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Structure of positive solution sets of differential boundary value problems

Xu Xian and Zhu Xunxia*

*Correspondence: zhuxunxia2006@16.com
Department of Mathematics, Jiangsu Normal University, Xuzhou, Jiangsu 221116, P.R. China

Abstract

In this paper, we first obtain some results on the structure of positive solution sets of differential boundary value problems. Then by using the results, we obtain an existence result for differential boundary value problems. The method used to show the main result is the global bifurcation theory.

Keywords: structure of positive solution sets; differential boundary value problems; bifurcation theory

1 Introduction

This paper considers the differential boundary value problem

$$\begin{cases} (p(t)\phi(u'))' + \lambda p(t)f(t,u) = 0, & a < t < b, \\ u(a) = u(b) = 0, \end{cases} \tag{1.1}$$

where f is ϕ-superlinear at ∞ and $f(t,0)$ maybe negative and p is a positive continuous function, $\lambda > 0$ is a parameter.

Equations of form (1.1) occur in the study for the p-Laplacian equation, non-Newtonial fluid theory and the turbulent flow of a gas in a porous medium. The case where

$$\alpha\left(|\nabla u|^2\right)|\nabla u| = |\nabla u|^{p-2}|\nabla u|, \quad p > 1,$$

i.e., perturbations of the p-Laplacian, has received much attention in the recent literature. Also, problem (1.1) with $f(t,0) > 0$ has been studied by several authors in recent years (see [1] and the references therein). Here, we are interested in the case when $f(t,0)$ may be negative (the so-called semipositone case) (see [2] and its references for a review). As pointed out by Lions in [3], semi-positone problems are mathematically very challenging. During the last ten years, finding positive solutions to semi-positone problems has been actively pursued and significant progress on semi-positone problems has taken place; see [4–8] and the references therein. For instance, Hai *et al.* [9] considered the existence positive solution of (1.1). Under some super-linear conditions on the non-linear term f, they proved that there exists $\lambda^* > 0$ such that (1.1) has one positive solution for $0 < \lambda < \lambda^*$. The main method in [9] used to show the main result are the fixed-point theorems.

The main purpose of this paper is going to study the structure of the positive set of (1.1). Rabinowitz [10] gave the first important results on the structure of the solution sets of non-linear equations and obtained by the degree theoretic method. Amamn [11] studied the structure of the positive solution set of non-linear equations; the reader is referred to [12, 13] for other results concerning the structure of solution sets of non-linear equations. In our paper, we will study the existence results for an unbounded connected component of a positive solution set for the differential boundary value problem of (1.1). This paper generalizes some results from the literature [9]. The paper is arranged as follows. In Section 2, we will give some preliminary lemmas. The main results will be given in Section 3.

2 Some lemmas

For convenience, we make the following assumptions:

(A1) ϕ is an odd, increasing homeomorphism on R with ϕ^{-1} concave on R^+.

(A2) For each $c > 0$, there exists $A_c > 0$ such that $\phi^{-1}(cu) \geq A_c\phi^{-1}(u)$, $u \in R^+$ and $\lim_{c\to\infty} A_c = +\infty$ (note that (A2) implies the existence of $B_c > 0$ such that $\phi^{-1}(cu) \leq B_c\phi^{-1}(u)$, $u \in R^+$ and $\lim_{c\to\infty} B_c = 0$).

(A3) $p : [a,b] \to (0,+\infty)$ is continuous.

(A4) $f : [a,b] \times R^1 \to R^1$ is continuous and

$$\lim_{u\to+\infty} \frac{f(t,u)}{\phi(u)} = +\infty$$

uniformly for $t \in [a,b]$.

Let $E = C[0,1]$, the usual real Banach space of continuous functions with the maximum norm $\|\cdot\|$. Let $e(t) = \frac{(t-a)(b-t)}{(b-a)^2}$ for $t \in [a,b]$ and $P = \{x \in E : u(t) \geq 0, \forall t \in [a,b]\}$. Then P is a cone of E. Define

$$f^*(t,u) = \begin{cases} f(t,u), & u \geq 0, \\ f(t,0), & u < 0. \end{cases}$$

Then $f^*(t,u) \geq -m$ for $u \in R^1$ (here $m > 0$ is a constant). For $u \in E$, $\lambda \in R^+$, define

$$A(\lambda,u) = \int_a^t \phi^{-1}\left(\frac{C}{p(s)} - \frac{\lambda}{p(s)}\int_a^s p(r)f^*(r,u)\,dr\right)ds,$$

here C is a constant such that

$$\int_a^b \phi^{-1}\left(\frac{C}{p(s)} - \frac{\lambda}{p(s)}\int_a^s p(r)f^*(r,u)\,dr\right)ds = 0.$$

We know that C exists and is unique for every $u \in P$ (see [14]). Then $u = A(\lambda,u)$ if and only if u is a solution of

$$\begin{cases} (p(t)\phi(u'))' + \lambda p(t)f^*(t,u) = 0, & a < t < b, \\ u(a) = u(b) = 0. \end{cases}$$

From [9], we have the following Lemmas 2.1 and 2.2.

Lemma 2.1 *Let ϕ satisfy* (A1) *and* (A2). *Then for each $d > 0$, there exist constants K_d and L_d such that*

$$K_d\phi(x) \geq \phi(dx) \geq L_d\phi(x), \quad \forall x \geq 0.$$

Further, $L_d \to 0$ as $d \to 0$ and $K_d \to \infty$ as $d \to \infty$.

Lemma 2.2 *Let ϕ be as in Lemma* 2.1. *Then there exist $0 < \alpha \leq 1$ and $\beta \geq 1$ such that $\phi^{-1}(x-y) \geq \alpha\phi^{-1}(x) - \beta\phi^{-1}(y)$ for $x \geq 0$, $y \geq 0$.*

Lemma 2.3 *Let $p_0, p_1 > 0$ such that $p_0 \leq p(t) \leq p_1$ for $t \in [a,b]$. Let $\lambda > 0$ and ω be a solution of*

$$\begin{cases} (p(t)\phi(\omega'))' + \lambda p(t)h(t) = 0, & t \in [a,b], \\ \omega(a) = \omega(b) = 0, \end{cases} \tag{2.1}$$

here $h(t)$ is continuous function with $h(t) \geq -m$ (here $m > 0$ is a constant) for $t \in [a,b]$, if $\|\omega\| \geq \frac{\beta(b-a)}{\alpha}\phi^{-1}(\lambda m\delta)$, then

$$\omega(t) \geq \bigl(\alpha\|\omega\| - \beta\phi^{-1}(\lambda m\delta)(b-a)\bigr)e(t) \quad \text{for } t \in [a,b],$$

where $\delta = \frac{p_1(b-a)}{p_0}$.

Proof By integrating, it follows that (3.1) has the unique solution given by

$$\omega(t) = \int_a^t \phi^{-1}\left\{\frac{1}{p(s)}\left(C - \lambda\int_a^s p(r)h(r)\,dr\right)\right\}ds,$$

where C is such that $\omega(b) = 0$. Let $\|\omega\|_\infty = |\omega(t_0)|$ for some $t_0 \in (a,b)$. Then

$$\omega(t) = \int_a^t \phi^{-1}\left\{\frac{\lambda}{p(s)}\left(\int_s^{t_0} p(r)\overline{h}(r)\,dr - \frac{\lambda m}{p(s)}\int_s^{t_0} p(r)\,dr\right)\right\}ds,$$

where $\overline{h}(t) = h(t) + m \geq 0$. By Lemma 2.2, we get

$$\omega(t) \geq \alpha\int_a^t \phi^{-1}\left[\frac{\lambda}{p(s)}\int_s^{t_0} p(r)\overline{h}(r)\,dr\right]ds - \beta\int_a^t \phi^{-1}\left(\frac{\lambda m}{p(s)}\int_s^{t_0} p(r)\,dr\right)ds.$$

Now

$$\begin{aligned} \int_a^t \phi^{-1}\left(\frac{\lambda m}{p(s)}\int_s^{t_0} p(r)\,dr\right)ds &\leq \int_a^t \phi^{-1}\left(\frac{\lambda m}{p_0}\int_s^{t_0} p_1\,dr\right)ds \\ &\leq \int_a^t \phi^{-1}\left(\frac{\lambda m p_1}{p_0}(b-a)\right)ds \\ &= \phi^{-1}(\lambda m\delta)(t-a). \end{aligned}$$

And so

$$\omega(t) \geq \alpha\overline{\omega}(t) - \beta\phi^{-1}(\lambda m\delta)(t-a) \geq -\beta\phi^{-1}(\lambda m\delta)(t-a) \quad \text{for } t \in [a,t_0], \tag{2.2}$$

here

$$\overline{\omega}(t) = \int_a^t \phi^{-1}\left[\frac{\lambda}{p(s)}\int_s^{t_0} p(r)\overline{h}(r)\,dr\right]ds \quad \text{for } t \in [a, t_0].$$

Note that $\overline{\omega}$ satisfies

$$\begin{cases} (p(t)\phi(\overline{\omega}'))' + \lambda p(t)\overline{h}(t) = 0, & t \in [a, t_0], \\ \overline{\omega}(a) = 0, & \overline{\omega}(t_0) \geq |\omega(t_0)|. \end{cases}$$

In fact, $\overline{\omega}(t) \geq \omega(t)$ for $t \in [a, t_0]$. We next prove that $\overline{\omega}(t) \geq \nu(t)$ for $t \in [a, t_0]$, here ν satisfies

$$\begin{cases} (p(t)\phi(\nu'))' = 0, & t \in [a, t_0], \\ \nu(a) = 0, & \nu(t_0) = \|\omega\|. \end{cases}$$

Suppose it is not true, then $\overline{\omega} - \nu$ has a negative absolute minimum at $\tau \in (a, t_0)$. Since

$$\overline{\omega}(a) - \nu(a) = 0, \qquad \overline{\omega}(t_0) - \nu(t_0) \geq 0,$$

there exist $\tau_0, \tau_1 \in [a, t_0]$ such that

$$\overline{\omega}(\tau_0) - \nu(\tau_0) = \overline{\omega}(\tau_1) - \nu(\tau_1) = 0,$$

and

$$\overline{\omega}(t) - \nu(t) < 0, \quad t \in (\tau_0, \tau_1).$$

Then

$$\big(p(t)\phi\big(\omega'(t)\big)\big)' - \big(p(t)\phi\big(\nu'(t)\big)\big)' = -\lambda p(t)\overline{h}(t) \leq 0 \quad \text{for } t \in (\tau_0, \tau_1).$$

Let $u(t) = \overline{\omega}(t) - \nu(t)$, $t \in (\tau_0, \tau_1)$, then

$$\int_{\tau_0}^{\tau_1} \big(\big(p(t)\phi\big(\overline{\omega}'(t)\big)\big)' - \big(p(t)\phi\big(\nu'(t)\big)\big)'\big)u(t)\,dt \geq 0.$$

On the other hand, using the inequality

$$\big(\phi(b) - \phi(a)\big)(b - a) \geq 0, \quad a, b \in R^1,$$

and the fact that there exists $\tau^* \in [\tau_0, \tau_1]$ such that $\overline{\omega}(\tau^*) \neq \nu(\tau^*)$, we have

$$\begin{aligned} &\int_{\tau_0}^{\tau_1} \big(\big(p(t)\phi\big(\overline{\omega}'(t)\big)\big)' - \big(p(t)\phi\big(\nu'(t)\big)\big)'\big)u(t)\,dt \\ &\quad = -\int_{\tau_0}^{\tau_1} \big(\big(p(t)\phi\big(\overline{\omega}'(t)\big)\big)' - \big(p(t)\phi\big(\nu'(t)\big)\big)'\big)\big(\overline{\omega}' - \nu'\big)\,dt < 0, \end{aligned}$$

which is a contradiction. So, $\overline{\omega}(t) \geq \nu(t)$ for $t \in [a, t_0]$. Obviously, $\nu(t) = \frac{\|\omega\|}{t_0 - a}(t - a)$, $t \in [a, t_0]$, since $\overline{\omega} \geq \theta$ for each $t \in [a, t_0]$. From (2.2), we have

$$\omega(t) \geq \big(\alpha\|\omega\| - \beta\phi^{-1}(\lambda m\delta)(b-a)\big)\frac{t-a}{t_0-a}, \quad t \in [a, t_0].$$

Similarly,

$$\omega(t) \geq \big(\alpha\|\omega\| - \beta\phi^{-1}(\lambda m\delta)(b-a)\big)\frac{b-t}{b-t_0}, \quad t \in [t_0, b].$$

If $\|\omega\| \geq \frac{\beta(b-a)}{\alpha}\phi^{-1}(\lambda m\delta)$, then

$$\omega(t) \geq \big(\alpha\|\omega\| - \beta\phi^{-1}(\lambda m\delta)(b-a)\big)e(t) \quad \text{for } t \in [a, b].$$

The proof is complete. □

Let $Q_\lambda = \{u \in E \mid u \geq (\alpha\|u\| - \beta\phi^{-1}(\lambda m\delta)(b-a))e(t), \forall t \in [a, b]\}$ for each $\lambda \in R^+$, where $\alpha, \beta > 0$ and $\delta > 0$ are defined as that in Lemma 2.2 and Lemma 2.3, respectively. Then Q_λ is also a cone of E. From Lemma 2.3, we know that $A : P \to Q_\lambda$ is completely continuous.

Let

$$L(P) = \overline{\big\{(\lambda, u) | \lambda \in R^+, u \in P\backslash\{\theta\} \text{ is a solution of (1.1)}\big\}}.$$

From [15, Lemma 29.1], we have Lemma 2.4.

Lemma 2.4 *Let X be a compact metric space. Assume that A and B are two disjoint closed subsets of X. Then either there exist a connected component of X meeting both A and B or $X = \Omega_A \cup \Omega_B$ where Ω_A, Ω_B are disjoint compact subsets of X containing A and B, respectively.*

Let U be an open and bounded subset of the metric space $[a, b] \times E$. We set $U(\lambda) = \{x \in E : (\lambda, x) \in U\}$, whose boundary is denoted by $\partial U(\lambda)$. Consider a map $h(\lambda, x) = x - k(\lambda, x)$, such that $k(\lambda, \cdot)$ is compact and $\theta \notin h(\partial U)$. Such a map h will also be called an admissible homotopy on U. If h is an admissible homotopy, for every $\lambda \in [a, b]$ and every $x \in \partial U(\lambda)$, one has that $h_\lambda(x) := h(\lambda, x) \neq \theta$ and it makes sense to evaluate $\deg(h_\lambda, U(\lambda), \theta)$.

Lemma 2.5 *If h is an admissible homotopy on $U \subset [a, b] \times E$, the* $\deg(h_\lambda, U(\lambda), \theta)$ *is constant for all $\lambda \in [a, b]$.*

Lemma 2.6 *Let $h \in E\backslash\{\theta\}$ such that $h \geq \alpha\|h\|e(t)$. Then for arbitrary* $\lambda \in (0, 1]$, *there exists $R_\lambda > 0$ such that for each $\lambda' \in [\lambda, 1]$, $R' \geq R_\lambda$ and $\mu \geq 0$,*

$$u \neq A(\lambda', u) + \mu h, \quad \forall u \in \partial B(\theta, R'),$$

where $B(\theta, R') = \{u : \|u\| < R'\}$.

Proof From (A4), for $m_1 > 0$, such that

$$\frac{\alpha(b-a)}{16}\phi^{-1}\left(\frac{m_1\lambda_0 p_0(b-a)}{16p_1}\right) - \frac{\alpha(b-a)}{8} > 1,$$

there exists $m_0 > 0$ such that $f^*(t,u) \geq m_1\phi(u)$ for $u \geq m_0$. Let

$$R_\lambda = \max\left\{\frac{1}{\alpha}\left[\frac{m_0}{\min_{t\in[\frac{b+3a}{4},\frac{3b+a}{4}]} e(t)} + \beta\phi^{-1}\big(\lambda_0 m\delta(b-a)\big)\right], \frac{\beta(b-a)}{\alpha}\phi^{-1}(\lambda_0 m\delta)\right\} + 1.$$

Assume by contradiction that

$$u_0 = A(\lambda_0, u_0) + \mu_0 h \tag{2.3}$$

for some $R' \geq R_\lambda$, $u_0 \in \partial B(\theta, R')$, $\lambda_0 \in [\lambda, 1]$ and $\mu_0 \geq 0$. Let

$$y_0(t) = A(\lambda_0, u_0) = \int_a^t \phi^{-1}\left(\frac{C}{p(s)} - \frac{\lambda_0}{p(s)}\int_a^s p(r)f^*(r,u_0)\,dr\right) ds.$$

From Lemma 2.3, we know that $y_0(t) \in Q_{\lambda_0}$. Namely,

$$y_0(t) \geq \big(\alpha\|y_0\| - \beta\phi^{-1}(\lambda_0 m\delta)(b-a)\big)e(t). \tag{2.4}$$

From (2.3) and (2.4), we have

$$\begin{aligned}
u_0 &= y_0(t) + \mu_0 h \\
&\geq \big(\alpha\|y_0\| - \beta\phi^{-1}\big(\lambda_0 m\delta(b-a)\big)\big)e(t) + \mu_0\alpha\|h\|e(t) \\
&\geq (\alpha\|y_0 + \mu_0 h\| - \beta\phi^{-1}\big(\lambda_0 m\delta(b-a)\big)e(t) \\
&= \big(\alpha\|u_0\| - \beta\phi^{-1}(\lambda_0 m\delta)(b-a)\big)e(t),
\end{aligned}$$

so $u_0 \in Q_{\lambda_0}$. For $\lambda_0 \in [\lambda, 1]$, we have

$$u_0(t) \geq \big(\alpha\|u_0\| - \beta\phi^{-1}\big(\lambda_0 m\delta(b-a)\big)\big)e(t) \geq m_0,$$

for $t \in (\frac{3a+b}{4}, \frac{a+b}{2})$. Therefore, let $\|u_0\| = u(t_{u_0})$, $t_{u_0} \in (\frac{3a+b}{4}, \frac{a+b}{2})$, assume that $t_{u_0} \geq \frac{5a+3b}{8}$, then

$$\begin{aligned}
\|u_0\| &= u(t_{u_0}) \\
&= \int_{\frac{3a+b}{4}}^{t_{u_0}} \phi^{-1}\left(\frac{\lambda_0}{p(s)}\int_s^{t_{u_0}} p(\tau)f^*(t,u_0)\,d\tau\right) ds + \mu_0 h \\
&\geq \alpha\int_{\frac{3a+b}{4}}^{t_{u_0}} \phi^{-1}\left(\frac{\lambda_0 p_0}{p_1}\int_s^{t_{u_0}} p(\tau)\big(f^*(t_0,u_0) + m\big)\,d\tau\right) ds \\
&\quad - \beta\int_{\frac{3a+b}{4}}^{t_{u_0}} \phi^{-1}\left(\frac{\lambda_0 p_1 m}{p_0}(t_{u_0} - s)\right) ds,
\end{aligned}$$

where

$$\begin{aligned}
&\int_{\frac{3a+b}{4}}^{t_{u_0}} \phi^{-1}\left(\frac{\lambda_0 p_0}{p_1}\int_s^{t_{u_0}} \big(f^*(\tau,u_0) + m\big)\,d\tau\right) ds \\
&\quad\geq \int_{\frac{3a+b}{4}}^{\frac{11a+5b}{16}} \phi^{-1}\left(\frac{\lambda_0 p_0}{p_1}\int_s^{t_{u_0}} \big(f^*(\tau,u_0) + m\big)\,d\tau\right) ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{\frac{3a+b}{4}}^{\frac{11a+5b}{16}} \phi^{-1}\left(\frac{\lambda_0 p_0}{p_1}\int_{\frac{11a+5b}{16}}^{\frac{5a+3b}{8}} m_1\phi\big(\|u_0\|\big)\,d\tau\right) ds \\
&= \int_{\frac{3a+b}{4}}^{\frac{11a+5b}{16}} \phi^{-1}\left(\frac{m_1\lambda_0 p_0(b-a)}{4p_1}\right)\|u_0\|\,ds \\
&= \frac{b-a}{16}\phi^{-1}\left(\frac{m_1\lambda_0 p_0(b-a)}{16p_1}\right)\|u_0\|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\frac{3a+b}{4}}^{t_{u_0}} \phi^{-1}\left(\frac{\lambda_0 p_1 m}{p_0}(t_{u_0}-s)\right) ds \\
&\quad\leq \int_{\frac{3a+b}{4}}^{t_{u_0}} \phi^{-1}\left(\frac{\lambda_0 p_1 m}{p_0}(b-a)\right) ds \\
&\quad= \phi^{-1}\left(\frac{\lambda_0 p_1 m}{p_0}(b-a)\right)\left(t_{u_0}-\frac{3a+b}{4}\right) \\
&\quad\leq \frac{(b-a)}{8}\phi^{-1}(\lambda_0 m\delta).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|u_0\| &\geq \frac{\alpha(b-a)}{16}\phi^{-1}\left(\frac{m_1\lambda_0 p_0(b-a)}{16p_1}\right)\|u_0\| - \frac{\beta(b-a)}{8}\phi^{-1}(\lambda_0 m\delta) \\
&\geq \frac{\alpha(b-a)}{16}\phi^{-1}\left(\frac{m_1\lambda_0 p_0(b-a)}{16p_1}\right)\|u_0\| - \frac{\alpha(b-a)}{8}\|u_0\| \\
&= \left(\frac{\alpha(b-a)}{16}\phi^{-1}\left(\frac{m_1\lambda_0 p_0(b-a)}{16p_1}\right) - \frac{\alpha(b-a)}{8}\right)\|u_0\|,
\end{aligned}
$$

so $\frac{\alpha(b-a)}{16}\phi^{-1}(\frac{m_1\lambda_0 p_0(b-a)}{16p_1}) - \frac{\alpha(b-a)}{8} \leq 1$, which is a contradiction. Then (2.3) holds. The proof is complete. □

3 Main results

For convenience, let us introduce the following symbols. For any r,

$$
\begin{aligned}
&M(r) = \big\{(\widetilde{\lambda}, u) \in [0,1] \times E : \|u\| = r\big\}, \\
&M[r,+\infty) = \big\{(\widetilde{\lambda}, u) \in [0,1] \times E : r \leq \|u\| < +\infty\big\}, \\
&M[0,r) = \big\{(\widetilde{\lambda}, u) \in [0,1] \times E : 0 \leq \|u\| < r\big\}.
\end{aligned}
$$

Now we give our main results of this paper.

Theorem 3.1 *Suppose* (A1) *to* (A4) *hold. Then* $L(P) \cap ([0,1] \times P)$ *possesses an unbounded connected component* D^* *such that* $\operatorname{Proj}_\lambda D^* \supset (0, \lambda^*]$ *for some* $\lambda^* > 0$ *and*

$$
\lim_{\lambda\to 0^+, (\lambda,u)\in D^*} \|u\| = +\infty,
$$

where $\operatorname{Proj}_\lambda D^*$ *denotes the projection of* D^* *onto the* λ*-axis.*

Proof We divide our proof into four steps.

Step 1. Let

$$T(\lambda,u)=\begin{cases}A(\lambda,u), & ([0,1]\times E)\cap M[2,+\infty),\\ \theta, & ([0,1]\times\overline{B(\theta,1)})\cup(\{0\}\times E),\end{cases}\tag{3.1}$$

where $\overline{B(\theta,1)}=\{u\in E:\|u\|\leq 1\}$. Obviously, $T(\cdot,\cdot)$ is a completely continuous operator.

Note that $T(\lambda,u)=A(\lambda,u)$ for all $\lambda\in[0,1]$ and $u\in E$ with $\|u\|\geq 2$. From Lemma 2.5, there exists $R_1>2$ large enough such that Fix $T(1,\cdot)\subset B(\theta,R_1)$ and

$$\deg\bigl(I-T(1,\cdot),B(\theta,R_1),\theta\bigr)=0.\tag{3.2}$$

Obviously,

$$\deg\bigl(I-T(1,\cdot),B(\theta,1),\theta\bigr)=1.\tag{3.3}$$

Therefore,

$$\begin{aligned}&\deg\bigl(I-T(1,\cdot),B(\theta,R_1)\backslash\overline{B(\theta,1)},\theta\bigr)\\&\quad=\deg\bigl(I-T(1,\cdot),B(\theta,R_1),\theta\bigr)-\deg\bigl(I-T(1,\cdot),B(\theta,1),\theta\bigr)\\&\quad=0-1=-1.\end{aligned}\tag{3.4}$$

So, Fix $T(1,\cdot)\neq\emptyset$ and Fix $T(1,\cdot)\subset U:=B(\theta,R_1)\backslash\overline{B(\theta,1)}$.

Step 2. Let

$$S=\bigl\{(\lambda,u)\in R^+\times E:u=T(\lambda,u)\bigr\}.$$

From Lemma 2.6, we have

$$L(P)\cap[0,1]\times\bigl\{u\in E:0<\|u\|\leq 1\bigr\}=\emptyset.$$

This implies that T has no bifurcation point on $[0,1]$. From step 1,we have $L(P)\cap(\{1\}\times E)\neq\emptyset$, then for each $(1,u)\in L(P)\cap(\{1\}\times E)$, denote by D_u the connected component of the metric space $L(P)\cap(\{1\}\times E)$ emitting from $(1,u)$. Now we will show that, there must exist $(1,u_0)\in L(P)\cap(\{1\}\times E)$ such that D_{u_0} is unbounded. Assume on the contrary that D_u is bounded for each $(1,u)\in L(P)\cap(\{1\}\times E)$. Take a bounded open neighborhood U_u^1 in $L(P)\cap(\{1\}\times E)$ for each D_u such that

$$Cl_{(\{1\}\times E)}U_u^1\cap\bigl([0,1]\times\{\theta\}\bigr)=\emptyset\tag{3.5}$$

and $U_u^1\cap(\{0\}\times E)=\emptyset$, where $Cl_{(\{1\}\times E)}U_u^1$ denotes the closure of U_u^1 in the metric space $[0,1]\times E$. Let $\partial_{[0,1]\times E}U_u^1$ denote the boundary of U_u^1 in the metric space $[0,1]\times E$. Obviously, $\partial_{[0,1]\times E}U_u^1\cap L(P)$ is a compact subset. Assume that $\partial_{[0,1]\times E}U_u^1\cap L(P)\neq\emptyset$. From the maximal connectedness of D_u, there is no connected subset of $\partial_{[0,1]\times E}U_u^1\cap L(P)$ meeting both $\partial_{[0,1]\times E}U_u^1\cap L(P)$ and D_u. From Lemma 2.4, there exist compact disjoint subsets

$\Omega_u^{(1)}$ and $\Omega_u^{(2)}$ of $Cl_{[0,1]\times E}\Omega_u^{(1)} \cap L(P)$ such that $D_u \subset \Omega_u^{(1)}$ and $\partial_{[0,1]\times E}U_u^{(1)} \cap L(P) \subset \Omega_u^{(2)}$, and $Cl_{[0,1]\times E}U_u^{(1)} \cap L(P) = \Omega_u^{(1)} \cup \Omega_u^{(2)}$. Let $d = d(\Omega_u^{(1)}, \Omega_u^{(2)}) > 0$ and $U_u^{(2)}$ be the $\frac{d}{3}$-neighborhood of $\Omega_u^{(1)}$ in the metric space $[0,1] \times E$. Set

$$U_u = \begin{cases} U_u^{(1)} \cap U_u^{(2)}, & \text{when } \partial_{[0,1]\times E}U_u^{(1)} \cap L(P) \neq \emptyset, \\ U_u^{(1)}, & \text{when } \partial_{[0,1]\times E}U_u^{(1)} \cap L(P) = \emptyset. \end{cases} \tag{3.6}$$

Then we have $D_u \subset U_u$ and

$$\partial_{[0,1]\times E}U_u \cap L(P) = \emptyset. \tag{3.7}$$

Obviously, the collection of the subsets

$$\{U_u \cap \{1\} \times E : (1,u) \in L(P) \cap (\{1\} \times E)\}$$

is an open cover of $L(P) \cap (\{1\} \times E)$. Since $L(P) \cap (\{1\} \times E)$ is compact, then there exist finite points, namely

$$(1,u_1), (1,u_2), \ldots, (1,u_n) \in (\{1\} \times E) \cap L(P),$$

such that

$$(\{1\} \times E) \cap L(P) \subset \bigcup_{i=1}^{n} (U_{u_i} \cap (\{1\} \times E)).$$

Let $U = \bigcup_{i=1}^{n} U_{u_i}$. Then U is a bounded open subset of $[0,1] \times E$. From (3.7), we have

$$\partial_{[0,1]\times E}U \cap L(P) \subset \bigcup_{i=i}^{n} ((\partial_{[0,1]\times E}U_{u_i}) \cap L(P)) = \emptyset. \tag{3.8}$$

Thus,

$$\partial_{[0,1]\times E}U \cap L(P) = \emptyset.$$

From (3.5) and (3.8), we have

$$\partial_{[0,1]\times E}U \cap S = \emptyset.$$

Then from Lemma 2.5, we have

$$\deg(I - T(\lambda,\cdot), U(1), \theta) = \deg(I - T(\lambda,\cdot), U(0), \theta). \tag{3.9}$$

Since $U(0) = \emptyset$, then

$$\deg(I - T(\lambda,\cdot), U(0), \theta) = 0. \tag{3.10}$$

Since Fix $T \subset B(\theta, R)$, then from (3.4) we have

$$\deg\bigl(I - T(\lambda, \cdot), U(1), \theta\bigr) = -1, \tag{3.11}$$

which contradicts to (3.10) and (3.11). Therefore, there must exist $(1, u_0) \in (\{1\} \times E) \cap L(P)$ such that D_{u_0} is bounded.

Step 3. Obviously, the projection of D_u is a interval, denote it by $[0, \lambda^*]$, then $\lambda^* \leq 1$. Then D_u is a bounded connected component of $([0, \lambda^*] \times E)$. Take $r_0 = \frac{\beta(b-a)}{\alpha}\phi(\lambda m\delta) + 2$, let

$$\begin{aligned}
Y_1 &= \bigl(\{1\} \times E\bigr) \cap M[r_0, +\infty),\\
Y_2 &= \bigl([0,1] \times E\bigr) \cap M(r_0),\\
Y^* &= \bigl([0,1] \times E\bigr) \cap M[r_0, +\infty).
\end{aligned}$$

Obviously, $D_u \cap (Y_1 \cup Y_2) \neq \emptyset$. For each $p \in D_u \cap (Y_1 \cup Y_2)$, denote by $E(p)$ the connected component of the metric space $D_u \cap Y^*$, which passes the point p. Now we shall prove that there must exist a $p_0 \in D_u \cap (Y_1 \cup Y_2)$ such that $E(p_0)$ is an unbounded connected component of the metric space $D_u \cap Y^*$. On the contrary, assume that $E(p)$ is bounded for each $p \in D_u \cap (Y_1 \cup Y_2)$. Then, for each $p \in D_u \cap (Y_1 \cup Y_2)$, in the same way as in the construction of U^* in (3.6) we can show that there exists a neighborhood $U^*(p)$ of $E(p)$ in Y^* such that

$$\partial_{Y^*} U^*(p) \cap D_u = \emptyset. \tag{3.12}$$

Obviously, the set of $\{U^*(p) \cap (Y_1 \cup Y_2) | p \in D_u \cap (Y_1 \cup Y_2)\}$ is an open cover of the set $D_u \cap (Y_1 \cup Y_2)$ and $D_u \cap (Y_1 \cup Y_2)$ is a compact set. Thus, there exist finite subsets of $\{U^*(p) \cap (Y_1 \cup Y_2) | p \in D_u \cap (Y_1 \cup Y_2)\}$, say

$$U^*(p_1) \cap (Y_1 \cup Y_2), U^*(p_2) \cap (Y_1 \cup Y_2), \ldots, U^*(p_n) \cap (Y_1 \cup Y_2),$$

which is also an open cover of $D_u \cap (Y_1 \cup Y_2)$, that is,

$$\bigcup_{i=1}^{n} \bigl(U^*(p_i) \cap (Y_1 \cup Y_2)\bigr) \supset D_u \cap (Y_1 \cup Y_2). \tag{3.13}$$

Let $U_0 = \bigcup_{i=1}^{n} U^*(p_i)$, then U_0 is a bounded open subset of Y^*. Since

$$\partial_{Y^*} U_0 \subset \bigcup_{i=1}^{n} \partial_{Y^*} U(p_i),$$

then by (3.12) we have

$$\partial_{Y^*} U_0 \cap D_u = \emptyset. \tag{3.14}$$

Let

$$W_1 = \bigl(\bigl([0,1] \times E\bigr) \cap M[0, r_0)\bigr) \cup U_0,$$

and $W_2 = ([0,1] \times E) \backslash Cl_{[0,1]\times E} W_1$. It is easy to see that

$$\partial_{[0,1]\times E} W_1 \subset \partial_{Y^*} U_0 \cup \bigl[\bigl(M(r_0) \cap \bigl([0,1] \times E\bigr)\bigr) \backslash \bigl(U_0 \cap (Y_1 \cup Y_2)\bigr)\bigr].$$

From (3.12) and (3.13), we see that $\partial_{[0,1]\times E} W_1 \cap D_u = \emptyset$. Obviously, $W_1 \cap D_u \neq \emptyset$ and $W_1 \cap W_2 = \emptyset$. Note the unboundedness of D_u, then $W_2 \cap D_u \neq \emptyset$. Now we have $D_u = (W_1 \cap D_u) \cup (W_2 \cap D_u)$, which is a contradiction of the connectedness of D_u. Therefore, there must exist $p_0 \in D_u \cap (Y_1 \cup Y_2)$ such that $E(p_0)$ is an unbounded connected component of $D_u \cap Y^*$.

Step 4. Since $u = T(\lambda, u) \in Q_\lambda$ for each $(\lambda, u) \in E(p_0)$, we have

$$u \geq \bigl(\alpha\|u\| - \beta\phi^{-1}(\lambda m\delta)\bigr)e(t) > \theta.$$

So, $u = A(\lambda, u)$ for each $(\lambda, u) \in E(p_0)$. This implies $E(p_0) \subset L(P)$ is an unbounded subset. Let D^* be the connected component of $L(P)$ containing $E(p_0)$. Obviously, there exists $\lambda^* > 0$ such that $\operatorname{Proj} D^* \supset (0, \lambda^*]$. As D^* is unbounded, we easily see that

$$\lim_{\lambda\to 0^+,(\lambda,u)\in D^*} \|u\| = +\infty.$$

The proof is complete. □

Corollary 3.1 *Let* (A1) *to* (A4) *hold. Then there exists* $\lambda^* > 0$ *such that problem* (1.1) *has a positive solution* u_λ *for* $0 < \lambda < \lambda^*$ *with* $\|u_\lambda\| \to \infty$ *as* $\lambda \to 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read, checked and approved the final manuscript.

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