fractional differential equation

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0,\ u(0)=0,\ u(1)=\beta u(\eta),\quad \eta\in(0,1).$$

By using the fixed point theorems on cones, Li et al. [19] established the existence of positive solutions for problem

$$D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0,\ u(0) = 0,\quad D_{0+}^{\beta}u(1) = aD_{0+}^{\beta}u(\xi),$$

where $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $0 \leq a \leq 1$, $\xi \in (0, 1)$ and $a\xi^{\alpha-\beta-2} \leq 1 - \beta$, $0 \leq \alpha - \beta - 1$ and $f: [0,1] \times [0, \infty) \to [0, \infty)$ satisfies Caratheodory type conditions.

Very recently, Moustafa and Nieto [20] considered the nontrivial solution for following higher order multi-point problem

$$D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0,\ n-1 \leq \alpha \leq n,\ n \in N \tag{1.1}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)} = 0,\quad u(1) = \sum_{i=1}^{m-2}\beta_i u(\eta_i), \tag{1.2}$$

where $n \geq 2$, $0 < \eta_i < 1$, $\beta_i > 0$, $i = 1, 2, \ldots, m - 2$, $\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha-1} < 1$, $f \in C([0,1] \times R, R)$. The existence of nontrivial solution was established by using the nonlinear alternative of Leray-Schauder. But, existence of positive solution for problem (1.1), (1.2), as far as we know, has not been considered before. Considering that problem (1.1) and (1.2) are more general than problems studied before, we believe that it is interesting to investigate the existence of positive solution for this problem.

In this article, we consider the existence and multiplicity of positive solutions for problem (1.1) and (1.2). We obtain some properties of the associated Green's function. By using these properties of Green's function and fixed point theorems on cones, we establish the existence and multiplicity of positive solutions.

2 Preliminaries

For the convenience of the reader, we present here the basic definitions and theory from fractional calculus theory. These definitions and theory can be founded in the literature [1].

Definition 2.1 The fractional integral of order $\alpha > 0$ of a function $u(t)$: $(0, \infty) \to R$ is given by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds$$

provided the right side is point-wise defined on $(0, \infty)$.

Definition 2.2 The fractional derivative of order $\alpha > 0$ of a continuous function $u(t)$: $(0, \infty) \to R$ is given by

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}ds$$

where $n = [\alpha] + 1$, provided that the right side is point-wise defined on $(0, \infty)$.

Lemma 2.1 Let $\alpha > 0$. If we assume $u \in C(0, 1) \cup L(0, 1)$, then problem $D_{0+}^{\alpha}u(t) = 0$ has solution

$$u(t) = C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_Nt^{\alpha-N}$$

for some $C_i \in R$, $i = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α.

Lemma 2.2 Assume that $u \in C(0, 1) \cup L(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cup L(0, 1)$. Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},$$

for some $C_i \in R$, $i = 1, 2, \ldots, N$.

Lemma 2.3 [21] Let E be a Banach space and $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let

$$A : K \cap (\Omega_2 \backslash \bar{\Omega}_1) \to K$$

be a completely continuous operator such that

$$||Au|| \le ||u||, \ u \in K \cap \partial\Omega_1, and ||Au|| \ge ||u||, \ u \in K \cap \partial\Omega_2 \ or$$
$$||Au|| \ge ||u||, \ u \in K \cap \partial\Omega_1, and \ ||Au|| \le ||u||, \ u \in K \cap \partial\Omega_2,$$

then A has a fixed point in $K \cap (\Omega_2 \backslash \bar{\Omega}_1)$.

Let $0 < a < b$ be given and let ψ be a nonnegative continuous concave functional on the cone C. Define the convex sets C_r and $C(\psi, a, b)$ by

$$C_r = \{u \in C | \ ||u|| < r\}$$
$$C(\psi, a, b) = \{u \in C | \ a \le \psi(u), \ ||u|| \le b\}.$$

Lemma 2.4 [22] Let $T : \bar{C}_r \to \bar{C}_r$ be a completely continuous operator and let ψ be a nonnegative continuous concave functional on C such that $\psi(u) \le ||u||$ for all $u \in \bar{C}_r$. Suppose that there exist $0 < a < b < d \le c$ such that

(S_1) $\{u \in C(\psi, b, d) | \ \psi(u) > b\} \neq \emptyset$ and $\psi(Tu) > b$ for $u \in C(\psi, b, d)$,

(S_2) $||Tu|| < a$ for $||u|| \le a$ and

(S_3) $\psi(Tu) > b$ for $u \in C(\psi, b, c)$ with $||Tu|| \ge d$.

Then T has at least three fixed points u_1, u_2, and u_3 such that

$$||u_1|| < a, \ b < \psi(u_2), \quad ||u_3|| > a, \ \ \psi(u_3) < b.$$

Lemma 2.5 Denote $\eta_0 = 0$, $\eta_{m-1} = 1$ and $\beta_0 = \beta_{m-1} = 0$. Given $y(t) \in C[0,1]$. The problem

$$D_{0+}^{\alpha} u(t) + y(t) = 0, \ u(0) = u'(0) = \cdots = u^{(n-2)} = 0, \ u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i), \tag{3.1}$$

is equivalent to

$$u(t) = \int_0^1 G(t, s) y(s) ds,$$

where

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)\left(1 - \sum\limits_{k=0}^{m-1} \beta_k \eta_k^{\alpha-1}\right)} \left[(1-s)^{\alpha-1} - \sum\limits_{k=i}^{m-1} \beta_k (\eta_k - s)^{\alpha-1}\right] & t \le s \\ -\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{t^{\alpha-1}}{\Gamma(\alpha)\left(1 - \sum\limits_{k=0}^{m-1} \beta_k \eta_k^{\alpha-1}\right)} \left[(1-s)^{\alpha-1} - \sum\limits_{k=i}^{m-1} \beta_k (\eta_k - s)^{\alpha-1}\right] & t \ge s \end{cases}$$

Furthermore, the function $G(t, s)$ is continuous on $[0,1] \times [0,1]$ and satisfies the condition

$$G(t,\ s) > 0,\ t,\ s \in [0,\ 1].$$

Proof. From Lemma 2.1, we get that problem (3.1) is equivalent to

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}.$$

The boundary conditions $u(0) = u'(0) = \ldots = u(n\text{-}2) = 0$ induce that $C_2 = C_3 = \ldots = C_n = 0$.. Considering the boundary condition $u(1) = \sum_{i=0}^{m-1} \beta_i u(\eta_i)$, we get

$$C_1 = \frac{1}{\left(1 - \sum_{i=0}^{m-1} \beta_i \eta_i^{\alpha-1}\right)\Gamma(\alpha)} \left[\int_0^1 (1-s)^{\alpha-1} y(s)ds - \sum_{i=0}^{m-1} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} y(s)ds\right].$$

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + \frac{t^{\alpha-1}}{\left(1 - \sum_{i=0}^{m-1} \beta_i \eta_i^{\alpha-1}\right)\Gamma(\alpha)} \\ &\times \left[\int_0^1 (1-s)^{\alpha-1} y(s)ds - \sum_{i=0}^{m-1} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} y(s)ds\right]. \end{aligned}$$

Then for $\eta_{i-1} < t < \eta_i$, $i = 1, 2, \ldots, m - 1$,

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds \\ &+ \frac{t^{\alpha-1}}{\left(1 - \sum_{i=1}^{m-2} \beta_i \eta_i^{\alpha-1}\right)\Gamma(\alpha)} \left[\int_0^1 (1-s)^{\alpha-1} y(s)ds - \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} y(s)ds\right] \\ = &\sum_{k=1}^{i-1} \int_{\eta_{k-1}}^{\eta_k} \left[-\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} + \frac{t^{\alpha-1}}{\left(1 - \sum_{i=1}^{m-2} \beta_i \eta_i^{\alpha-1}\right)\Gamma(\alpha)} \left((1-s)^{\alpha-1} - \sum_{j=k}^{m-1} \beta_j (\eta_j - s)^{\alpha-1}\right)\right] y(s)ds \\ &+ \int_{\eta_{i-1}}^{t} \left[-\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} + \frac{t^{\alpha-1}}{\left(1 - \sum_{i=1}^{m-2} \beta_i \eta_i^{\alpha-1}\right)\Gamma(\alpha)} \left((1-s)^{\alpha-1} - \sum_{j=i}^{i-1} \beta_j (\eta_j - s)^{\alpha-1}\right)\right] y(s)ds \\ &+ \int_t^{\eta_i} \frac{t^{\alpha-1}}{\left(1 - \sum_{i=1}^{m-2} \beta_i \eta_i^{\alpha-1}\right)\Gamma(\alpha)} \left[(1-s)^{\alpha-1} - \sum_{j=i}^{m-1} \beta_j (\eta_j - s)^{\alpha-1}\right] y(s)ds \\ &+ \sum_{k=i}^{m-1} \int_{\eta_{k-1}}^{\eta_k} \frac{t^{\alpha-1}}{\left(1 - \sum_{i=1}^{m-2} \beta_i \eta_i^{\alpha-1}\right)\Gamma(\alpha)} \left[(1-s)^{\alpha-1} - \sum_{j=k}^{m-1} \beta_j (\eta_j - s)^{\alpha-1}\right] y(s)ds = \int_0^1 G(t,s) y(s)ds \end{aligned}$$

Furthermore, for $\eta_{i-1} \le s \le \eta_i$, $i = 1, 2, \ldots, m\text{-}1$ and $t \le s$

$$\left(\Gamma(\alpha)\left(1 - \sum_{k=0}^{m-1} \beta_k \eta_k^{\alpha-1}\right)\right) G(t,s) \ge t^{\alpha-1} \sum_{k=i}^{m-2} \left((1-s)^{\alpha-1} - (\eta_k - s)^{\alpha-1}\right) > 0.$$

For $\eta_{i-1} \le s \le \eta_i$, $i = 1, 2, \ldots, m\text{-}1$ and $t \ge s$

$$\left(\Gamma(\alpha)\left(1-\sum_{k=0}^{m-1}\beta_k\eta_k^{\alpha-1}\right)\right)G(t,s) \ge t^{\alpha-1}\left[(1-s)^{\alpha-1}-\left(1-\frac{s}{t}\right)^{\alpha-1}\right]$$
$$+t^{\alpha-1}\sum_{k=0}^{i-1}\beta_k\eta_k^{\alpha-1}\left[\left(1-\frac{s}{t}\right)^{\alpha-1}-\left(1-\frac{s}{\eta_k}\right)^{\alpha-1}\right] > 0.$$

□

Lemma 2.6 The function $G(t, s)$ satisfies the following conditions:

(1) $G(t, s) \le G(s, s)$, $t, s \in [0, 1]$,

(2) There exists function $\gamma(s)$ such that $\min_{\eta_{m-2}\le s\le 1} G(t,\ s) \ge \gamma(s)G(s,\ s),\ 0 < s < 1$.

Proof (1) For $\eta_{i-1} < s < \eta_i$, $i = 1, 2, \ldots, m\text{-}1$, Denote

$$g_1(t,\ s) = \frac{t^{\alpha-1}}{\Gamma(\alpha)\left(1-\sum\limits_{k=0}^{m-1}\beta_k\eta_k^{\alpha-1}\right)}\left[(1-s)^{\alpha-1}-\sum_{k=i}^{m-1}\beta_k(\eta_k-s)^{\alpha-1}\right],$$

$$g_2(t,\ s) = -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-1}}{\Gamma(\alpha)\left(1-\sum\limits_{k=0}^{m-1}\beta_k\eta_k^{\alpha-1}\right)}\left[(1-s)^{\alpha-1}-\sum_{k=i}^{m-1}\beta_k(\eta_k-s)^{\alpha-1}\right].$$

The facts that

$$\frac{\partial g_1(t,\ s)}{\partial t} = \frac{(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha)\left(1-\sum\limits_{k=0}^{m-1}\beta_k\eta_k^{\alpha-1}\right)}\left[(1-s)^{\alpha-1}-\sum_{k=i}^{m-1}\beta_k(\eta_k-s)^{\alpha-1}\right] > 0,$$
$$\frac{\partial g_2(t,\ s)}{\partial t} = -\frac{(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}+\frac{(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha)\left(1-\sum\limits_{k=0}^{m-1}\beta_k\eta_k^{\alpha-1}\right)}\left[(1-s)^{\alpha-1}-\sum_{k=i}^{m-1}\beta_k(\eta_k-s)^{\alpha-1}\right] < 0$$

imply that $g_1(t, s)$ is decreasing with respect to t for $[\eta_{i-1}, s]$ and g_2 (t, s) is increasing with respect to t for $[s, \eta_i]$, $i = 1, 2, \ldots, m$ - 1. Thus one can easily check that

$$G(t,s) \le G(s,s),\ \ t,s \in [0,1].$$

(2) For $\eta_{m-2} < t < 1$, denote

$$\gamma_i(t,\ s) = -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-1}}{\Gamma(\alpha)\left(1-\sum\limits_{k=0}^{m-1}\beta_k\eta_k^{\alpha-1}\right)}\left[(1-s)^{\alpha-1}-\sum_{k=i}^{m-1}\beta_k(\eta_k-s)^{\alpha-1}\right],$$

$$\gamma(s) = \min\left\{\gamma_i(\eta_{m-2},\ s),\ \frac{1}{\Gamma(\alpha)\left(1-\sum\limits_{k=0}^{m-1}\beta_k\eta_k^{\alpha-1}\right)}\left[(1-s)^{\alpha-1}-\sum_{k=i}^{m-1}\beta_k(\eta_k-s)^{\alpha-1}\right]\right\},\ i = 1,\ 2,\ \ldots,\ m-2$$

$$\gamma_i(t,\ s) = -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-1}}{\Gamma(\alpha)\left(1-\sum\limits_{k=0}^{m-1}\beta_k\eta_k^{\alpha-1}\right)}\left[(1-s)^{\alpha-1}-\sum_{k=i}^{m-1}\beta_k(\eta_k-s)^{\alpha-1}\right],$$

$$\gamma(s) = \min\left\{\gamma_i(\eta_{m-2},\ s),\ \frac{1}{\Gamma(\alpha)\left(1-\sum\limits_{k=0}^{m-1}\beta_k\eta_k^{\alpha-1}\right)}\left[(1-s)^{\alpha-1}-\sum_{k=i}^{m-1}\beta_k(\eta_k-s)^{\alpha-1}\right]\right\},\ i = 1,\ 2,\ \ldots,\ m-2$$

Thus we have

$$\gamma(s) > 0,\ \min_{\eta_{m-2}<t<1} G(t,s) \ge \gamma(s)G(s,\ s) = \gamma(s)\max_{0\le t\le 1} G(t,\ s).$$

3 Main results

Let $X = C[0,1]$ be a Banach space endowed with the norm

$$||u|| = \max_{0 \le t \le 1} |u(t)|, \ u \in X.$$

Define the cone $P \subset E$ by $P = \{u \in X \mid u(t) \ge 0\}$.

Theorem 3.1 Define the operator $T : P \to X$,

$$Tu(t) = \int_0^1 G(t, s) f(s, u(s)) ds,$$

then $T : P \to P$ is completely continuous.

Proof From the nonnegative and continuous properties of function f and $G(t, s)$, one can obtain easily that the operator $T : P \to P$ and T is continuous. Let Ω be a bounded subset of cone P. That is, there exists a positive constant $M_1 > 0$ such that $||u|| \le M_1$ for all $u \in \Omega$. Thus for each $u \in \Omega$, t_1, $t_2 \in [0,1]$, one has

$$\begin{aligned} |Tu(t_1) - Tu(t_2)| &= \left| \int_0^1 (G(t_1, s) - G(t_2, s)) f(s, u(s)) ds \right| \\ &\le \int_0^1 \left| G(t_1, s) - G(t_2, s) \right| f(s, u(s)) ds \\ &\le M_2 \int_0^1 \left| G(t_1, s) - G(t_2, s) \right| ds \end{aligned}$$

Then the continuity of function $G(t, s)$ implies that T is equicontinuity on the bounded subset of P . On the other hand, for $u \in \Omega$, there exist constant M_2>0 such that

$$f(t, u) \le M_2, \ t \in [0, 1], \ u \in \Omega.$$

Then

$$Tu(t) = \int_0^1 G(t, s) f(s, u(s)) ds \le M_2 \int_0^1 G(s, s) ds.$$

which implies that T is uniformly bounded on the bounded subset of P . Then an application of Ascoli-Arezela ensures that $T : P \to P$ is completely continuous.

Theorem 3.2 Assume that there exist two positive constant $r_2 > \frac{N}{M} r_1 > 0$ such that

(A1) $f(t, u) \le Mr_2$, $(t, u) \in [0, 1] \times [0, r_2]$

(A2) $f(t, u) \ge Nr_1$, $(t, u) \in [0, 1] \times [0, r_1]$

where

$$M = \left(\int_0^1 G(s, s) ds \right)^{-1}, \quad N = \left(\int_{\eta_{m-2}}^1 \gamma(s) G(s, s) ds \right)^{-1},$$

Then problem (1.1) and (1.2) has at least one positive solution u such that $r_1 \le ||u|| \le r_2$.

Proof Let $\Omega_2 = \{u \in P \mid ||u|| \le r_2\}$. For $u \in \partial\Omega_2$, considering assumption ($A1$), we have

$$0 \le u(t) \le r_1,\ and\ f(t,\ u) \ge Mr_2,\ t \in [0, 1],$$

$$Tu(t) = \int_0^1 G(t,\ s)f(s,\ u(s))ds \le Mr_2 \int_0^1 G(s,\ s)ds \le r_2.$$

Thus $||Tu|| \le ||u||$, $u \in \partial\Omega_2$.

Let $\Omega_1 = \{u \in P \mid ||u|| \le r_1\}$. For $u \in \partial\Omega_1$, considering assumption ($A2$), we have

$$0 \le u(t) \le r_1,\ and\ f(t,\ u) \ge Nr_2,\ t \in [0, 1]$$

Thus for $t \in [\eta_{m-2}, 1]$, we get

$$Tu(t) = \int_0^1 G(t,\ s)f(s,\ u(s))ds \ge \int_{\eta_{m-2}}^1 \gamma(s)G(s,\ s)f(s,\ u(s))ds \ge r_1,$$

which gives that $||Tu|| \ge ||u||$, $u \in \partial\Omega_1$. An application of Lemma (2.5) ensures the existence of positive solution $u(t)$ of problem (1.1) and (1.2).

Theorem 3.3 Suppose that there exist constants $0 < a < b < c$ such that

($A3$) $f(t, u) < Ma$, for $(t, u) \in [0,1] \times [0, a]$,

($A4$) $f(t, u) \ge Nb$, for $(t, u) \in [\eta_{m-2}, 1] \times [b, c]$,

($A5$) $f(t, u) \le Mc$, for $(t, u) \in [0,1] \times [0, c]$,

then problem (1.1) and (1.2) has at least three positive solution u_1, u_2, u_3 with

$$\max_{0\le t\le 1} |u_1| \le a,\ b < \min_{\eta_{m-2}\le t\le 1} |u_2| < \max_{0\le t\le 1} |u_2| \le c, a < \max_{0\le t\le 1} |u_3| \le c,\ \min_{\eta_{m-2}\le t\le 1} |u_3| < b.$$

Proof Let the nonnegative continuous concave functional θ on the cone P defined by

$$\theta(u) = \min_{\eta_{m-2}\le t\le 1} |u(t)|.$$

If $u \in \bar{P}_c$, then $||u|| \le c$. Then by condition (A5), we have

$$f(t,\ u) \le Mc, \text{for } (t,\ u) \in [0,\ 1] \times [0,\ c].$$

Thus

$$|T(u)(t)| = \left|\int_0^1 G(t,\ s)f(s,\ u(s))ds\right| \le Mc \int_0^1 G(s,\ s)ds = c.$$

which yields that $T : \bar{P}_c \to \bar{P}_c$. In the same way, we get that

$$\|Tu\| < a,\ for\ u \le a.$$

We chose the function $u(t) = \dfrac{b+c}{2}\ t \in [0, 1]$. We claim that $\dfrac{b+c}{2} \in \{u \in P(\theta,\ b,\ c)|\theta(u) > b\}$, which ensures that $\{u \in P\ (\theta,\ b,\ c)|\theta(u) > b\} \ne \emptyset$.

And for $u \in P(\theta, b, c)$, we have

$$f(t, u(t)) \geq Nb, \; t \in [\eta_{m-2}, 1]$$

Then

$$\theta(Tu) = \min_{\eta_{m-2} \leq t \leq 1} \left| \int_0^1 G(t, s) f(s, u(s)) ds \right| > Nb \int_{\eta_{m-2}}^1 \gamma(s) G(s, s) ds = b,$$

which yields that $\theta(Tu) > b$, for $u \in P(\theta, b, c)$.

An application of Lemma (2.6) ensures that problem (1.1) and (1.2) has at least three positive solutions with

$$\max_{0 \leq t \leq 1} |u_1| \leq a, \; b < \min_{\eta_{m-2} \leq t \leq 1} |u_2| < \max_{0 \leq t \leq 1} |u_2| \leq c, a < \max_{0 \leq t \leq 1} |u_3| \leq c, \; \min_{\eta_{m-2} \leq t \leq 1} |u_3| < b.$$

Acknowledgements
This study was supported by the Anhui Provincial Natural Science Foundation (10040606Q50) and the Natural Science Foundation of Anhui Department of Education (KJ2010A285).

Authors' contributions
The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 23 October 2011 Accepted: 15 May 2012 Published: 15 May 2012

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doi:10.1186/1687-2770-2012-57
Cite this article as: Liu: Existence results for higher order fractional differential equation with multi-point boundary condition. *Boundary Value Problems* 2012 **2012**:57.