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Existence of solutions for quasilinear elliptic equations with superlinear nonlinearities

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Abstract

Working in a weighted Sobolev space, a new result involving superlinear nonlinearities for a quasilinear elliptic boundary value problem in a domain in R^N is established. The proofs rely on the Galerkin method, Brouwer's theorem and a new weighted compact Sobolev-type embedding theorem due to V.L. Shapiro.

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1 Introduction

Consider the following quasilinear elliptic problem:

$$\begin{cases} \mathcal{M}u = [\lambda_1 u + f(x, u)]\rho - G, & x \in \Omega, \\ u \in H_{p,q,\rho}^1(\Omega, \Gamma), \end{cases} \quad (1.1)$$

where Ω is an open (possibly unbounded) set in R^N ($N \geq 1$), λ_1 is the first eigenvalue of \mathcal{L} ((2.3) below), and \mathcal{M} is a singular quasilinear elliptic operator defined by

$$\mathcal{M}u = - \sum_{i,j=1}^N D_i [p_i^{\frac{1}{2}}(x) p_j^{\frac{1}{2}}(x) \sigma_i^{\frac{1}{2}}(u) \sigma_j^{\frac{1}{2}}(u) b_{ij}(x) D_j u] + b_0 \sigma_0 q u. \quad (1.2)$$

The nonlinear part $f(x, u)$ in Eq. (1.1) satisfies certain superlinear conditions.

There have been many results for quasilinear elliptic equations under the conditions of which the nonlinearities satisfy sublinear or linear growth in a weighted Sobolev space. One can refer to [1–6].

However, there seem to be relatively few papers that consider the quasilinear elliptic equations with superlinearity, because the compactly embedding theorem cannot be obtained easily.

The aim of this paper is to obtain an existence result for problem (1.1). Our methods combine the Galerkin-type techniques, Brouwer's fixed-point theorem, and a new compactly embedding theorem established by V.L. Shapiro in [7].

This paper is organized as follows. In Section 2, we introduce some necessary assumptions and main results. In Section 3, four fundamental lemmas are established. In Section 4, the proofs of the main results are given.

2 Assumptions and main results

In this section, we introduce some assumptions and give the main results in this paper.

Let $\Gamma \subset \partial\Omega$ be a fixed closed set (it may be the empty set) and $\rho(x), p_i(x), q(x) \in C^0(\Omega)$ be weight functions. $q(x)$ is nonnegative (maybe identically zero). Denote $p(x)$ as the vector function $(p_1(x), p_2(x), \dots, p_N(x))$.

Consider the following pre-Hilbert spaces

$$C_\rho^0(\Omega) = \left\{ u \in C^0(\Omega) \mid \int_\Omega |u|^2 \rho < \infty \right\}$$

with inner product $\langle u, v \rangle_\rho = \int_\Omega uv\rho, \forall u, v \in C_\rho^0(\Omega)$, and

$$C_{p,q,\rho}^1(\Omega, \Gamma) = \left\{ u \in C^0(\bar{\Omega}) \cap C^2(\Omega) \mid u(x) = 0, \forall x \in \Gamma; \int_\Omega \left[\sum_{i=1}^N |D_i u|^2 p_i + u^2(q + \rho) \right] < \infty \right\}$$

with the inner product

$$\langle u, v \rangle_{p,q,\rho} = \int_\Omega \left(\sum_{i=1}^N p_i D_i u D_i v + (q + \rho) uv \right) \tag{2.1}$$

$\forall u, v \in C_{p,q,\rho}^1(\Omega)$ where $D_i u = \frac{\partial u}{\partial x_i}, i = 1, \dots, N$. Let $L_\rho^2(\Omega)$ be the Hilbert space obtained through the completion of $C_\rho^0(\Omega)$ by using the method of Cauchy sequences with respect to the norm $\|u\|_\rho = \langle u, u \rangle_\rho^{\frac{1}{2}}$, and $H_{p,q,\rho}^1(\Omega, \Gamma)$ be the completion of the space $C_{p,q,\rho}^1(\Omega, \Gamma)$ with the norm $\|u\|_{p,q,\rho} = \langle u, u \rangle_{p,q,\rho}^{\frac{1}{2}}$. Similarly, we may have $L_{p_i}^2(\Omega) (i = 1, \dots, N)$ and $L_q^2(\Omega)$. Consequently, (2.1) may lead to

$$\langle u, v \rangle_{p,q,\rho} = \sum_{i=1}^N \langle D_i u, D_i v \rangle_{p_i} + \langle u, v \rangle_\rho + \langle u, v \rangle_q. \tag{2.2}$$

Definition 2.1 For the quasilinear differential operator \mathcal{M} , the two-form is

$$\mathcal{M}(u, v) = \int_\Omega \sum_{i,j=1}^N p_i^{\frac{1}{2}} p_j^{\frac{1}{2}} \sigma_i^{\frac{1}{2}}(u) \sigma_j^{\frac{1}{2}}(u) b_{ij}(x) D_j u D_i v + \langle b_0 \sigma_0(u) u, v \rangle_q, \\ \forall u, v \in H_{p,q,\rho}^1(\Omega, \Gamma).$$

For the linear differential operator,

$$\mathcal{L}u = - \sum_{i,j=1}^N D_i \left[p_i^{\frac{1}{2}} p_j^{\frac{1}{2}} a_{ij}(x) D_j u \right] + a_0 q u, \tag{2.3}$$

the two-form is

$$\mathcal{L}(u, v) = \int_\Omega \sum_{i,j=1}^N p_i^{\frac{1}{2}} p_j^{\frac{1}{2}} a_{ij}(x) D_j u D_i v + \langle a_0 u, v \rangle_q, \quad \forall u, v \in H_{p,q,\rho}^1(\Omega, \Gamma).$$

Definition 2.2 (Ω, Γ) is a simple- V_L region if the following conditions $(V_L - 1)$ - $(V_L - 5)$ hold:

$(V_L - 1)$ There exists a complete orthonormal system $\{\varphi_n\}_{n=1}^\infty$ in L^2_ρ . Also, $\varphi_n \in H^1_{p,q,\rho}(\Omega, \Gamma) \cap C^2(\Omega)$, $\forall n$;

$(V_L - 2)$ There exists a sequence of eigenvalues $\{\lambda_n\}_{n=1}^\infty$, corresponding to the orthonormal sequence $\{\varphi_n\}_{n=1}^\infty$, and satisfying $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, such that $\mathcal{L}(\varphi_n, v) = \lambda_n \langle \varphi_n, v \rangle_\rho$, $\forall v \in H^1_{p,q,\rho}(\Omega, \Gamma)$;

$(V_L - 3)$ $\Omega = \Omega_1 \times \dots \times \Omega_N$, where $\Omega_i \subset R$ is an open set for $i = 1, \dots, N$;

$(V_L - 4)$ For each $p_i(x)$ and $\rho(x)$ in $(V_L - 1)$ - $(V_L - 2)$, associated with each Ω_i there are positive functions $p_i^*(s), \rho_i^*(s) \in C^0(\Omega_i)$ satisfying $\int_{\Omega_i} [p_i^*(s) + \rho_i^*(s)] ds < \infty$, and $\rho(x) = \rho_1^*(x_1) \dots \rho_N^*(x_N)$, $p_i(x) = \rho_1^*(x_1) \dots \rho_{i-1}^*(x_{i-1}) p_i^*(x_i) \rho_{i+1}^*(x_{i+1}) \dots \rho_N^*(x_N)$, for $i = 1, \dots, N$;

$(V_L - 5)$ For each $\Omega_i, p_i^*, \rho_i^*$ ($i = 1, \dots, N$), there exists $h_i \in C^0(\Omega_i) \cap L^\theta_{\rho_i^*}(\Omega_i)$ for $2 < \theta < \infty$ with the property

$$|\Phi(t)| \leq h_i(t) \|\Phi\|_{p_i^*, \rho_i^*}, \quad \forall \Phi \in C^1(\Omega_i), \forall t \in \Omega_i,$$

where $\|\Phi\|_{p_i^*, \rho_i^*}^2 = \int_{\Omega_i} [p_i^*(t) |\frac{d\Phi(t)}{dt}|^2 + \rho_i^*(t) \Phi^2(t)] dt$.

There are many examples to illustrate the Simple- V_L region. One can refer to [7] and [8].

Remark 2.1 From $(V_L - 3)$ and $(V_L - 4)$, it is easy to see that $\rho(x), p_i(x)$ are positive and

$$\int_\Omega \rho(x) dx < \infty, \quad \int_\Omega p_i(x) dx < \infty, \quad i = 1, \dots, N. \tag{2.4}$$

Definition 2.3 \mathcal{M} is near-related to \mathcal{L} if the following condition holds:

$$\lim_{\|u\|_{p,q,\rho} \rightarrow \infty} \frac{\mathcal{M}(u, v) - \mathcal{L}(u, v)}{\|u\|_{p,q,\rho}} = 0, \quad \text{uniformly for } \|v\|_{p,q,\rho} \leq 1.$$

We make the following assumptions concerning the operators \mathcal{M} and \mathcal{L} : $a_{ij}(x)$ ($i, j = 1, \dots, N$) and $a_0(x)$ satisfy (so do $b_{ij}(x)$ and $b_0(x)$):

$$\begin{cases} (1) a_0(x), a_{ij}(x) \in C^0(\Omega) \cap L^\infty(\Omega), i, j = 1, \dots, N; \\ (2) a_{ij}(x) = a_{ji}(x), \quad \forall x \in \Omega, i, j = 1, \dots, N; \\ (3) a_0(x) \geq \beta_0 > 0 \quad (b_0(x) \geq \beta_1 > 0), \quad \forall x \in \Omega; \\ (4) \sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \geq c_0 |\xi|^2 \quad (\sum_{i,j=1}^N b_{ij}(x) \xi_i \xi_j \geq c_1 |\xi|^2), \\ \quad \forall x \in \Omega, \xi \in R^N \text{ where } c_0 > 0 (c_1 > 0), |\xi|^2 = \sum_{i=1}^N \xi_i^2. \end{cases} \tag{2.5}$$

It is assumed throughout the paper that $\sigma_i(u)$ ($i = 0, 1, \dots, N$) meets:

$(\sigma - 1)$ $\sigma_i(u) : H^1_{p,q,\rho}(\Omega, \Gamma) \rightarrow R$ is weakly sequentially continuous;

$(\sigma - 2)$ $\exists \eta_0, \eta_1 > 0$, s.t. $\eta_0 \leq \sigma_i(u) \leq \eta_1$, $\forall u \in H^1_{p,q,\rho}(\Omega, \Gamma)$.

$f(x, s)$ meets the following conditions:

$(f - 1)$ $f(x, s)$ satisfies the Caratheodory conditions;

$(f - 2)$ (superlinear growth condition) There exists θ with $2 < \theta < \frac{2N}{N-1}$, such that

$$|f(x, s)| \leq h_0(x) + K|s|^{\theta-1}, \quad \forall s \in R, \text{ a.e. } x \in \Omega,$$

where $h_0(x) \in L^{\theta^*}_\rho(\Omega)$. K is a nonnegative constant and $\theta^* = \frac{\theta}{\theta-1}$.

($f - 3$) There exists a nonnegative function $\tilde{h}_0(x) \in L^{\theta^*}_{\rho}(\Omega)$ and a constant $\beta^* > 0$, such that

$$sf(x, s) \leq -\beta^* |s|^2 + \tilde{h}_0(x)|s|, \quad \forall s \in R, \text{ a.e. } x \in \Omega.$$

Remark 2.2 Observing that for $N = 2$, $f(x, s) = -g(x)s|s|^{\frac{5}{3}} - \beta^*s$, where $g(x) \in C^0(\Omega) \cap L^{\infty}(\Omega)$ is a positive function, and meets both ($f - 2$) and ($f - 3$).

Now we state our main results in this paper.

Theorem 2.1 Assume that (Ω, Γ) is a simple- V_L region, the operator \mathcal{M} satisfies $(\sigma - 1)$ - $(\sigma - 2)$, and is near-related to the operator \mathcal{L} , (2.5) holds for both \mathcal{M} and \mathcal{L} , f meets ($f - 1$)- $(f - 3)$, and $G \in [H^1_{p,q,\rho}(\Omega, \Gamma)]'$ (the dual of $H^1_{p,q,\rho}(\Omega, \Gamma)$). Then the problem (1.1) has at least one nontrivial weak solution, that is, there exists a $u^* \in H^1_{p,q,\rho}(\Omega, \Gamma)$ such that

$$\mathcal{M}(u^*, v) = \lambda_1 \langle u^*, v \rangle_{\rho} + \int_{\Omega} f(x, u^*)v\rho - G(v) \quad \forall v \in H^1_{p,q,\rho}(\Omega, \Gamma).$$

To derive out Theorem 2.1, we first discuss the problem in S_n , which is the subspace of $H^1_{p,q,\rho}(\Omega, \Gamma)$ spanned by $\varphi_1, \dots, \varphi_n$. Then by virtue of the Galerkin method, the results will be extended to $H^1_{p,q,\rho}(\Omega, \Gamma)$.

3 Fundamental lemmas

In this section, we introduce and establish four fundamental lemmas. Lemmas 3.1 and 3.2 give two useful embedding theorems. Lemma 3.3 constructs some approximation solutions in S_n . Lemma 3.4 studies the properties of the approximation solutions.

Lemma 3.1 ([7]) Assume that \mathcal{L} is given by (2.3) and (Ω, Γ) is a simple- V_L region. For $N \geq 2$, then $H^1_{p,q,\rho}(\Omega, \Gamma)$ is compactly imbedded in $L^{\theta}_{\rho}(\Omega)$ for $\forall \theta$ ($2 < \theta < \frac{2N}{N-1}$); for $N = 1$, then $H^1_{p,q,\rho}(\Omega, \Gamma)$ is compactly imbedded in $L^{\theta}_{\rho}(\Omega)$ for $\forall \theta$ ($2 < \theta < \infty$).

Lemma 3.2 ([7]) Assume that \mathcal{L} is given by (2.3) and (Ω, Γ) is a simple- V_L region. Then $H^1_{p,q,\rho}(\Omega, \Gamma)$ is compactly imbedded in $L^2_{\rho}(\Omega)$.

Lemma 3.3 Let all the assumptions in Theorem 2.1 hold. Then for $n \geq 2$, there exists a $u_n \in S_n$ such that

$$\mathcal{M}(u_n, v) = \left(\lambda_1 - \frac{1}{n} \right) \langle u_n, v \rangle_{\rho} + \int_{\Omega} f(x, u_n)v\rho - G(v), \quad \forall v \in S_n. \tag{3.1}$$

Proof For fixed n ($n \geq 2$) and $\forall \alpha = (\alpha_1, \dots, \alpha_n) \in R^n$, set $u = \sum_{k=1}^n \alpha_k \varphi_k$. From simple- V_L conditions, we obtain

$$\mathcal{L}(u, u) = \sum_{k=1}^n \lambda_k \alpha_k^2, \quad \|u\|_{\rho}^2 = \sum_{k=1}^n \alpha_k^2 = |\alpha|^2, \tag{3.2}$$

$$\mathcal{L}(u, u) + \|u\|_{\rho}^2 \leq (\lambda_n + 1) \|u\|_{\rho}^2. \tag{3.3}$$

From (3) and (4) of (2.5), we have

$$\mathcal{L}(u, u) + \|u\|_\rho^2 \geq c_0 \int_\Omega \sum_{i=1}^N p_i |D_i u|^2 + \beta_0 \|u\|_q^2 + \|u\|_\rho^2 \geq l \|u\|_{p,q,\rho}^2, \quad (3.4)$$

where $l = \min\{c_0, \beta_0, 1\} > 0$. Combining (3.3) with (3.4), we get

$$\|u\|_\rho^2 \leq \|u\|_{p,q,\rho}^2 \leq \frac{\lambda_n + 1}{l} \|u\|_\rho^2. \quad (3.5)$$

For $m \geq 2$, a positive integer, we put

$$f_m(x, s) = \begin{cases} f(x, m), & m \leq s; \\ f(x, s), & -m \leq s \leq m; \\ f(x, -m), & s \leq -m. \end{cases} \quad (3.6)$$

Note from (f-2) that $|f_m(x, s)| \leq h_0(x) + K|m|^{\theta-1}$ for $\forall s \in \mathbb{R}$, a.e. $x \in \Omega$. Also, from $h_0(x) \in L_\rho^{\theta^*}$, the Hölder inequality, Minkowski inequality, and (2.4), for $\forall v \in S_n$, we get

$$\int_\Omega |f_m(x, u)v\rho| \leq \|f_m(x, u)\|_{L_\rho^{\theta^*}} \|v\|_{L_\rho^\theta} \leq T_m \|v\|_{L_\rho^\theta}, \quad (3.7)$$

where T_m is a positive constant depending on m .

The remaining proof is separated into two parts. The first part is to prove the claim (3.8) for $f_m(x, s)$. The second part is to get the conclusion by leaving $m \rightarrow \infty$ based on (3.8).

Part 1. Fix m ($m \geq 2$). To show there exists $u_{n,m}^*$ such that

$$\mathcal{M}(u_{n,m}^*, v) = \left(\lambda_1 - \frac{1}{n}\right) \langle u_{n,m}^*, v \rangle_\rho + \int_\Omega f_m(x, u_{n,m}^*)v\rho - G(v), \quad \forall v \in S_n, \quad (3.8)$$

we set

$$F_k(\alpha) = \mathcal{M}(u, \varphi_k) - \left(\lambda_1 - \frac{1}{n}\right) \langle u, \varphi_k \rangle_\rho - \int_\Omega f_m(x, u)\varphi_k\rho + G(\varphi_k), \quad k = 1, \dots, n.$$

It is clear that $\sum_{k=1}^n F_k(\alpha)\alpha_k = I(\alpha) + II(\alpha)$, where

$$I(\alpha) = \mathcal{M}(u, u) - \mathcal{L}(u, u) - \int_\Omega f_m(x, u)u\rho + G(u), \quad (3.9)$$

$$II(\alpha) = \mathcal{L}(u, u) - \left(\lambda_1 - \frac{1}{n}\right) \langle u, u \rangle_\rho. \quad (3.10)$$

For (3.9), observing the fact that the operator \mathcal{M} is near-related to \mathcal{L} , $G \in [H_{p,q,\rho}^1(\Omega, \Gamma)]'$, (3.2), (3.5), (3.7), and Lemma 3.1, we conclude that

$$\lim_{|\alpha| \rightarrow \infty} \frac{\mathcal{M}(u, u) - \mathcal{L}(u, u)}{|\alpha|^2} = 0, \quad \lim_{|\alpha| \rightarrow \infty} \frac{|\int_\Omega f_m(x, u)u\rho|}{|\alpha|^2} = 0, \quad \lim_{|\alpha| \rightarrow \infty} \frac{|G(u)|}{|\alpha|^2} = 0,$$

and

$$\lim_{|\alpha| \rightarrow \infty} \frac{|I(\alpha)|}{|\alpha|^2} = 0.$$

For (3.10), by (3.2), we obtain

$$H(\alpha) = \sum_{k=1}^n \left(\lambda_k - \lambda_1 + \frac{1}{n} \right) \alpha_k^2 \geq \frac{1}{n} |\alpha|^2. \tag{3.11}$$

Consequently, $\sum_{k=1}^n F_k(\alpha) \alpha_k \geq \frac{|\alpha|^2}{2n}$ where $|\alpha| \geq s_0$ (here s_0 is a large enough constant). By virtue of the generalized Brouwer's theorem [9], there exists $\gamma_{n,m} = (\gamma_{n,m}^{(1)}, \dots, \gamma_{n,m}^{(n)})$, such that $F_k(\gamma_{n,m}) = 0$, $k = 1, \dots, n$. Taking $u_{n,m}^* = \sum_{k=1}^n \gamma_{n,m}^{(k)} \varphi_k$, then (3.8) holds.

Part 2. We claim that $\{\|u_{n,m}^*\|_\rho\}_{m=2}^\infty$ (n fixed) is uniformly bounded according to m .

Arguing by contradiction, without loss of generality, suppose that

$$\lim_{m \rightarrow \infty} \|u_{n,m}^*\|_\rho = \infty. \tag{3.12}$$

Taking $v = u_{n,m}^*$ in (3.8),

$$\begin{aligned} & \mathcal{L}(u_{n,m}^*, u_{n,m}^*) - \lambda_1 \langle u_{n,m}^*, u_{n,m}^* \rangle_\rho + \frac{1}{n} \langle u_{n,m}^*, u_{n,m}^* \rangle_\rho \\ &= \int_\Omega f_m(x, u_{n,m}^*) u_{n,m}^* \rho - G(u_{n,m}^*) + \mathcal{L}(u_{n,m}^*, u_{n,m}^*) - \mathcal{M}(u_{n,m}^*, u_{n,m}^*) \end{aligned} \tag{3.13}$$

holds, that is,

$$\begin{aligned} & \sum_{k=1}^n (\lambda_k - \lambda_1) |\widehat{u}_{n,m}^*(k)|^2 + \frac{1}{n} \|u_{n,m}^*\|_\rho^2 \\ &= \int_\Omega f_m(x, u_{n,m}^*) u_{n,m}^* \rho - G(u_{n,m}^*) + \mathcal{L}(u_{n,m}^*, u_{n,m}^*) - \mathcal{M}(u_{n,m}^*, u_{n,m}^*), \end{aligned} \tag{3.14}$$

where $\widehat{u}_{n,m}^*(k) = \langle \varphi_k, u_{n,m}^* \rangle_\rho$.

On the other hand, using (f-3), for $s \geq m$, we have

$$sf_m(x, s) = \frac{s}{m} \cdot mf(x, m) \leq \widetilde{h}_0(x) |s|, \quad \text{a.e. } x \in \Omega. \tag{3.15}$$

Similarly, we can also obtain the same conclusion where $-m \leq s \leq m$ or $s \leq -m$. As a result,

$$sf_m(x, s) \leq \widetilde{h}_0(x) |s|, \quad \forall s \in R, \text{ a.e. } x \in \Omega. \tag{3.16}$$

(3.14) and (3.16) imply that

$$\frac{1}{n} \|u_{n,m}^*\|_\rho^2 \leq \|\widetilde{h}_0\|_{L^\theta} \|u_{n,m}^*\|_{L^\theta} + |G(u_{n,m}^*)| + \mathcal{L}(u_{n,m}^*, u_{n,m}^*) - \mathcal{M}(u_{n,m}^*, u_{n,m}^*). \tag{3.17}$$

Dividing both sides of (3.17) by $\|u_{n,m}^*\|_\rho^2$ and leaving $m \rightarrow \infty$, we obtain from the fact that \mathcal{M} is near-related to \mathcal{L} , $G \in [H_{p,q,\rho}^1(\Omega, \Gamma)]'$, $\widetilde{h}_0(x) \in L_\rho^\theta(\Omega)$, and (3.5) together with Lemma 3.1 that $\frac{1}{n} \leq 0$. However, n is a positive integer. So, we have arrived at a contradiction. (3.12) does not hold. Then

$$\exists K_1 > 0, \quad \|u_{n,m}^*\|_\rho \leq K_1, \quad \forall m \geq 2. \tag{3.18}$$

(3.5) and (3.18) imply that there is a subsequence of $\{u_{n,m}^*\}_{m=2}^\infty$ (for ease of notation take the full sequence) and a $u_n \in S_n$ [10] such that

$$\begin{cases} \lim_{m \rightarrow \infty} \|u_{n,m}^* - u_n\|_{p,q,\rho} = 0; \\ \lim_{m \rightarrow \infty} u_{n,m}^*(x) = u_n(x), \quad \text{a.e. } x \in \Omega; \\ \lim_{m \rightarrow \infty} D_i u_{n,m}^*(x) = D_i u_n(x), \quad \text{a.e. } x \in \Omega, i = 1, \dots, N. \end{cases} \quad (3.19)$$

Therefore, from (3.19), we obtain

$$\lim_{m \rightarrow \infty} \mathcal{M}(u_{n,m}^*, v) = \mathcal{M}(u_n, v), \quad \forall v \in S_n. \quad (3.20)$$

And recall Lemma 3.1 that

$$\lim_{m \rightarrow \infty} \int_{\Omega} |u_{n,m}^* - u_n|^\theta \rho = 0. \quad (3.21)$$

Moreover, there exists a $W(x) \in L^\theta_\rho$ and a subsequence $\{u_{n,m_j}^*\}_{j=1}^\infty \subset \{u_{n,m}^*\}_{m=2}^\infty$ such that $|u_{n,m_j}^*(x)| \leq W(x)$, a.e. $x \in \Omega$ for $\forall j$.

Since by virtue of the Hölder inequality, (f - 1), (f - 2) and the Lebesgue dominated convergence theorem, we get

$$\lim_{j \rightarrow \infty} \int_{\Omega} f_{m_j}(x, u_{n,m_j}^*) v \rho = \int_{\Omega} f(x, u_n) v \rho, \quad \forall v \in S_n.$$

Now replacing m by m_j in (3.8) and taking the limit as $j \rightarrow \infty$ on both sides of the equation, we consequently obtain that (3.1) holds and Lemma 3.3 is completed. \square

Lemma 3.4 *Let all the assumptions in Theorem 2.1 hold. Then the sequence $\{u_n\}$ obtained in Lemma 3.3 is uniformly bounded in $H^1_{p,q,\rho}(\Omega, \Gamma)$.*

Proof For $\{u_n\}_{n=2}^\infty \subset S_n$ in Lemma 3.3, set $u_n = \sum_{k=1}^n \hat{u}_n(k) \varphi_k$ where $\hat{u}_n(k) = \langle \varphi_k, u_n \rangle_\rho$.

We suppose that Lemma 3.4 is false. Without loss of generality, suppose that

$$\lim_{n \rightarrow \infty} \|u_n\|_{p,q,\rho} = \infty. \quad (3.22)$$

To lead to a contradiction, taking $v = u_n$ in (3.1), then

$$\mathcal{M}(u_n, u_n) = \left(\lambda_1 - \frac{1}{n} \right) \langle u_n, u_n \rangle_\rho + \int_{\Omega} f(x, u_n) u_n \rho - G(u_n). \quad (3.23)$$

And we can get

$$\frac{1}{n} \|u_n\|_\rho^2 \leq \int_{\Omega} f(x, u_n) u_n \rho + |G(u_n)| + \mathcal{L}(u_n, u_n) - \mathcal{M}(u_n, u_n). \quad (3.24)$$

In view of $G \in [H^1_{p,q,\rho}(\Omega, \Gamma)]'$, (f - 3), and (3.24),

$$\left(\frac{1}{n} + \beta^* \right) \|u_n\|_\rho^2 \leq \|\tilde{u}_0\|_{L^\theta_\rho} \|u_n\|_{L^\theta_\rho} + |G(u_n)| + \mathcal{L}(u_n, u_n) - \mathcal{M}(u_n, u_n). \quad (3.25)$$

Dividing both sides of (3.25) by $\|u\|_{p,q,\rho}^2$ and leaving $n \rightarrow \infty$, from the fact that \mathcal{M} is near-related to $\mathcal{L}, \tilde{h}_0(x) \in L_\rho^{\theta^*}$ and Lemma 3.1, we get

$$\lim_{n \rightarrow \infty} \frac{\|u_n\|_\rho}{\|u_n\|_{p,q,\rho}} = 0. \tag{3.26}$$

Apply (3) and (4) of (2.5) in conjunction with $(\sigma - 2)$ to show

$$\mathcal{M}(u_n, u_n) \geq c_1 \eta_0 \int_\Omega \sum_{i=1}^N p_i |D_i u_n|^2 + \beta_1 \eta_0 \|u_n\|_q^2, \tag{3.27}$$

also, (1) and (2) of (2.5) to show

$$\mathcal{L}(u_n, u_n) \leq K_2 \int_\Omega \sum_{i=1}^N p_i |D_i u_n|^2 + K_3 \|u_n\|_q^2, \tag{3.28}$$

where K_2 and K_3 are positive constants. Setting $c_2 = \min\{c_1 \eta_0, \beta_1 \eta_0\}$ and $c_3 = \max\{K_2, K_3\}$, it is obvious $c_2, c_3 > 0$. By (3.27) and (3.28), we conclude that

$$\frac{c_2}{c_3} \mathcal{L}(u_n, u_n) \leq \mathcal{M}(u_n, u_n). \tag{3.29}$$

Using (3.29) and (f - 3), from (3.25), we obtain

$$\frac{c_2}{c_3} \mathcal{L}(u_n, u_n) + \beta^* \|u_n\|_\rho^2 \leq \lambda_1 \|u_n\|_\rho^2 + \|\tilde{h}_0\|_{L_\rho^{\theta^*}} \|u_n\|_{L_\rho^\theta} + |G(u_n)|. \tag{3.30}$$

From (3.4), we get $\mathcal{L}(u_n, u_n) + \|u_n\|_\rho^2 \geq l \|u_n\|_{p,q,\rho}^2$. Set $c_4 = \min\{\frac{c_2}{c_3}, \beta^*\}$. It is easy to obtain

$$c_4 l \|u_n\|_{p,q,\rho}^2 \leq \lambda_1 \|u_n\|_\rho^2 + \|\tilde{h}_0\|_{L_\rho^{\theta^*}} \|u_n\|_{L_\rho^\theta} + |G(u_n)|. \tag{3.31}$$

Dividing both sides of (3.31) by $\|u\|_{p,q,\rho}^2$ and leaving $n \rightarrow \infty$, by (3.26), we get

$$c_4 l \leq 0. \tag{3.32}$$

So, we have arrived at a contradiction. Thus, there holds

$$\|u_n\|_{p,q,\rho} \leq K_4, \quad \forall n \geq 2 \text{ and some } K_4 > 0. \tag{3.33}$$

Lemma 3.4 is completed. □

4 Proof of Theorem 2.1

Proof Since $H_{p,q,\rho}^1(\Omega, \Gamma)$ is a separable Hilbert space, from Lemma 3.1 and Lemma 3.2, we conclude that there exist a subsequence of $\{u_n\}_{n=2}^\infty$ (which for ease of notation we take the full sequence) and a function $u^* \in H_{p,q,\rho}^1(\Omega, \Gamma)$ with the following properties [10]:

$$\lim_{n \rightarrow \infty} \left[\|u_n - u^*\|_\rho + \int_\Omega |u_n - u^*|^\theta \rho \right] = 0; \tag{4.1}$$

$$\exists W'(x) \in L^2_\rho(\Omega) \cap L^\theta_\rho(\Omega), \quad \text{s.t. } |u_n(x)| \leq W'(x), \quad \text{a.e. } x \in \Omega, n \geq 2; \tag{4.2}$$

$$\lim_{n \rightarrow \infty} u_n(x) = u^*(x), \quad \text{a.e. } x \in \Omega; \tag{4.3}$$

$$\lim_{n \rightarrow \infty} \langle D_i u_n, v \rangle_{p_i} = \langle D_i u^*, v \rangle_{p_i}, \quad \forall v \in L^2_{p_i}, i = 1, \dots, N; \tag{4.4}$$

$$\lim_{n \rightarrow \infty} \langle u_n, v \rangle_q = \langle u^*, v \rangle_q, \quad \forall v \in L^2_q; \tag{4.5}$$

$$\lim_{n \rightarrow \infty} G(u_n) = G(u^*). \tag{4.6}$$

Let $v_j \in S_j$ where $J \geq 2$ is a fixed but arbitrary positive integer. In fact, for $n \geq J$, we have

$$\begin{aligned} \mathcal{M}(u_n, v_j) - \mathcal{M}(u^*, v_j) &= \int_\Omega \sum_{i,j=1}^N \{ p_i^{\frac{1}{2}} p_j^{\frac{1}{2}} \sigma_i^{\frac{1}{2}}(u_n) \sigma_j^{\frac{1}{2}}(u_n) b_{ij}(x) D_j(u_n - u^*) D_i v_j \\ &\quad + p_i^{\frac{1}{2}} p_j^{\frac{1}{2}} [\sigma_i^{\frac{1}{2}}(u_n) \sigma_j^{\frac{1}{2}}(u_n) - \sigma_i^{\frac{1}{2}}(u^*) \sigma_j^{\frac{1}{2}}(u^*)] b_{ij}(x) D_j u^* D_i v_j \} \\ &\quad + \langle b_0 [\sigma_0(u_n) u_n - \sigma_0(u^*) u^*], v_j \rangle_q. \end{aligned} \tag{4.7}$$

Observing (1) of (2.5), $(\sigma - 1) - (\sigma - 2)$, (4.3)-(4.5) and $D_i v_j \in L^2_{p_i}$, we have

$$\lim_{n \rightarrow \infty} \mathcal{M}(u_n, v_j) = \mathcal{M}(u^*, v_j). \tag{4.8}$$

On the other hand, applying $(f - 1)$, $(f - 2)$, (4.2), (4.3), the Hölder inequality, and the Lebesgue dominated convergence theorem, we obtain that

$$\lim_{n \rightarrow \infty} \int_\Omega f(x, u_n) v_j \rho = \int_\Omega f(x, u^*) v_j \rho, \quad \text{a.e. } x \in \Omega. \tag{4.9}$$

Also, by (3.1), we have

$$\mathcal{M}(u_n, v_j) = \left(\lambda_1 - \frac{1}{n} \right) \langle u_n, v_j \rangle_\rho + \int_\Omega f(x, u_n) v_j \rho - G(v_j). \tag{4.10}$$

For (4.10), leaving $n \rightarrow \infty$, from (4.1), (4.8), and (4.9), we have

$$\mathcal{M}(u^*, v_j) = \lambda_1 \langle u^*, v_j \rangle_\rho + \int_\Omega f(x, u^*) v_j \rho - G(v_j). \tag{4.11}$$

Next, given $v \in H^1_{p,q,\rho}(\Omega, \Gamma)$, we define a projection $P_j : H^1_{p,q,\rho}(\Omega, \Gamma) \rightarrow S_j$, that is,

$$P_j v = \sum_{k=1}^J \widehat{v}(k) \varphi_k \in S_j, \tag{4.12}$$

where $\widehat{v}(k) = \langle \varphi_k, v \rangle_\rho$. It is easy to get $\lim_{j \rightarrow \infty} \|P_j v - v\|_{p,q,\rho} = 0$. As a result, there hold

$$\begin{cases} \lim_{j \rightarrow \infty} \mathcal{M}(u^*, P_j v) = \mathcal{M}(u^*, v), \\ \lim_{j \rightarrow \infty} \langle u^*, P_j v \rangle_\rho = \langle u^*, v \rangle_\rho, \\ \lim_{j \rightarrow \infty} \int_\Omega f(x, u^*) P_j v \rho = \int_\Omega f(x, u^*) v \rho, \\ \lim_{j \rightarrow \infty} G(P_j v) = G(v). \end{cases} \tag{4.13}$$

Replacing v_j by $P_j v$ in (4.11), passing to the limit as $J \rightarrow \infty$ on both sides, and using (4.13), we can obtain

$$\mathcal{M}(u^*, v) = \lambda_1 \langle u^*, v \rangle_\rho + \int_{\Omega} f(x, u^*) v \rho - G(v), \quad \forall v \in H_{p,q,\rho}^1(\Omega, \Gamma).$$

Hence, the proof of Theorem 2.1 is complete. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The paper is the result of joint work of all authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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