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A Cauchy-type problem with a sequential fractional derivative in the space of continuous functions

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Abstract

A Cauchy-type nonlinear problem for a class of fractional differential equations with sequential derivatives is considered in the space of weighted continuous functions. Some properties and composition identities are derived. The equivalence with the associated integral equation is established. An existence and uniqueness result of the global continuous solution is proved.
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1 Introduction

We consider a Cauchy-type problem associated with the equation

$$D_a^\alpha \left[(x-a)^r D_a^\beta y(x) \right] = f(x, y), \quad x > a, \quad 0 < \alpha < 1, \quad 0 \le \beta \le 1, \quad r < \alpha, \qquad (1)$$

where D_a^α and D_a^β are the Riemann-Liouville fractional derivatives.

In recent years there has been a considerable interest in the theory and applications of fractional differential equations. As for the theory, the investigations include the existence and uniqueness of solutions, asymptotic behavior, stability, etc. See for example the books [1-3] and the articles [4-10] and the references therein.

As for the applications, fractional models provide a tool for capturing and understanding complex phenomena in many fields. See for example the surveys in [1,11] and the collection of applications in [12].

Some recent applications include control systems [13,14], viscoelasticity [15-18], and nanotechnology [19]. Also fractional models are used to model a vibrating string [20], and anomalous transport [21], anomalous diffusion [22-24].

Another field of applications is in random walk and stochastic processes [25-27] and its applications in financial modeling [28-30]. Other physical and engineering processes are given in [31,32]

In a series of articles, [33-35], Glushak studied the uniform well-posedness of a Cauchy-type problem with two fractional derivatives and bounded operator. He also proposed a criterion for the uniform correctness of unbounded operator.

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In this article we prove an existence and uniqueness result for a nonlinear Cauchy-type problem associated with the Equation (1) in the space of weighted continuous functions.

We start with some preliminaries in Section 2. In Section 3 we define the sequential derivative and develop some properties and composition identities. In Section 4 we set up the Cauchy-type problem and establish the equivalence with the associated integral equation. Finally, in Section 5 we prove the existence and uniqueness of the solution.

2 Preliminaries

In this section we present some definitions, lemmas, properties and notation which we use later. For more details please see [1].

Let $-\infty < a < b < \infty$. Let $C[a, b]$ denote the spaces of continuous functions on $[a, b]$. We denote by $L(a, b)$ the spaces of Lebesgue integrable functions on (a, b). Let $CL(a, b) = L(a, b) \cap C(a, b)$.

We introduce the weighted spaces of continuous functions

$$C_\gamma[a, b] = \left\{ f : (a, b] \to \mathbb{R} : (x - a)^\gamma f(x) \in C[a, b] \right\}, \quad \gamma \in \mathbb{R}, \tag{2}$$

with the norm

$$\|f\|_{C_\gamma[a,b]} = \left\| (x - a)^\gamma f(x) \right\|_{C[a,b]'} \tag{3}$$

where

$$\|f\|_{C[a,b]} = \max_{x \in [a,b]} \left| f(x) \right|. \tag{4}$$

In the case f is not defined at $x = a$ or $\gamma < 0$ we let $(x - a)^\gamma f(x)\big|_{x=a} = \lim_{x - a^+} (x - a)^\gamma f(x)$. The spaces $C_\gamma[a, b]$ satisfy the following properties.

- $C_0[a,b] = C[a,b]$.
- $C_{\gamma_1}[a, b] \subset C_{\gamma_2}[a, b], \gamma_1 < \gamma_2$.
- $C_\gamma[a,b] \subset CL(a,b), \gamma < 1$.
- $f \in C_\gamma[a, b]$ if and only if $f \in C(a, b)$ and $\lim_{x \to a^+} (x - a)^\gamma f(x)$ exists and is finite.

The left-sided Riemann-Liouville fractional integrals and derivatives are defined as follows.

Definition 1 *Let* $f \in L(a,b)$. *The integral*

$$I_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(s)}{(x - s)^{1-\alpha}} ds, \quad x > a, \quad \alpha > 0, \tag{5}$$

is called the left-sided Riemann-Liouville fractional integral of order α of the function f.

Definition 2 *The expression*

$$D_a^\alpha f(x) := D I_a^{\alpha-1} f(x), \quad x > a, \quad 0 < \alpha < 1, \quad D = \frac{d}{dx}, \tag{6}$$

is called the left-sided Riemann-Liouville fractional derivative of order α of f provided the right-hand side exists.

For power functions we have the following formulas.

Lemma 3 *For x >a we have*

$$\left[I_a^\alpha (t-a)^{\beta-1} \right](x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}, \quad \alpha \geq 0, \quad \beta > 0. \tag{7}$$

$$\left[D_a^\alpha (t-a)^{\alpha-1} \right](x) = 0, \quad 0 < \alpha < 1. \tag{8}$$

Next we present some mapping properties of the operator I_a^α.

Lemma 4 *For $\alpha > 0$, I_a^α maps L(a, b) into L(a, b).*

The proof of Lemma 4 is given in [36]. The following lemma is proved in [37].

Lemma 5 *For $\alpha > 0$, I_a^α maps C[a, b] into C[a, b].*

The following lemma is proved in [38].

Lemma 6 *Let $\alpha \geq 0$. If $f \in CL(a, b)$ then $I_a^\alpha f \in CL(a, b)$.*

The mapping properties of I_a^α in the spaces $C_\gamma[a, b]$, $0 \leq \alpha \leq \gamma < 1$, are given in [1], Lemma 2.8 which is proved in [39] in Russian. For completeness we present here a more general result for $\alpha > 0$ and $\gamma < 1$. First we prove the necessity condition at the left end.

Lemma 7 *Let $\alpha \geq 0$ and $\gamma < 1$. If $f \in C_\gamma[a, b]$ then*

$$\lim_{x \to a^+} (x-a)^{\gamma-\alpha} I_a^\alpha f(x) = \frac{c\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}, \tag{9}$$

where $c = \lim_{x \to a^+} (x-a)^\gamma f(x)$.

Proof. Note that from Lemma 3 we have

$$I_a^\alpha (x-a)^{-\gamma} = \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}(x-a)^{\alpha-\gamma}.$$

Thus

$$\left| (x-a)^{\gamma-\alpha} I_a^\alpha f(x) - \frac{c\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} \right| = \left| (x-a)^{\gamma-\alpha} I_a^\alpha f(x) - c(x-a)^{\gamma-\alpha} I_a^\alpha (x-a)^{-\gamma} \right|$$

$$= (x-a)^{\gamma-\alpha} \left| I_a^\alpha f(x) - c I_a^\alpha (x-a)^{-\gamma} \right|$$

$$= (x-a)^{\gamma-\alpha} \left| I_a^\alpha \left[(x-a)^{-\gamma} \{(x-a)^\gamma f(x) - c\} \right] \right|.$$

Now, given $\epsilon > 0$ there exists $\delta > 0$ such that $x - a < \delta$ implies that

$$\left| (x-a)^\gamma f(x) - c \right| < \varepsilon \frac{\Gamma(1+\alpha-\gamma)}{\Gamma(1-\gamma)}.$$

Thus

$$\left| (x-a)^{\gamma-\alpha} I_a^\alpha f(x) - \frac{c\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} \right| = (x-a)^{\gamma-\alpha} \left| I_a^\alpha \left[(x-a)^{-\gamma} \{(x-a)^\gamma f(x) - c\} \right] \right|$$

$$< \varepsilon \frac{\Gamma(1+\alpha-\gamma)}{\Gamma(1-\gamma)}(x-a)^{\gamma-\alpha} I_a^\alpha \left[(x-a)^{-\gamma} \right]$$

$$= \varepsilon \frac{\Gamma(1+\alpha-\gamma)}{\Gamma(1-\gamma)}(x-a)^{\gamma-\alpha} \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}(x-a)^{\alpha-\gamma}$$

$$= \varepsilon.$$

This yields the limit (9).

Next we present the mapping properties of I_a^α in the spaces $C_\gamma[a, b]$.

Lemma 8 *Let* $\alpha > 0$ *and* $\gamma < 1$. *If* $f \in C_\gamma[a, b]$ *then* $I_a^\alpha f \in C_{\gamma-\alpha}[a, b]$ *and for* $x \in (a, b]$ *we have*

$$\left| I_a^\alpha f(x) \right| \leq \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} (x-a)^{\alpha-\gamma} \|f\|_{C_\gamma[a,b]}. \tag{10}$$

Proof. From Lemmas 6 and 7 we have $I^\alpha f \in C(a, b)$ and $\lim_{x\to a^+} (x-a)^{\gamma-\alpha} I_a^\alpha f(x)$ exists and is finite. Thus $I_a^\alpha f \in C_{\gamma-\alpha}[a, b]$. Now for $x \in (a, b]$ we have

$$\left| I_a^\alpha f(x) \right| = \frac{1}{\Gamma(\alpha)} \left| \int_a^x (x-t)^{\alpha-1} (t-a)^{-\gamma} (t-a)^\gamma f(t) dt \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \|f\|_{C_\gamma[a,b]} \int_a^x (x-t)^{\alpha-1} (t-a)^{-\gamma} dt.$$

The relation (10) follows by applying Lemma 3.

Consequently, from Lemma 8 we have the following property.

Lemma 9 *Let* $\alpha > 0$, $\gamma < 1$, *and* $r \in \mathbb{R}$. *If* $f \in C_\gamma[a, b]$ *then* $(x-a)^{-r} I_a^\alpha f \in C_{\gamma+r-\alpha}[a, b]$. *In particular, if* $\gamma + r - \alpha < 1$ *then* $I^\alpha f \in CL(a, b)$.

Later, the following observation is important.

Lemma 10 *Let* $\alpha > 0$ *and* $r < \alpha$. *If* $f \in CL(a, b)$ *then* $(x-a)^{-r} I_a^\alpha f \in CL(a, b)$.

Proof. When $r \leq 0$ the result follows clearly from Lemma 6. When $r > 0$ it follows from Lemma 6 that $(x-a)^{-r} I_a^\alpha f \in C(a, b)$ and we only need to show that $(x-a)^{-r} I_a^\alpha f \in L(a, b)$.

For any $x \in (a, b]$ we have the following inequality.

$$\left| I_a^\alpha f(x) \right| = \frac{1}{\Gamma(\alpha)} \left| \int_a^x (x-t)^{\alpha-1} f(t) dt \right|$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-r-1} (x-t)^r \left| f(t) \right| dt$$

$$\leq \frac{(x-a)^r}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-r-1} \left| f(t) \right| dt$$

$$= \frac{\Gamma(\alpha-r)}{\Gamma(\alpha)} (x-a)^r I_a^{\alpha-r} \left| f(x) \right|.$$

Or,

$$(x-a)^{-r} \left| I_a^\alpha f(x) \right| \leq \frac{\Gamma(\alpha-r)}{\Gamma(\alpha)} I_a^{\alpha-r} \left| f(x) \right|.$$

From Lemma 4 the right-hand side is in $L(a, b)$ and thus $(x-a)^{-r} I_a^\alpha f \in L(a, b)$. This completes the proof.

The following lemma follows by direct calculations using Dirichlet formula, [36].

Lemma 11 *Let $\alpha \geq 0$, $\beta \geq 0$, and $f \in CL(a, b)$. Then*

$$I_a^\alpha I_a^\beta f(x) = I_a^{\alpha+\beta} f(x),$$ (11)

for all $x \in (a, b]$.

Lemma 11 leads to the left inverse operator.

Lemma 12 *Let $\alpha > 0$ and $f \in CL(a, b)$. Then*

$$D_a^\alpha I_a^\alpha f(x) = f(x),$$ (12)

for all $x \in (a, b]$.

Now we present a version of the fundamental theorem of fractional calculus.

Lemma 13 *Let $0 < \alpha < 1$. If $f \in C(a, b)$ and $D_a^\alpha f \in CL(a, b)$, then $f \in CL(a, b)$, $I_a^\alpha f(a^+)$ exists and is finite, and*

$$I_a^\alpha D_a^\alpha f(x) = f(x) - \frac{I_a^{1-\alpha} f(a^+)}{\Gamma(\alpha)} (x-a)^{\alpha-1},$$ (13)

for all $x \in (a, b]$.

Proof. From Lemma 12 we have for all $x \in (a, b]$ the relation

$$D_a^\alpha I_a^\alpha D_a^\alpha f(x) = D_a^\alpha f(x),$$

which we can write as

$$D_a^\alpha \left[f - I_a^\alpha D_a^\alpha f \right](x) = 0.$$

This implies that

$$f(x) - I_a^\alpha D_a^\alpha f(x) = c(x-a)^{\alpha-1},$$ (14)

for some constant c. Since Lemma 6 implies that $I_a^\alpha D_a^\alpha f \in CL(a, b)$, we also have $f \in CL(a, b)$. Also, if we apply $I_a^{1-\alpha}$ to both sides of (14) we obtain

$$I_a^{1-\alpha} f(x) = I_a D_a^\alpha f(x) = c\Gamma(\alpha).$$

Taking the limit yields $I_a^{1-\alpha} f(a^+) = c\Gamma(\alpha)$ and (13) is obtained.

In the proof of our existence and uniqueness result we will use the following results.

Lemma 14 *Let $\gamma \in \mathbb{R}$, $a < c < b$, $g \in C_\gamma[a, c]$, $g \in C[c, b]$ and g is continuous at c. Then $g \in C_\gamma[a, b]$.*

Theorem 15 ([1], **Banach Fixed Point Theorem**) *Let (U, d) be a nonempty complete metric space. Let $T : U \to U$ be a map such that for every $u, v \in U$, the relation*

$$d(Tu, Tv) \leq w d(u, v), \quad 0 \leq w < 1,$$

holds. Then the operator T has a unique fixed point $u^* \in U$.

3 Sequential derivative

In this section we define the sequential derivative and integral that we consider and develop some of their properties. In particular, we derive the composition identities.

Definition 16 *Let $\alpha > 0$, $\beta > 0$, $r \in \mathbb{R}$. Let $f \in CL(a, b)$. Define the sequential integral $\mathcal{J}_{r,a}^{\alpha,\beta} f$ and the sequential derivative $\mathcal{D}_{r,a}^{\alpha,\beta} f$ by*

$$\mathcal{J}_{r,a}^{\alpha,\beta}f(x) = I_a^\alpha(x-a)^{-r}I_a^\beta f(x),\tag{15}$$

and

$$\mathcal{D}_{r,a}^{\alpha,\beta}f(x) = D_a^\alpha(x-a)^r D_a^\beta f(x),\tag{16}$$

if the right-hand sides exist.

From Lemma 3 we have the following formula for the power function.

Lemma 17 *Let* $\alpha > 0$, $\beta > 0$, $r \in \mathbb{R}$. *If*

$$\rho > \max\{-1, \beta - r - 1\},$$

then for $x > a$,

$$\mathcal{D}_{r,a}^{\alpha,\beta}(x-a)^\rho = \frac{\Gamma(\rho+1)}{\Gamma(\rho-\beta+1)}\frac{\Gamma(\rho+r-\beta+1)}{\Gamma(\rho+r-\beta-\alpha+1)}(x-a)^{\rho+r-\beta-\alpha}.\tag{17}$$

Moreover, from Lemmas 3 and 17 we have the following vanishing derivatives.

Lemma 18

(a) *Let* $\alpha > 0$, $0 < \beta < 1$, $r \in \mathbb{R}$. *Then for* $x > a$,

$$\mathcal{D}_{r,a}^{\alpha,\beta}(x-a)^{\beta-1} = 0.\tag{18}$$

(b) *Let* $0 < \alpha < 1$ *and* $\beta > 0$. *Let* $r \in \mathbb{R}$ *be such that* $r < \alpha + \beta$. *Then for* $x > a$,

$$\mathcal{D}_{r,a}^{\alpha,\beta}(x-a)^{\alpha+\beta-r-1} = 0.\tag{19}$$

Lemma 19 (Left inverse) *Let* $\alpha > 0$, $\beta > 0$, *and* $r \in \mathbb{R}$. *If* $f \in CL(a, b)$ *such that* $(x-a)^{-r}I_a^\alpha f \in CL(a,b)$ *then*

$$\mathcal{D}_{r,a}^{\alpha,\beta}\mathcal{J}_{r,a}^{\beta,\alpha}f(x) = f(x),\tag{20}$$

for all $x \in (a, b]$.

Proof. Relation (20) follows directly by applying Lemma 12 twice.

From Lemmas 8 and 9 we have the following mapping property of the operator $\mathcal{J}_{r,a}^{\beta,\alpha}$.

Lemma 20 *Let* $\alpha > 0$, $\beta > 0$, *and* $r < 1 + \alpha$. *Let* $0 \le \gamma < \min\{1, 1 + \alpha - r\}$. *If* $f \in C_\gamma[a, b]$ *then* $\mathcal{J}_{r,a}^{\beta,\alpha}f \in C_{\gamma+r-\alpha-\beta}[a,b]$ *and for* $x \in (a, b]$ *we have*

$$\left|\mathcal{J}_{r,a}^{\beta,\alpha}f(x)\right| \le k\|f\|_{C_\gamma[a,b]}(x-a)^{\alpha+\beta-r-\gamma},\tag{21}$$

where

$$k = \frac{\Gamma(1-\gamma)\Gamma(1+\alpha-r-\gamma)}{\Gamma(1-\gamma+\alpha)\Gamma(1+\alpha+\beta-r-\gamma)}.\tag{22}$$

Lemma 20 implies the following.

Lemma 21 *Let* $\alpha > 0$, $\beta > 0$, *and* $r < 1 + \alpha$. *Let* $0 \le \gamma < \min\{1, 1 + \alpha - r\}$. *If* $r \le \alpha + \beta$, *then* $\mathcal{J}_{r,a}^{\beta,\alpha}$ *is bounded in* $C_\gamma[a, b]$ *and*

$$\left\| \mathcal{J}_{r,a}^{\beta,\alpha} f \right\|_{C_\gamma[a,b]} \le k(b-a)^{\alpha+\beta-r} \left\| f \right\|_{C_\gamma[a,b]}, \tag{23}$$

where k is given by (22).

Proof. Since $\gamma + r - \alpha - \beta \le \gamma$, then from Lemma 20 we have

$$\mathcal{J}_{r,a}^{\beta,\alpha} f \in C_{\gamma+r-\alpha-\beta}[a,b] \subset C_\gamma[a,b].$$

The bound in (23) follows by multiply (21) by $(x-a)^\gamma$ and taking the maximum.

As a special case of Lemma 21, we have

Lemma 22 *Let* $\alpha > 0$, $\beta > 0$, *and* $r < \min\{\alpha + 1, \alpha + \beta\}$. *Then* $\mathcal{J}_{r,a}^{\beta,\alpha}$, *maps* C[a, b] *into* C[a, b] *and*

$$\left\| \mathcal{J}_{r,a}^{\beta,\alpha} f \right\|_{C[a,b]} \le L(b-a)^{\alpha+\beta-r} \left\| f \right\|_{C[a,b]}, \tag{24}$$

where

$$L = \frac{\Gamma(1+\alpha-r)}{\Gamma(1+\alpha)\Gamma(1+\alpha+\beta-r)}. \tag{25}$$

The following is an analogous result to the result for the Riemann-Liouville integral proved in [10].

Lemma 23 *Let* $\alpha > 0$, $\beta > 0$, *and* $r < \alpha$. *Let* $f \in CL(a, c)$. *Let*

$$g(x) = \frac{1}{\Gamma(\beta)} \int_a^c (x-t)^{\beta-1} (t-a)^{-r} I_a^\alpha f(t) \, dt.$$

Then

$$\lim_{x \to c^+} g(x) = \mathcal{J}_{r,a}^{\beta,\alpha} f(c).$$

Proof. Since $r < \alpha$, Lemma 10 implies that $(x-a)^{-r} I_a^\alpha f(x) \in CL(a, c)$. Thus $\mathcal{J}_{r,a}^{\beta,\alpha} f(c)$ is finite and

$$\left| g(x) - \mathcal{J}_{r,a}^{\beta,c} f(c) \right| \le \frac{1}{\Gamma(\beta)} \int_a^c k(x,t)(t-a)^{-r} I_a^\alpha \left| f(t) \right| \, dt,$$

where

$$k(x,t) = \left| (c-t)^{\beta-1} - (x-t)^{\beta-1} \right|.$$

Since $\lim_{x \to c^+} k(x,t) = 0$, the limit of the right-hand side vanishes and the proof is complete.

The following lemma relates the fractional derivative $\mathcal{D}_{r,a}^{\alpha,\beta}$ to the Riemann-Liouville derivative D_a^β.

Lemma 24 *Let* $0 < \alpha < 1$, $\beta \ge 0$, *and* $r \in \mathbb{R}$. *If* $D_a^\beta \gamma(x) \in C(a, b)$ *and* $D_{r,a}^{\alpha,\beta} \gamma \in CL(a, b)$ *then* $(x-a)^r D_a^\beta \gamma \in CL(a, b)$, $I_a^{1-\alpha} \left[(x-a)^r D_a^\beta \gamma \right] (a^+)$ *exists and finite, and*

$$D_a^\beta \gamma(x) = (x-a)^{-r} I_a^\alpha \mathcal{D}_{r,a}^{\alpha,\beta} \gamma(x) + \frac{I^{1-\alpha}\left[(x-a)^r D_a^\beta \gamma\right](a^+)}{\Gamma(\alpha)}(x-a)^{\alpha-r-1}, \tag{26}$$

for all $x \in (a, b]$. If in addition, $r < \alpha$ then $D_a^\beta \gamma \in CL(a,b)$.

Proof. Clearly $D_a^\beta \gamma \in C(a,b)$ implies that $(x-a)^r D_a^\beta \gamma \in C(a,b)$. Thus we can apply Lemma 13 to $(x-a)^r D_a^\beta \gamma$ and obtain

$$I_a^\alpha \mathcal{D}_{r,a}^{\alpha,\beta} \gamma(x) = I_a^\alpha D_a^\alpha (x-a)^r D_a^\beta \gamma(x)$$

$$= (x-a)^r D_a^\beta \gamma(x) - \frac{I_a^{1-\alpha}\left[(x-a)^r D_a^\beta \gamma\right](a^+)}{\Gamma(\alpha)}(x-a)^{\alpha-1}.$$

By multiplying both sides by $(x-a)^r$ we obtain (26). If $r < \alpha$ then Lemma 10 implies that $(x-a)^{-r} I_a^\alpha \mathcal{D}_{r,a}^{\alpha,\beta} \gamma(x) \in CL(a,b)$ and thus from (26) we have $D_a^\beta \gamma \in CL(a,b)$. This proves the result.

The Next lemma gives an analogous result to the fundamental theorem of calculus in terms of the operators $\mathcal{D}_{r,a}^{\alpha,\beta}$ and $\mathcal{J}_{r,a}^{\alpha,\beta}$.

Lemma 25 *Let* $0 < \alpha < 1$ *and* $0 < \beta < 1$. *Let* $y \in C(a, b)$ *be such that* $\mathcal{D}_{r,a}^{\alpha,\beta} \gamma \in CL(a,b)$ *and* $D_a^\beta \gamma \in CL(a,b)$. *Then both* $I_a^{1-\alpha}\left[(x-a)^r D_a^\beta \gamma\right](a^+)$ *and* $I_a^{1-\beta} \gamma(a^+)$ *exist,* $y \in CL(a, b)$, *and*

$$\mathcal{J}_{r,a}^{\beta,\alpha} \mathcal{D}_{r,a}^{\alpha,\beta} \gamma(x) = \gamma(x) - \frac{\left[I_a^{1-\beta} \gamma\right](a^+)}{\Gamma(\beta)}(x-a)^{\beta-1}$$
$$- \frac{\left[I_a^{1-\alpha}(x-a)^r D_a^\beta \gamma\right](a^+)}{\Gamma(\alpha)} \frac{\Gamma(\alpha+r)}{\Gamma(\alpha+\beta-r)}(x-a)^{\alpha+\beta-r-1}, \tag{27}$$

for all $x \in (a, b]$.

Proof. By applying Lemma 13 twice we obtain

$$\mathcal{J}_{r,a}^{\beta,\alpha} \mathcal{D}_{r,a}^{\alpha,\beta} \gamma(x) = I_a^\beta (x-a)^{-r} I_a^\alpha D_a^\alpha (x-a)^r D_a^\beta \gamma(x)$$

$$= I^\beta (x-a)^{-r}\left[(x-a)^r D_a^\beta \gamma(x) - \frac{\left[I_a^{1-\alpha}(x-a)^r D_a^\beta \gamma\right](a^+)}{\Gamma(\alpha)}(x-a)^{\alpha-1}\right]$$

$$= I_a^\beta D_a^\beta \gamma - \frac{\left[I_a^{1-\alpha}(x-a)^r D_a^\beta \gamma\right](a^+)}{\Gamma(\alpha)} I_a^\beta (x-a)^{\alpha-1-r}$$

$$= \gamma(x) - \frac{\left[I_a^{1-\beta} \gamma\right](a^+)}{\Gamma\beta}(x-a)^{\beta-1}$$

$$- \frac{\left[I_a^{1-\alpha}(x-a)^r D_a^\beta \gamma\right](a^+)}{\Gamma(\alpha)} \frac{\Gamma(\alpha+r)}{\Gamma(\alpha+\beta-r)}(x-a)^{\alpha+\beta-r-1}.$$

4 Cauchy-type problem and equivalency

Consider the Cauchy-type problem

$$\mathcal{D}_{r,a}^{\alpha,\beta} \gamma(x) = f(x, \gamma(x)), \quad a < x \le b, \quad 0 < \alpha < 1, \quad 0 < \beta < 1, r \in \mathbb{R}, \tag{28}$$

$$\lim_{x \to a^+} I_a^{1-\alpha} \left[(x-a)^r D_a^\beta y \right](x) = c_1,$$ (29)

$$\lim_{x \to a^+} I_a^{1-\beta} y(x) = c_0,$$ (30)

where c_0 and c_1 are real numbers.

In this problem there are two conditions even when $0 < \alpha + \beta < 1$. The two initial conditions are based on the composition (27). The condition (29) is of one order less than that in the differential Equation (28) while the condition (30) is one order less than the equation for $D_a^\beta y$.

In addition, from [1, Lemma 3.2], the condition (30) follows from the condition

$$\lim_{x \to a^+} \left[(x-a)^{1-\beta} y(x) \right] = \frac{c_0}{\Gamma(\beta)},$$ (31)

and if $0 < \alpha - r < 1$ then (29) follows from the condition

$$\lim_{x \to a^+} \left[(x-a)^{1-\alpha+r} D_a^\beta y(x) \right] = \frac{c_1}{\Gamma(\alpha)}.$$ (32)

Consequently, the results below hold under conditions of the type (31) and (32).

Now, Based on the composition in Lemma 24, in the next theorem we establish an equivalence with the following fractional integro-differential equation:

$$D_a^\beta y = (x-a)^{-r} I_a^\alpha f[x, y(x)] + \frac{c_1}{\Gamma(\alpha)} (x-a)^{\alpha-r-1},$$ (33)

$$\lim_{x \to a^+} I_a^{1-\beta} y(x) = c_0.$$ (34)

Theorem 26 *Let $0 < \alpha < 1$, $\beta > 0$ and $r \in \mathbb{R}$. Let $f : (a, b] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(., y(.)) \in C_{1-\alpha}[a, b]$ for any $y \in C_{1-\alpha}[a, b]$. Then we have the following.*

(a) If $y \in C_{1-\alpha}[a, b]$ satisfies (33) and (34) then $y(x)$ satisfies (28-30).

(b) If $y \in C_{1-\alpha}[a, b]$ with $D_a^\beta y \in C(a, b)$ satisfy (28-30), then $y(x)$ satisfies (33-34).

Proof.

For assertion (a), let $y \in C_{1-\alpha}[a, b]$ satisfy (33-34). We multiply (33) by $(x - a)^r$ to obtain

$$(x-a)^r D_a^\beta y = I_a^\alpha f(x, y(x)) + \frac{c_1}{\Gamma(\alpha)} (x-a)^{\alpha-1}.$$ (35)

Next we apply D_a^α to both sides of (35) to obtain (28). As for the initial condition, apply $I_a^{1-\alpha}$ to both sides of (35) and then take the limit to obtain (29).

For assertion (b), let $y \in C_{1-\alpha}[a, b]$ satisfy (28-30). Since $f(x, y(x)) \in C_{1-\alpha}[a, b]$, then from (28) we have $\mathcal{D}_{r,a}^{\alpha,\beta} y \in C_{1-\alpha}[a, b]$. Since also by hypothesis $D_a^\beta y \in C(a, b)$, we can apply Lemma 24 and the formula (26) holds. By substituting the initial condition we obtain (33). This completes the proof.

The composition in Lemma 25 leads to the nonlinear integral equation,

$$y(x) = \mathcal{J}_{r,a}^{\beta,\alpha} f(x, y(x)) + \frac{c_0}{\Gamma(\beta)}(x-a)^{\beta-1} + \frac{c_1 \Gamma(\alpha-r)}{\Gamma(\alpha)\Gamma(\alpha+\beta-r)}(x-a)^{\alpha+\beta-r-1}. \quad (36)$$

The following theorem establishes an equivalence with this equation.

Theorem 27 *Let* $0 < \alpha < 1$, $0 < \beta < 1$ *and* $r < \alpha$. *Let* $f : (a, b] \times \mathbb{R} \to \mathbb{R}$ *be a function such that* $f(., y(.)) \in C_{1-\beta}[a, b]$ *for any* $y \in C_{1-\beta}[a, b]$. *Then the following statements hold.*

(a) If $y \in C_{1-\beta}[a, b]$ *satisfies the integral Equation (36) then* $y(x)$ *satisfies the Cauchy-type problem (28-30).*

(b) If $y \in C_{1-\beta}[a, b]$ *with* $D_a^\beta y \in C(a, b)$ *satisfies the Cauchy-type problem (28-30), then* $y(x)$ *satisfies the integral Equation (36).*

Proof. (a). Let $y \in C_{1-\beta}[a, b]$ satisfy the integral Equation (36). By hypothesis we have $f \in C_{1-\beta}[a, b]$. Moreover, from Lemma 9 and the hypothesis $r < \alpha$, we have

$$(x-a)^{-r} I_a^\alpha f \in C_{1+r-\alpha-\beta}[a, b] \subset CL(a, b).$$

Thus the hypothesis of Lemmas 18 and 19 are satisfied. Applying the operator $\mathcal{D}_{r,a}^{\alpha,\beta}$ to both sides of (36) and using Lemmas 18 and 19 yields (28) as follows.

$$\begin{aligned}
\mathcal{D}_{r,a}^{\alpha,\beta} y(x) &= \mathcal{D}_{r,a}^{\alpha,\beta} \mathcal{J}_{r,a}^{\beta,\alpha} f(x, y(x)) + \\
&\quad \mathcal{D}_{r,a}^{\alpha,\beta}\left[\frac{c_0}{\Gamma(\beta)}(x-a)^{\beta-1} + \frac{c_1 \Gamma(\alpha-r)}{\Gamma(\alpha)\Gamma(\alpha+\beta-r)}(x-a)^{\alpha+\beta-r-1}\right] \\
&= f(x, y(x)).
\end{aligned}$$

Next, applying $I_a^{1-\beta}$ to both sides of (36) yields

$$I_a^{1-\beta} y(x) = \mathcal{J}_{r,a}^{1,\alpha} f(x, y(x)) + c_0 + \frac{c_1}{\Gamma(\alpha)}\frac{\Gamma(\alpha-r)}{\Gamma(\alpha+\beta-r)}\frac{\Gamma(\alpha+\beta-r)}{\Gamma(\alpha-r+1)}(x-a)^{\alpha-r}. \quad (37)$$

Since $r < \alpha$, taking the limit we obtain the initial condition (30).

Applying $I_a^{1-\alpha}(x-a)^r D_a^\beta$ to both sides of (36) and using Lemmas 3, 11, and 12 yields

$$\begin{aligned}
I_a^{1-\alpha}(x-a)^r D_a^\beta y(x) &= I_a^{1-\alpha}(x-a)^r D_a^\beta \left[\mathcal{J}_{r,a}^{\beta,\alpha} f(x, y(x))\right] + \\
&\quad I_a^{1-\alpha}(x-a)^r D_a^\beta\left[\frac{c_1 \Gamma(\alpha-r)}{\Gamma(\alpha)\Gamma(\alpha+\beta-r)}(x-a)^{\alpha+\beta-r-1}\right] \\
&= I_a^1 f(x, y(x)) + c_1.
\end{aligned}$$

Again, taking the limit we obtain the initial condition (29).

(b). Let $y \in C_{1-\beta}[a, b]$ satisfy (28-30). Since $f(x, y(x)) \in CL(a, b)$ then from (28), $\mathcal{D}_{r,a}^{\alpha,\beta} y \in CL(a, b)$. Since $r < \alpha$ then from Lemma 24 we have $D_a^\beta y \in CL(a, b)$. Thus we can apply Lemma 25 and the formula (27) holds. By using the initial conditions we obtain (36). This completes the proof.

In the next section we use this equivalence to prove the existence and uniqueness of solutions.

5 Existence and uniqueness of the solution of the Cauchy-type problem

In this section we prove an existence and uniqueness result for the Cauchy-type problem (28-30) using the integral Equation (36). For this purpose we introduce the following lemma.

Lemma 28 *Let $0 < r < \alpha < 1$, $0 < \beta < 1$, then the fractional differentiation operator $\mathcal{J}_{r,a}^{\beta,\alpha}$ is bounded in $C_{1-\beta}[a, b]$ and*

$$\left\| \mathcal{J}_{r,a}^{\beta,\alpha} f \right\|_{C_{1-\beta}[a,b]} \leq K(b-a)^{\alpha+\beta-r} \|f\|_{C_{1-\beta}[a,b]}, \tag{38}$$

where

$$K = \frac{\Gamma(\beta)\Gamma(\alpha+\beta-r)}{\Gamma(\alpha+\beta)\Gamma(\alpha+2\beta-r)}. \tag{39}$$

Proof. Clearly from the hypothesis we have $r < \alpha + \beta$ and $0 < 1 - \beta < \min\{1, 1 + \alpha - r\}$. Thus the result follows by taking $\gamma = 1 - \beta$ in Lemma 21.

Theorem 29 *Let $0 < r < \alpha < 1$, $0 \leq \beta < 1$. Let $f : (a, b] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(.,y(.)) \in C_{1-\beta}[a, b]$ for any $y \in C_{1-\beta}[a, b]$ and the condition:*

$$\left| f(x, y_1) - f(x, y_2) \right| \leq A \left| y_1 - y_2 \right|, \quad A > 0, \tag{40}$$

is satisfied for all $x \in (a, b]$ and for all $y_1, y_2 \in \mathbb{R}$.

Then the Cauchy-type problem (28-30) has a solution $y \in C_{1-\beta}[a, b]$. Furthermore, if for this solution $D_a^\beta y \in C(a, b)$, then this solution is unique.

Proof.

According to Theorem 27(a), we can consider the existence of an $C_{1-\beta}[a, b]$ solution for the integral Equation (36). This equation holds in any interval $(a, x_1) \subseteq (a, b]$, $a < x_1 < b$. Choose x_1 such that

$$w_1 := AK(x_1 - a)^{\alpha+\beta-r} < 1,$$

where K is given by (39). We rewrite the integral equation in the form $y(x) = Ty(x)$, where

$$Ty(x) = v_0(x) + \mathcal{J}_{r,a}^{\beta,\alpha} f(x, y(x)),$$

and

$$v_0(x) = \frac{c_0}{\Gamma(\beta)}(x-a)^{\beta-1} + \frac{c_1 \Gamma(\alpha-r)}{\Gamma(\alpha)\Gamma(\alpha+\beta-r)}(x-a)^{\alpha+\beta-r-1}.$$

Since $r < \alpha$ then $v_0 \in C_{1-\beta}[a, b]$. Thus, it follows from Lemma 28 that if $y \in C_{1-\beta}[a, x_1]$ then $Ty \in C_{1-\beta}[a, x_1]$. Also, for any y_1, y_2 in $C_{1-\beta}[a, x_1]$, we have

$$\begin{aligned}
\left\| Ty_1 - Ty_2 \right\|_{C_{1-\beta}[a,x_1]} &\leq \left\| \mathcal{J}_{r,a}^{\beta,\alpha} \left\{ |f(x, y_1(x)) - f(x, y_2(x))| \right\} \right\|_{C_{1-\beta}[a,x_1]} \\
&\leq A \left\| \mathcal{J}_{r,a}^{\beta,\alpha} \left\{ |y_1(x) - y_2(x)| \right\} \right\|_{C_{1-\beta}[a,x_1]} \\
&\leq AK(x_1 - a)^{\alpha+\beta-r} \left\| y_1 - y_2 \right\|_{C_{1-\beta}[a,x_1]} \\
&\leq w_1 \left\| y_1 - y_2 \right\|_{C_{1-\beta}[a,x_1]}, \quad 0 < w_1 < 1.
\end{aligned}$$

Hence by Theorem 15 there exists a unique solution $y^* \in C_{1-\beta}[a, x_1]$ to the Equation (36) on the interval $(a, x_1]$.

If $x_1 \neq b$ then we consider the interval $[x_1, b]$. On this interval we consider solutions $y \in C[x_1, b]$ for the equation

$$y(x) = Ty(x) := v_{01}(x) + \mathcal{J}_{r,x_1}^{\beta,\alpha} f(x, y(x)), \quad x \in [x_1, b], \tag{41}$$

where

$$v_{01}(x) = \frac{y_0}{\Gamma(\beta)} (x - a)^{\beta-1} +$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{x_1} \left[(x - t)^{\beta-1} (t - a)^{-r} \left\{ \int_a^t (t - s)^{\alpha-1} f(s, y(s)) ds \right\} \right] dt.$$

Now we select $x_2 \in (x_1, b]$ such that

$$w_2 := AL(x_2 - x_1)^{\alpha+\beta-r} < 1,$$

where L is given by (25). Since the solution is uniquely defined on the interval $(a, x_1]$, we can consider $v_{01}(x)$ to be a known function. For $y_1, y_2 \in C[x_1, x_2]$, it follows from the Lipschitz condition and Lemma 22 that

$$\begin{aligned}
\left\| Ty_1 - Ty_2 \right\|_{C[x_1,x_2]} &\leq \left\| \mathcal{J}_{r,x_1}^{\beta,\alpha} \left\{ |f(x, y_1(x)) - f(x, y_2(x))| \right\} \right\|_{C[x_1,x_2]} \\
&\leq A \left\| \mathcal{J}_{r,x_1}^{\beta,\alpha} \left\{ |y_1(x) - y_2(x)| \right\} \right\|_{C[x_1,x_2]} \\
&\leq AL(x_2 - x_1)^{\alpha+\beta-r} \left\| y_1 - y_2 \right\|_{C[x_1,x_2]} \\
&\leq w_2 \left\| y_1 - y_2 \right\|_{C[x_1,x_2]}.
\end{aligned}$$

Since $0 < w_2 < 1$, T is a contraction. Since $f(x, y(x)) \in C[x_1, x_2]$ for any $y \in C[x_1, x_2]$, then $\mathcal{J}_{r,x_1}^{\beta,\alpha} f \in C[x_1, x_2]$. Moreover, clearly $v_{01}(x)$ is in $C[x_1, x_2]$. Thus the right-hand side of (41) is in $C[x_1, x_2]$. Therefore T maps $C[x_1, x_2]$ into itself. By Theorem 15, there exists a unique solution $y_1^* \in C[x_1, x_2]$ to the equation on the interval $[x_1, x_2]$. Moreover, it follows from Lemma 23 that $y_1^*(x_1) = y_0^*(x_1)$. Therefore if

$$y^*(x) = \begin{cases} y_0^*(x), & a < x \leq x_1, \\ y_1^*(x), & x_1 < x \leq x_2, \end{cases}$$

then by Lemma 14, $y^* \in C_{1-\beta}[a, x_2]$. So y^* is the unique solution of (36) in $C_{1-\beta}[a, x_2]$ on the interval $(a, x_2]$.

If $x_2 \neq b$, we repeat the process as necessary, say M - 2 times, to obtain the unique solutions $y_k^* \in C_{1-\beta}[x_k, x_{k+1}]$, $k = 2, 3, \ldots, M$, where $a = x_0 < x_1 < \cdots < x_M = b$, such that

$$w_{k+1} = AL(x_{k+1} - x_k)^{\alpha+\beta-r} < 1.$$

As a result we have the unique solution $y^* \in C_{1-\beta}[a, b]$ of (36) given by

$$y^*(x) = y_k^*(x), \quad x \in (x_k, x_{k+1}], \quad k = 0, 1, \ldots, M - 1. \tag{42}$$

This solution is also a solution for (28-30).

If $D_a^\beta \gamma \in C(a, b)$ then the uniqueness follows from part (b) of Theorem 27. This completes the proof.

Competing interests
The author declares that they have no competing interests.

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