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A bifurcation problem for a class of periodically perturbed autonomous parabolic equations

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Abstract

The paper deals with the problem of the existence of a branch of T-periodic solutions originating from the isolated limit cycle of an autonomous parabolic equation in a Banach space when it is perturbed by a nonlinear T-periodic term of small amplitude.

We solve this problem by first introducing a novel integral operator, whose fixed points are T-periodic solutions of the considered equation and *vice versa*. Then we compute the Malkin bifurcation function associated to this integral operator and we provide conditions under which the well-known assumption of the existence of a simple zero of the Malkin bifurcation function guarantees the existence of the branch.

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1 Introduction

In recent years, bifurcation problems for smooth and nonsmooth dynamical systems have received a renewed attention and interest from different fields of engineering, physics and mathematics. We mention here, among others, the monographs [1–5] and the review papers [6, 7]. Of particular interest is the study of the bifurcation of periodic solutions for periodically perturbed autonomous systems of the form:

$$\dot{x} = \phi(x) + \varepsilon\psi(t, x, \varepsilon), \tag{1}$$

where $\phi \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, $\psi \in C^1(\mathbb{R} \times \mathbb{R}^n \times [0, 1], \mathbb{R}^n)$, ψ is T-periodic with respect to time and $\varepsilon \geq 0$ is a small parameter. Precisely, one seeks for the existence of a family of T-periodic solutions originating from a limit cycle x_0 of the autonomous unperturbed system.

Existence, uniqueness and asymptotic stability of bifurcating periodic solutions for system (1) are classical problems; see [8, 9]. The main tool employed in these papers is the so-called Malkin bifurcation function:

$$f_0(\theta) = \int_0^T \langle z_0(\tau), \psi(\tau - \theta, x_0(\tau), 0)\rangle \, d\tau,$$

where z_0 is a T-periodic solution of

$$\dot{z} = -\left(\phi'\left(x_0(t)\right)\right)^* z$$

the adjoint system of the linearized system

$$\dot{y} = \phi'\left(x_0(t)\right)y.$$

It is assumed that the linearized system has only one characteristic multiplier with absolute value 1.

Since the pioneering papers [8, 9], a relevant bibliography devoted to this subject has been developed. From this bibliography, we quote in the sequel some of the papers more related to the present paper. In [10], the case when the cycle x_0 is not isolated was considered. By means of suitably defined bifurcation functions $f_{m,n}$, $m, n \in \mathbb{N}$, called Melnikov subharmonic functions, the existence of periodic solutions near to x_0 was proved. The periods of the solutions are in $m : n$ ratio with respect to the period of the perturbation term. The case when 1 is not a simple multiplier of the linearized system was treated in [11]. The existence of at least two branches of T-periodic solutions originating from x_0 is shown in [12, 13] and their stability, in the sense of Lyapunov, follows from the results of [14, 15]. Developments of the Malkin's and Melnikov's approaches have permitted to prove several results about the existence of bifurcating solutions in [16–20]. Furthermore, the use of a Melnikov function permits to detect chaotic behavior of a suitable iterate of the Poincaré map Π_ε associated to the differential equation (1), which is assumed to have a homoclinic orbit. Indeed, the existence of a simple zero of the considered Melnikov function ensures such a chaotic behavior; see [21, 22].

Very recently, in [23], a new method to prove bifurcation of a branch of asymptotically stable periodic solutions to (1) has been proposed. The method consists first in converting the problem of finding fixed points of the singular Poincaré map Π_ε, $\varepsilon \geq 0$, associated to (1) into the problem of the existence of zeros of an equation of the form:

$$P(x) + \varepsilon Q(x, \varepsilon) = 0,$$

where $P : \mathbb{R}^n \to \mathbb{R}^n$ and $Q : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ are given by $P(x) = \Pi_0(x) - x$ and $Q(x, \varepsilon) = \frac{\Pi_\varepsilon(x) - \Pi_0(x)}{\varepsilon}$ with singular $P'(x_0)$. Then, by a convenient scaling of the variable x, we introduce an equivalent equation $\Psi(w, \varepsilon) = 0$. For this equation, under the usual assumption of the existence of a simple zero of the Malkin bifurcation function associated to (1), the classical implicit function theorem can be applied to prove the existence of a branch of solutions originating from x_0.

The same approach has been employed in [24] for a class of systems for which the resulting operators P and Q satisfy regularity conditions, which permit to apply the implicit function theorem, only along certain directions at the point $x_0(\cdot)$ of the limit cycle. Conditions to ensure the existence of several branches of T-periodic solution emanating from x_0 are provided by means of suitably defined Malkin bifurcation functions.

In all the papers cited before, the existence of periodic solutions for $\varepsilon \geq 0$ small is a consequence of the application of a convenient version of the implicit function theorem. This requires, as assumed for system (1), that $\phi \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $\psi \in C^1(\mathbb{R} \times \mathbb{R}^n \times [0,1], \mathbb{R}^n)$.

Under less restrictive regularity conditions, by using topological tools such as the coincidence degree [25], the Leray-Schauder degree and the related continuation principles [26–28], existence results for $\varepsilon \geq 0$ small have been proved in [29] when the autonomous system is Hamiltonian and in [30, 31] when the limit cycle is isolated. More precisely, in [29, 31] the existence of two branches of T-periodic solutions was proved. Roughly speaking, in these papers, the bifurcation functions are employed to guarantee that the topological degree of certain operators is different from zero, rather than for the application of an implicit function theorem.

Topological degree arguments have been also employed in [32] to show the existence of periodic solutions for $\varepsilon \geq 0$ small in the case when the unperturbed system is nonautonomous and the perturbation consists of two nonlinear periodic terms with multiplicative different powers of $\varepsilon \geq 0$. Finally, the behavior of the bifurcating periodic solutions when the perturbation vanishes has been studied in [33] for a nonsmooth system of the form (1) having an isolated limit cycle and in [34] for nonsmooth planar Hamiltonian systems.

A first attempt to extend to infinite dimensional bifurcation problem the approach outlined in [23] has been presented in [35], with the aim of studying the bifurcation of periodic solutions for a functional differential equation of neutral type. In the present paper, we precise and generalize the idea of how to use a suitable abstract Malkin bifurcation function to deal with infinite dimensional bifurcation problems. To this aim, we consider the following autonomous differential equation of parabolic type periodically perturbed by a nonlinear term of small amplitude:

$$\dot{x} = Ax + \phi(x) + \varepsilon \psi(t, x, \varepsilon), \quad t \geq 0, \varepsilon > 0, \tag{2}$$

where A is the infinitesimal generator of a strongly continuous semigroup e^{At}, $t \geq 0$, acting in the Banach space E, satisfying the Radon-Nikodym property, $\phi : E \to E$ is twice continuously Frechét differentiable and $\psi : \mathbb{R} \times E \times [0, 1] \to E$ is continuously Frechét differentiable with respect to x, ε and T-periodic with respect to time. The functions ϕ and ψ satisfy suitable condensivity conditions with respect to the Hausdorff measure of noncompactness. The crucial assumption is that the unperturbed equation at $\varepsilon = 0$ has a continuous T-periodic isolated solution $x_0 : \mathbb{R} \to E$, i.e., $x_0 \in C_T(E)$.

The paper is organized as follows. In Section 2, we precise the conditions under which there is at least a branch of T-periodic solutions to (2) emanating from x_0. This existence result follows from the application of [36, Theorem 2]; this theorem relies on the method introduced in [23]. Precisely, to solve the bifurcation problem for (2), we introduce an equivalent integral equation whose zeros are the T-periodic solutions to (2) and that we rewrite in the following form:

$$\tilde{P}(x) + \varepsilon \tilde{Q}(x, \varepsilon) = 0,$$

where $\tilde{P} : C_T(E) \to C_T(E)$ and $\tilde{Q} : C_T(E) \times [0, 1] \to C_T(E)$. This equation has a branch of solutions originating from $x_0(\theta_0)$ if the Malkin bifurcation function given by

$$M(\theta) = \int_0^T \langle \tilde{Q}(x_0(\theta), 0)(t), z_0(t + \theta) \rangle \, dt, \quad \theta \in [0, T],$$

has $\theta_0 \in [0, T]$ as simple zero. Here, for any $\theta \in [0, T], x_0(\theta) \in C_T(E)$ is given by $x_0(\theta)(\cdot) := x_\theta(\cdot) = x(\cdot + \theta)$, $z_0(\theta)$ is the eigenvector corresponding to the simple eigenvalue 0 of $(\tilde{P}'(x_0(\theta)))^*$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing of E with its dual E'. The main difficulty to verify the conditions of [36, Theorem 2] consists in proving that the zero eigenvalue of $\tilde{P}'(x_0(\theta))$ is simple. In fact, in this case, the assumption that the linearized equation, around the limit cycle $x_0(\theta)$, $\theta \in [0, T]$, of the autonomous system at $\varepsilon = 0$, does not have neither T-periodic solutions linearly independent with $x_0'(\theta)$ nor Floquet adjoint solutions to $x_0'(\theta)$ does not guarantee that the zero eigenvalue of $\tilde{P}'(x_0(\theta))$ is simple. To overcome this difficulty, we define in Section 3 a novel integral operator, equivalent to that associated to (2) with the property that for the resulting equation $P(x) + \varepsilon Q(x, \varepsilon) = 0$, the operator $P'(x_0(\theta))$ has 0 as simple eigenvalue.

Furthermore, in Section 4, we calculate the Malkin bifurcation function associated to the integral operator introduced in Section 3, and we formulate in Theorem 2 the result of the existence of a branch of T-periodic solutions parameterized by $\varepsilon \geq 0$ small. Proposition 1 of Section 5 states a somewhat surprising result: the Malkin functions associated to the two integral operators coincide and they have the common form of the classical Malkin function introduced for ordinary differential equations in finite dimensional spaces of the form (1). Finally, in Section 6, we provide a concrete example of a system of partial differential equations to which our abstract bifurcation result applies.

2 Assumptions and statement of the problem

The paper deals with the problem of the existence of bifurcation of T-periodic solutions for the T-periodically perturbed autonomous equation of the form

$$\dot{x} = Ax + \phi(x) + \varepsilon \psi(t, x, \varepsilon), \quad t \geq 0, \varepsilon \geq 0, \tag{3}$$

from a T-periodic limit cycle x_0 of the unperturbed system corresponding to $\varepsilon = 0$. Here, A is the infinitesimal generator of a strongly continuous semigroup e^{At}, $t \geq 0$, acting in the Banach space E, which satisfies the Radon-Nikodym property; see [37, Theorem 23, p.276]; $\phi : E \to E$, $\psi : \mathbb{R} \times E \times [0, 1] \to E$ is T-periodic and $x \in C_T(E)$, the space of T-periodic continuous functions $x : \mathbb{R} \to E$.

Throughout the paper, we assume the following conditions on A, ϕ and ψ.

(H1) $(e^{At})^* = e^{A^*t}$ and there exists $\alpha_0 > 0$ such that

$$\left\| e^{At} \right\|_E \leq e^{-\alpha_0 t}, \quad t \geq 0.$$

(H2) ϕ is twice continuously Fréchet differentiable, ψ is continuously Fréchet differentiable with respect to the pair (x, ε). Moreover, for any nonempty, bounded set $\Omega \subset E$ we have

$$\chi\big(\phi(\Omega)\big) \leq k\chi(\Omega);$$

$$\chi\big(\psi\big([0, T] \times \Omega \times [0, 1]\big)\big) \leq l\chi(\Omega),$$

where $0 < k/\alpha_0 < 1$, $l > 0$ and $\chi(\cdot)$ is the Hausdorff measure of noncompactness [38].

(H3) The unperturbed equation

$$\dot{x} = Ax + \phi(x) \tag{4}$$

has a T-periodic isolated solution $x_0 \in C_T(E)$, hence the set of shifts $x_\theta(\cdot) = x_0(\cdot + \theta)$, for any $\theta \in [0, T]$, represents a family of T-periodic solutions to (4). Moreover, for $\theta \in [0, T]$, we have that $y_\theta(t) := x'_\theta(t)$ is a T-periodic solution to the linearized equation

$$\dot{y} = Ay + \phi'(x_0(\theta))y, \tag{5}$$

where $x_0(\theta) := x_\theta$. We assume that $x'_0(\theta) \neq 0$ for any $\theta \in [0, T]$ and that (5) does not possess neither T-periodic solution linearly independent with $y_\theta(t)$ nor Floquet adjoint solution to $y_\theta(t)$, whenever $\theta \in [0, T]$, *i.e.*, (5) does not have solutions of the form

$$y(t) = v(t) + \frac{t}{T} y_\theta(t),$$

where $v(t)$ is a T-periodic function.

We pose the following.

Problem To find conditions to ensure the existence of a branch of T-periodic solutions to (3) parameterized by $\varepsilon \geq 0$, originating, for some $\theta_0 \in [0, T]$, from the family of T-periodic solutions $x_0(\theta)$.

To solve this problem, we first reduce the existence of T-periodic solutions to (3) to the problem of finding fixed points of an integral equation. For this, we introduce the linear operator $J : C_T(E) \to C_T(E)$ as follows:

$$(Jy)(t) := e^{At}\left(I - e^{AT}\right)^{-1} \int_0^T e^{A(T-s)} y(s) \, ds + \int_0^t e^{A(t-s)} y(s) \, ds.$$

Therefore, if we let

$$\left(Jr_\varepsilon(y)\right)(t) := J\left(\phi(y(t)) + \varepsilon \psi\left(t, y(t), \varepsilon\right)\right),$$

then a function $y \in C_T(E)$ satisfying

$$J\left(r_\varepsilon(y)\right) = y \tag{6}$$

is a solution to (3) and *vice versa*. Moreover, it is easy to verify that the equation

$$\left(J'\left(r_0(x_\theta)\right)\right)y = y$$

is equivalent to the linearized unperturbed equation

$$\dot{y} = Ay + a_\theta(t)y,$$

where $a_\theta(t) := \phi'(x_0(t + \theta))$. Hence, we can rewrite (6) in the following form:

$$\tilde{P}(y) + \varepsilon \tilde{Q}(y, \varepsilon) = 0, \tag{7}$$

where $\tilde{P}: C_T(E) \to C_T(E)$ and $\tilde{Q}: C_T(E) \times [0,1] \to C_T(E)$ are defined as follows:

$$\tilde{P}(y)(t) := y(t) - e^{At}\left(I - e^{AT}\right)^{-1} \int_0^T e^{A(T-s)}\phi\big(y(s)\big)\,ds - \int_0^t e^{A(t-s)}\phi\big(y(s)\big)\,ds,$$

and

$$\tilde{Q}(y,\varepsilon)(t) := -e^{At}\left(I - e^{AT}\right)^{-1} \int_0^T e^{A(T-s)}\psi\big(s, y(s), \varepsilon\big)\,ds - \int_0^t e^{A(t-s)}\psi\big(s, y(s), \varepsilon\big)\,ds.$$

In conclusion, our problem will be solved if we show that for $\varepsilon \geq 0$ sufficiently small, equation (7) has a solution. To this end, it would be sufficient to verify the conditions of the following result.

Theorem 1 ([36, Theorem 2]) *Let B be a Banach space, let $P: B \to B$ be a twice continuously Fréchet differentiable map and $Q: B \times [0,1] \to B$ continuously Fréchet differentiable with respect to both the variables.*

Assume that the equation $P(x) = 0$ has one-dimensional set of solutions $x_0(\theta) \in B$, parameterized by $\theta \in [0,T]$, such that there exists $x_0''(\theta)$ for any $\theta \in [0,T]$ and $x_0'(\theta) \neq 0$ for any $\theta \in [0,T]$. Assume that the eigenvalue $0 \in \sigma(P'(x_0(\theta)))$ is simple and the operator $I - P'(x_0(\theta)): B \to B$ is compact, whenever $\theta \in [0,T]$. Consider the function defined by

$$M(\theta) := \int_0^T \big\langle Q\big(x_0(\theta), 0\big)(t), z_0(t + \theta)\big\rangle\,dt,$$

where $z_0(\theta)$ is the eigenvector corresponding to the simple eigenvalue $0 \in \sigma(P'(x_0(\theta)))^*$. Here, $*$ denotes the adjoint operator.

Then, for each $\theta_0 \in [0,T]$ such that $M(\theta_0) = 0$ and $M'(\theta_0) \neq 0$ the equation $P(x) + \varepsilon Q(x, \varepsilon) = 0$ is solvable, for $\varepsilon \geq 0$ sufficiently small, in a neighborhood of the point $x_0(\theta_0)$ and the solution has the form

$$x(\varepsilon) = x_0(\theta_0) + \varepsilon w + o(\varepsilon),$$

where $w \in B$ can be determined in explicit form as shown in [35, Theorem 2.3] and [36, Lemma 3].

As it has been observed in [39], the compactness of the operator $I - P'(x_0(\theta))$ can be replaced by the condensivity of $I - P'(x_0(\theta))$ with respect to the Hausdorff measure of non-compactness. Indeed, as it is shown in [40], under assumptions (H1)-(H2), the operator $I - P$ and thus $I - P'(x_0(\theta))$, see [38, Theorem 1.5.4], are condensing with constant $k/\alpha_0 < 1$. Furthermore, [38, Theorem 2.6.11] ensures that zero is an eigenvalue of $P'(x_0(\theta))$ of finite multiplicity. Therefore, under assumptions (H1)-(H3), one can easily verify that the conditions of the previous Theorem 1 are satisfied for (7) except the condition of the simplicity of the zero eigenvalue of $\tilde{P}'(x_0(\theta))$, $\theta \in [0,T]$. In fact, the assumption that (5) does not possess neither T-periodic solutions linearly independent with $y_\theta(t)$, nor Floquet adjoint solutions to $y_\theta(t)$, whenever $\theta \in [0,T]$, does not imply that the zero eigenvalue of $\tilde{P}'(x_0(\theta))$ is simple, despite the existence of a bijection between the T-periodic solutions to (5) and

the T-periodic solutions to $\tilde{P}'(x_0(\theta)) = 0$. Moreover, as it shown in [41] the simplicity of the zero eigenvalue of $\tilde{P}'(x_0(\theta))$ does not imply that the T-periodic solution $y_\theta(t)$ to (5) has the property mentioned above.

In conclusion, in order to apply Theorem 1, we will introduce a novel integral operator whose fixed points are also fixed points of (6) and *vice versa*, and thus T-periodic solutions to (3). Moreover, we will show that the zero eigenvalue of the corresponding operator $P'(x_0(\theta))$ is simple. This is the aim of the next section.

3 A novel equivalent integral operator

Let $F_\varepsilon(y) := \tilde{P}(y) + \varepsilon\tilde{Q}(y, \varepsilon)$, hence equation (7) reads as $F_\varepsilon(y) = 0$. Consider the integral equation

$$F_\varepsilon(y)(t) - \xi(t)\int_0^\tau F_\varepsilon(y)(s)\,ds = 0, \quad t \in [0, T], \tag{8}$$

where $\xi : E \to C_T(E)$ is defined in the next lemma and $\tau \in [0, T]$ is a given point.

For any fixed $\theta \in [0, T]$, let $e_0 := x_0'(\theta) \neq 0$, $\gamma_0 := \int_0^\tau e_0(s)\,ds$ and $\beta := \langle f, \int_0^\tau (Je_0)(s)\,ds\rangle$ where $f \in E'$.

We can now formulate the following result.

Lemma 1 *Assume* (H1)-(H3) *and that* $\tau \in [0, T]$ *and* $f \in E'$ *satisfy the conditions*:

(H4) $\gamma_0 \neq 0$.

(H5) $\beta \neq 0$.

(H6) $\langle f, \gamma_0\rangle = 1$.

Define $\xi : E \to C_T(E)$ *as follows*:

$$\forall x \in E, \quad \xi(t)x := \langle f, x\rangle g(t), \quad t \in [0, T],$$

where $g := e_0 + \frac{1}{\beta}(Je_0) \in C_T(E)$. *Then* (8) *is equivalent to* (6). *Moreover, the zero eigenvalue of* $P'(x_0(\theta))$ *is simple, where* $P : C_T(E) \to C_T(E)$ *is given by*

$$P(y)(t) := \tilde{P}(y)(t) - \xi(t)\int_0^\tau \tilde{P}(y)(s)\,ds, \quad t \in [0, T].$$

Proof First of all observe that, under our assumptions, we have that

$$1 \notin \sigma\left(\int_0^\tau \xi(s)\,ds\right).$$

Indeed, arguing by contradiction assume that $\hat{y} \in E$ is such that

$$\left(\int_0^\tau \xi(s)\,ds\right)\hat{y} = \hat{y},$$

then, by the definition of ξ, we obtain

$$\left(\int_0^\tau \xi(s)\,ds\right)\hat{y} = \langle f, \hat{y}\rangle\int_0^\tau g(s)\,ds,$$

hence

$$\hat{y} = \langle f, \hat{y} \rangle \int_0^\tau g(s)\,ds,$$

thus $\hat{y}$ and $\int_0^\tau g(s)\,ds$ are linearly dependent, *i.e.*, $\int_0^\tau g(s)\,ds$ is also an eigenvector of $\int_0^\tau \xi(s)\,ds$, that is,

$$\int_0^\tau \xi(s)\,ds \int_0^\tau g(s)\,ds = \int_0^\tau g(s)\,ds.$$

In conclusion, we should have

$$\left\langle f, \int_0^\tau g(s)\,ds \right\rangle = 1.$$

On the other hand, as it is easy to verify our conditions imply that

$$\left\langle f, \int_0^\tau g(s)\,ds \right\rangle = 2. \tag{9}$$

We now prove the equivalence between (6) and (8). Clearly, if $F_\varepsilon(y) = 0$ for some $y \in C_T(E)$ then (8) is satisfied. Conversely, assume that $y \in C_T(E)$ is a solution to (8), hence

$$F_\varepsilon(y)(t) - \xi(t) \int_0^\tau F_\varepsilon(y)(s)\,ds = 0, \quad t \in [0, T].$$

Integrating on the interval $[0, \tau]$, we obtain

$$\int_0^\tau F_\varepsilon(y)(s)\,ds - \int_0^\tau \xi(s)\,ds \int_0^\tau F_\varepsilon(y)(s)\,ds = 0,$$

or equivalently,

$$\left(I - \int_0^\tau \xi(s)\,ds\right) \int_0^\tau F_\varepsilon(y)(s)\,ds = 0. \tag{10}$$

Since $1 \notin \sigma\left(\int_0^\tau \xi(s)\,ds\right)$ from (10), it follows that $\int_0^\tau F_\varepsilon(y)(s)\,ds = 0$ and so from (8) we get $F_\varepsilon(y) = 0$.

It remains to prove the second part of the lemma. For this, given $\theta \in [0, T]$, let $a_\theta(t) := \phi'(x_0(t + \theta))$, $t \in [0, T]$, and let $(\alpha_\theta y)(t) := a_\theta(t)y(t)$. To simplify the notation in the sequel, we omit the subscript θ. Observe that

$$F_0'\big(x_0(\theta)\big)y = \tilde{P}'\big(x_0(\theta)\big)y = y - J(\alpha y).$$

Then the equation

$$\big(F_0'\big(x_0(\theta)\big)y\big)(t) - \xi(t) \int_0^\tau \big(F_0'\big(x_0(\theta)\big)y\big)(s)\,ds = 0$$

can be rewritten as follows:

$$y(t) - (J(\alpha y))(t) - \xi(t) \int_0^\tau (y(s) - (J(\alpha y))(s)) \, ds = 0. \tag{11}$$

For $t \in [0, T]$, define

$$P(y)(t) := \tilde{P}(y)(t) - \xi(t) \int_0^\tau \tilde{P}(y)(s) \, ds$$

and

$$Q(y, \varepsilon)(t) := \tilde{Q}(y, \varepsilon)(t) - \xi(t) \int_0^\tau \tilde{Q}(y, \varepsilon)(s) \, ds,$$

then the equation (11) takes the form

$$P'(x_0(\theta))y = 0.$$

Clearly, e_0 is an eigenvector of $P'(x_0(\theta))$ corresponding to the zero eigenvalue, *i.e.*, $P'(x_0(\theta))e_0 = 0$. Assume now that there exists an adjoint vector e_1 to e_0, namely

$$P'(x_0(\theta))e_1 = e_0,$$

or

$$e_1(t) - (J(\alpha e_1))(t) - \xi(t) \int_0^\tau (e_1(s) - (J(\alpha e_1))(s)) \, ds = e_0(t) \tag{12}$$

for any $t \in [0, T]$. By assumption, there are no adjoint Floquet solutions to (5), thus

$$\dot{e}_1(t) = A e_1(t) + a(t) e_1(t) - e_0(t) \tag{13}$$

does not possess T-periodic solution $e_1(t)$. The integral form of (13) is given by

$$e_1(t) = (J(\alpha e_1))(t) - (J e_0)(t). \tag{14}$$

Therefore, it remains to show that (12) and (14) coincide, namely

$$\xi(t) \int_0^\tau (e_1(s) - (J(\alpha e_1))(s)) \, ds + e_0(t) = -(J e_0)(t). \tag{15}$$

For this, integrating (12) on the interval $[0, \tau]$, we obtain

$$\int_0^\tau (e_1(s) - (J(\alpha e_1))(s)) \, ds - \int_0^\tau \xi(s) \, ds \int_0^\tau (e_1(s) - (J(\alpha e_1))(s)) \, ds = \int_0^\tau e_0(s) \, ds,$$

that is

$$\left(I - \int_0^\tau \xi(s) \, ds\right) \int_0^\tau (e_1(s) - (J(\alpha e_1))(s)) \, ds = \int_0^\tau e_0(s) \, ds.$$

On the other hand $1 \notin \sigma(\int_0^\tau \xi(s)\,ds)$, thus

$$\int_0^\tau \big(e_1(s) - (J(\alpha e_1))(s)\big)\,ds = \left(I - \int_0^\tau \xi(s)\,ds\right)^{-1} \int_0^\tau e_0(s)\,ds. \tag{16}$$

From (16), we get the following form for (15)

$$\xi(t)\left(I - \int_0^\tau \xi(s)\,ds\right)^{-1} \int_0^\tau e_0(s)\,ds = -e_0(t) - (Je_0)(t). \tag{17}$$

Then

$$\left(I - \int_0^\tau \xi(s)\,ds - I\right)\left(I - \int_0^\tau \xi(s)\,ds\right)^{-1} \int_0^\tau e_0(s)\,ds = \int_0^\tau \big(e_0(s) + (Je_0)(s)\big)\,ds,$$

hence

$$\int_0^\tau e_0(s)\,ds - \left(I - \int_0^\tau \xi(s)\,ds\right)^{-1} \int_0^\tau e_0(s)\,ds = \int_0^\tau \big(e_0(s) + (Je_0)(s)\big)\,ds,$$

and

$$\left(I - \int_0^\tau \xi(s)\,ds\right)^{-1} \int_0^\tau e_0(s)\,ds = -\int_0^\tau (Je_0)(s)\,ds. \tag{18}$$

Finally substituting (18) into (17), we obtain

$$\xi(t)\int_0^\tau (Je_0)(s)\,ds = e_0(t) + (Je_0)(t). \tag{19}$$

By our definition $\xi(t)x = \langle f, x\rangle g(t)$, $g(t) = \frac{1}{\beta}(e_0(t) + (Je_0)(t))$, with $\langle f, \int_0^\tau e_0(s)\,ds\rangle = 1$ and $\langle f, \int_0^\tau (Je_0)(s)\,ds\rangle = \beta$. Therefore, (19) is satisfied and this concludes the proof. □

Remark 1 Observe that a little though convinces of the existence of $\tau \in [0, T]$ and $f \in E'$ satisfying the conditions (H4)-(H6) of Lemma 1.

Furthermore, recall that assumptions (H1)-(H2) ensure that the operator $I - P$ is condensing with constant $k/\alpha_0 < 1$ (see [40]); moreover, [38, Theorem 1.5.4] guarantees that $I - P'(x_0(\theta))$ is also condensing with the same constant. Finally, by [38, Theorem 2.6.11], zero turns out to be an eigenvalue of $P'(x_0(\theta))$ of finite multiplicity. The second part of the proof of Lemma 1 shows that it is simple.

4 The Malkin bifurcation function

In the previous section, Lemma 1 states that the operator P, associated to the integral equation (8) satisfies the conditions of Theorem 1. This section is devoted to the computation of the following Malkin bifurcation function $M_\xi(\theta)$ associated to (8)

$$M_\xi(\theta) := \int_0^T \langle \tilde{Q}(x_0(\theta), 0)(t), z_0(t + \theta)\rangle\,dt - \int_0^T \left\langle \xi(t)\int_0^\tau \tilde{Q}(x_0(\theta), 0)(s)\,ds, z_0(t + \theta)\right\rangle dt,$$

where $z_0(\theta)$ is an eigenvector of $(P'(x_0(\theta)))^*$, *i.e.*,

$$\left(P'\left(x_0(\theta)\right)\right)^* z_0(\theta) = 0.$$

For notational convenience, we simply denote $z_0(\theta) \in C_T(E)$ by z_0. In order to compute $M_\xi(\theta)$, it is necessary to determine z_0 in explicit form. The following result solves the problem.

Lemma 2 *Assume* (H1)-(H6), *we have that*

$$z_0(t) = \gamma(t) - \mathbb{1}_{[0,\tau]}(t) \int_0^T \langle g(t), \gamma(t) \rangle f \, dt,$$

where γ is an eigenvector of $(J\alpha)^*$ corresponding to the eigenvalue 1 and $\mathbb{1}_{[0,\tau]}$ is the characteristic function of the interval $[0, \tau]$.

Proof By assumption, E has the Radon-Nikodym property, then the eigenvalue of $(P'(x_0(\theta)))^*$ can be determined, without loss of generality in the Hilbert space H_T^1. Hence, for any $x, y \in C_T(E)$ we have

$$\int_0^T \langle P'\left(x_0(\theta)\right)x(t), y(t) \rangle \, dt$$

$$= \int_0^T \left\langle x(t) - \left(J(\alpha x)\right)(t) - \xi(t) \int_0^\tau \left(x(s) - \left(J(\alpha x)\right)(s)\right) ds, y(t) \right\rangle dt.$$

First, by using assumption (H1), we calculate

$$\int_0^T \langle \left(J(\alpha x)\right)(t), y(t) \rangle \, dt$$

$$= \int_0^T \left\langle e^{At}\left(I - e^{AT}\right)^{-1} \int_0^T e^{A(T-s)} a(s)x(s) \, ds + \int_0^t e^{A(t-s)} a(s)x(s) \, ds, y(t) \right\rangle dt$$

$$= \int_0^T \left\langle \int_0^T e^{A(T-s)} a(s)x(s) \, ds, \left(I - e^{A^*T}\right)^{-1} e^{A^*t} y(t) \right\rangle dt$$

$$\quad + \int_0^T \left\langle \int_0^t e^{A(t-s)} a(s)x(s) \, ds, y(t) \right\rangle dt$$

$$= \int_0^T \int_0^T \langle e^{A(T-s)} a(s)x(s), \left(I - e^{A^*T}\right)^{-1} e^{A^*t} y(t) \rangle \, ds \, dt$$

$$\quad + \int_0^T \int_0^t \langle e^{A(t-s)} a(s)x(s), y(t) \rangle \, ds \, dt$$

$$= \int_0^T \int_0^T \langle e^{-As} a(s)x(s), e^{A^*T}\left(I - e^{A^*T}\right)^{-1} e^{A^*t} y(t) \rangle \, dt \, ds$$

$$\quad + \int_0^T \int_s^T \langle e^{-As} a(s)x(s), e^{A^*t} y(t) \rangle \, dt \, ds$$

$$= \int_0^T \left\langle x(s), \int_0^T a^*(s) e^{A^*(T-s)}\left(I - e^{A^*T}\right)^{-1} e^{A^*t} y(t) \, dt \right\rangle ds$$

$$\quad + \int_0^T \left\langle x(s), \int_s^T a^*(s) e^{A^*(t-s)} y(t) \, dt \right\rangle ds.$$

Therefore, we have

$$((J\alpha)^*y)(t) = \int_0^T a^*(t)e^{A^*(T-t)}\big(I - e^{A^*T}\big)^{-1}e^{A^*s}y(s)\,ds + \int_t^T a^*(t)e^{A^*(s-t)}y(s)\,ds. \qquad (20)$$

We now calculate the adjoint operator for $\xi \int_0^\tau (x(s) - (J(\alpha x))(s))\,ds$. For this, consider

$$\int_0^T \Big\langle \xi(t) \int_0^\tau (x(s) - (J(\alpha x))(s))\,ds, y(t) \Big\rangle dt$$

$$= \int_0^T \Big\langle \int_0^\tau (x(s) - (J(\alpha x))(s))\,ds, \xi^*(t)y(t) \Big\rangle dt$$

$$= \int_0^T \Big\langle \int_0^T \mathbb{1}_{[0,\tau]}(s)\big(x(s) - (J(\alpha x))(s)\big)\,ds, \xi^*(t)y(t) \Big\rangle dt$$

$$= \int_0^T \Big\langle \mathbb{1}_{[0,\tau]}(s)\big(x(s) - (J(\alpha x))(s)\big), \int_0^T \xi^*(t)y(t)\,dt \Big\rangle ds$$

$$= \int_0^T \Big\langle x(s) - (J(\alpha x))(s), \mathbb{1}_{[0,\tau]}(s) \int_0^T \xi^*(t)y(t)\,dt \Big\rangle ds$$

$$= \int_0^T \Big\langle x(s), \mathbb{1}_{[0,\tau]}(s) \int_0^T \xi^*(t)y(t)\,dt \Big\rangle ds - \int_0^T \Big\langle x(s), (J\alpha)^* \mathbb{1}_{[0,\tau]}(s) \int_0^T \xi^*(t)y(t)\,dt \Big\rangle ds$$

$$= \int_0^T \Big\langle x(s), \mathbb{1}_{[0,\tau]}(s) \int_0^T \xi^*(t)y(t)\,dt - (J\alpha)^* \mathbb{1}_{[0,\tau]}(s) \int_0^T \xi^*(t)y(t)\,dt \Big\rangle ds.$$

Thus,

$$\Big(\xi(t) \int_0^\tau (y(s) - ((J\alpha)^*y)(s))\,ds \Big)^*$$

$$= \mathbb{1}_{[0,\tau]}(t) \int_0^T \xi^*(t)y(t)\,dt - (J\alpha)^* \mathbb{1}_{[0,\tau]}(t) \int_0^T \xi^*(t)y(t)\,dt.$$

Finally, we calculate $\xi^*(t)$, for this consider

$$\langle \xi(t)x, y \rangle = \langle \langle f, x\rangle g(t), y \rangle = \langle f, x\rangle\langle g(t), y \rangle = \langle x, \langle g(t), y\rangle f \rangle,$$

i.e.

$$\xi^*(t)y = \langle g(t), y\rangle f.$$

Now, we are in the position to determine the eigenvector z_0 of $(P'(x_0(\theta)))^*$. We have that

$$z_0(t) = (J\alpha)^* z_0(t) + \mathbb{1}_{[0,\tau]}(t) \int_0^T \langle g(s), z_0(s)\rangle f\,ds - (J\alpha)^* \mathbb{1}_{[0,\tau]}(t) \int_0^T \langle g(s), z_0(s)\rangle f\,ds.$$

Then

$$z_0(t) - \mathbb{1}_{[0,\tau]}(t) \int_0^T \langle g(s), z_0(s)\rangle f\,ds = (J\alpha)^* \Big(z_0(t) - \mathbb{1}_{[0,\tau]}(t) \int_0^T \langle g(s), z_0(s)\rangle f\,ds \Big).$$

Let $\gamma(t) := z_0(t) - \mathbb{1}_{[0,\tau]}(t) \int_0^T \langle g(s), z_0(s) \rangle f \, ds$, then the previous equation takes the form

$$\gamma = (J\alpha)^* \gamma, \tag{21}$$

namely γ is an eigenvector of the linear operator $(J\alpha)^* : C_T(E) \to C_T(E)$ corresponding to the simple eigenvalue 1. Therefore, by replacing y with γ in (20), we obtain

$$\gamma(t) = \int_0^T a^*(t) e^{A^*(T-t)} \left(I - e^{A^*T} \right)^{-1} e^{A^*s} \gamma(s) \, ds + \int_t^T a^*(t) e^{A^*(s-t)} \gamma(s) \, ds.$$

The aim now is to find γ, for this consider the adjoint equation to (5)

$$\dot{v}(t) = -A^* v(t) - a^*(t) v(t) \tag{22}$$

and the solution of (22), defined for $t \in [0, T]$, given by

$$v(t) = e^{-A^*(t-T)} v(T) + \int_T^t e^{-A^*(t-s)} \left(-a^*(s) \right) v(s) \, ds. \tag{23}$$

For $t = 0$, we have

$$v(0) = e^{A^*T} v(T) + \int_0^T e^{A^*s} a^*(s) v(s) \, ds.$$

Since v is T-periodic, we obtain

$$v(T) = \left(I - e^{A^*T} \right)^{-1} \int_0^T e^{A^*s} a^*(s) v(s) \, ds. \tag{24}$$

By using (24) into (23), we get

$$v(t) = e^{A^*(T-t)} \left(I - e^{A^*T} \right)^{-1} \int_0^T e^{A^*s} a^*(s) v(s) \, ds + \int_t^T e^{-A^*(t-s)} a^*(s) v(s) \, ds. \tag{25}$$

Put $\omega(t) := -\dot{v}(t) - A^* v(t)$, then $\omega(t) = a^*(t) v(t)$ and (25) becomes

$$v(t) = e^{A^*(T-t)} \left(I - e^{A^*T} \right)^{-1} \int_0^T e^{A^*s} \omega(s) \, ds + \int_t^T e^{A^*(s-t)} \omega(s) \, ds.$$

Therefore,

$$\omega(t) = \int_0^T a^*(t) e^{A^*(T-t)} \left(I - e^{A^*T} \right)^{-1} e^{A^*s} \omega(s) \, ds + \int_t^T a^*(t) e^{A^*(s-t)} \omega(s) \, ds,$$

i.e., $\omega(t)$ is a solution to (21). Hence,

$$\gamma(t) = -\dot{v}(t) - A^* v(t),$$

where $v(t)$ is the T-periodic solution to the adjoint equation (22). Finally, from

$$z_0(t) - \mathbb{1}_{[0,\tau]}(t) \int_0^T \langle g(s), z_0(s) \rangle f \, ds = \gamma(t),$$

we obtain

$$\int_0^T \langle g(t), z_0(t) \rangle \, dt - \int_0^T \mathbb{1}_{[0,\tau]}(t) \int_0^T \langle g(s), z_0(s) \rangle \, ds \langle g(t), f \rangle \, dt$$

$$= \int_0^T \langle g(t), \gamma(t) \rangle \, dt,$$

and

$$\int_0^T \langle g(t), z_0(t) \rangle \, dt - \int_0^T \langle g(s), z_0(s) \rangle \, ds \left\langle \int_0^\tau g(t) \, dt, f \right\rangle = \int_0^T \langle g(t), \gamma(t) \rangle \, dt.$$

By (9), we get

$$\int_0^T \langle g(t), z_0(t) \rangle \, dt = - \int_0^T \langle g(t), \gamma(t) \rangle \, dt.$$

In conclusion,

$$z_0(t) = \gamma(t) - \mathbb{1}_{[0,\tau]}(t) \int_0^T \langle g(t), \gamma(t) \rangle f \, dt. \qquad \square$$

Lemmas 1 and 2, together with the fact that (H1)-(H2) ensure the condensivity of $I - P'(x_0(\theta))$, $\theta \in [0, T]$, of constant $0 < k/\alpha_0 < 1$ (see [40]) allow to apply Theorem 1 to state the following.

Theorem 2 *Assume* (H1)-(H6). *If there exists* $\theta_0 \in [0, T]$ *such that* $M_\xi(\theta_0) = 0$ *and* $M_\xi'(\theta_0) \neq 0$. *Then there exists a branch of T-periodic solutions to* (3) *of the form*

$$x(\varepsilon) = x_0(\theta_0) + \varepsilon w + o(\varepsilon)$$

for $\varepsilon \geq 0$ *sufficiently small and* $w \in C_T(E)$.

Remark 2 The function w can be calculated in an explicit form as shown in [35, Theorem 2.3] and [36, Lemma 3].

5 An invariance property of the Malkin bifurcation function

In what follows, we state an interesting property of the Malkin bifurcation functions introduced before. Precisely, we can prove the following result.

Proposition 1 *Let* $\theta \in [0, T]$, *assume that* $0 \in \sigma(\tilde{P}'(x_0(\theta)))$ *is simple. Then the Malkin bifurcation function* $M(\theta)$ *associated to system* (6) *coincide with the Malkin bifurcation function* $M_\xi(\theta)$ *associated to system* (8).

Proof Consider

$$M_\xi(\theta) = \int_0^T \langle \tilde{Q}(x_0(\theta), 0)(t), z_0(t) \rangle \, dt - \int_0^T \left\langle \xi(t) \int_0^\tau \tilde{Q}(x_0(\theta), 0)(s) \, ds, z_0(t) \right\rangle \, dt$$

$$= \int_0^T \langle \tilde{Q}(x_0(\theta), 0)(t), \gamma(t) \rangle \, dt$$

$$- \int_0^T \left\langle \tilde{Q}(x_0(\theta), 0)(t), \mathbb{1}_{[0,\tau]}(t) \int_0^T \langle g(s), \gamma(s) \rangle \, ds f \right\rangle dt$$

$$- \int_0^T \left\langle \xi(t) \int_0^\tau \tilde{Q}(x_0(\theta), 0)(s) \, ds, \gamma(t) - \mathbb{1}_{[0,\tau]}(t) \int_0^T \langle g(s), \gamma(s) \rangle \, ds f \right\rangle dt.$$

Let $\mu := \int_0^T \langle g(s), \gamma(s) \rangle \, ds$, then by (9) we have

$$M_\xi(\theta) = \int_0^T \langle \tilde{Q}(x_0(\theta), 0)(t), \gamma(t) \rangle \, dt - \int_0^T \langle \tilde{Q}(x_0(\theta), 0)(t), \mathbb{1}_{[0,\tau]}(t) \mu f \rangle \, dt$$

$$- \int_0^T \left\langle \left\langle f, \int_0^\tau \tilde{Q}(x_0(\theta), 0)(s) \, ds \right\rangle g(t), \gamma(t) - \mathbb{1}_{[0,\tau]}(t) \mu f \right\rangle dt$$

$$= \int_0^T \langle \tilde{Q}(x_0(\theta), 0)(t), \gamma(t) \rangle \, dt - \int_0^T \langle \tilde{Q}(x_0(\theta), 0)(t), \mathbb{1}_{[0,\tau]}(t) \mu f \rangle \, dt$$

$$- \int_0^T \left\langle \left\langle f, \int_0^\tau \tilde{Q}(x_0(\theta), 0)(s) \, ds \right\rangle g(t), \gamma(t) \right\rangle dt$$

$$+ \int_0^T \left\langle \left\langle f, \int_0^\tau \tilde{Q}(x_0(\theta), 0)(s) \, ds \right\rangle g(t), \mathbb{1}_{[0,\tau]}(t) \mu f \right\rangle dt$$

$$= \int_0^T \langle \tilde{Q}(x_0(\theta), 0)(t), \gamma(t) \rangle \, dt - 2 \int_0^T \langle \tilde{Q}(x_0(\theta), 0)(t), \mathbb{1}_{[0,\tau]}(t) \mu f \rangle \, dt$$

$$+ \int_0^T \left\langle f, \int_0^\tau \tilde{Q}(x_0(\theta), 0)(s) \, ds \right\rangle \langle g(t), \mathbb{1}_{[0,\tau]}(t) \mu f \rangle \, dt$$

$$= \int_0^T \langle \tilde{Q}(x_0(\theta), 0)(t), \gamma(t) \rangle \, dt - 2 \int_0^\tau \langle \tilde{Q}(x_0(\theta), 0)(t), \mu f \rangle \, dt$$

$$+ 2\mu \left\langle f, \int_0^\tau \tilde{Q}(x_0(\theta), 0)(s) \, ds \right\rangle = \int_0^T \langle \tilde{Q}(x_0(\theta), 0)(t), \gamma(t) \rangle \, dt = M(\theta). \qquad \Box$$

Remark 3 $M(\theta)$ can be rewritten in the classical form of the Malkin bifurcation function $f_0(\theta)$ for ordinary differential equations as (1) of the Introduction. In fact, consider

$$M(\theta) = \int_0^T \langle \tilde{Q}(x_0(\theta), 0)(t), \gamma(t) \rangle \, dt = \int_0^T \langle -(J\Psi(x_0(\theta)))(t), -\dot{v}(t) - A^* v(t) \rangle \, dt$$

$$= \int_0^T \langle (J\Psi(x_0(\theta)))(s), \dot{v}(s) \rangle \, ds + \int_0^T \langle (J\Psi(x_0(\theta)))(s), A^* v(s) \rangle \, ds$$

$$= \langle (J\Psi(x_0(\theta)))(t), v(t) \rangle \big|_0^T - \int_0^T \langle A(J\Psi(x_0(\theta)))(t) + \Psi(x_0(\theta))(t), v(t) \rangle \, dt$$

$$+ \int_0^T A \langle (J\Psi(x_0(\theta)))(t), v(t) \rangle \, dt.$$

Since $(J\Psi(x_0(\theta)))(0) = (J\Psi(x_0(\theta)))(T)$, $v(0) = v(T)$, Ψ is the superposition operator generated by ψ and v solves (22) we have

$$M(\theta) = \int_0^T \langle \psi(t, x_0(t+\theta), 0), v(t+\theta) \rangle \, dt.$$

6 An example

In order to introduce an example consistent with the general approach of the paper that requires the employ of the theory of condensing operators, we are led to consider partial differential equations of hyperbolic type, whose abstract formulation in Banach spaces gives rise to infinitesimal generators of noncompact C_0-semigroups; see, *e.g.*, [42].

Precisely, following [43] and [44], we present a concrete, not academic example concerning the existence of periodic solutions of a system of two autonomous damped wave equations in a thin domain with Neumann boundary conditions. The study of the dynamics of partial differential equations in thin domains has received many attention in the past few years; see [45] and the extensive references therein. The system has the form

$$
\begin{aligned}
&\frac{\partial^2 u_1}{\partial t^2} = \Delta_x u_1 + \frac{1}{\lambda^2}\frac{\partial^2 u_1}{\partial y^2} - \beta_1\frac{\partial u_1}{\partial t} - \alpha_1 u_1 + \phi_1(x,\lambda y,u_1,u_2), \\
&\frac{\partial^2 u_2}{\partial t^2} = \Delta_x u_2 + \frac{1}{\lambda^2}\frac{\partial^2 u_2}{\partial y^2} - \beta_2\frac{\partial u_2}{\partial t} - \alpha_2 u_2 + \phi_2(x,\lambda y,u_1,u_2), \\
&\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0,
\end{aligned}
\tag{26}
$$

where $(x,y) \in Q := \Omega \times (0,1) \subset \mathbb{R}^{n+1}$, $n \geq 2$, Ω is a C^2-smooth bounded domain in $\mathbb{R}^n$, ν denotes the outward unit normal vector to Q, λ is a small positive parameter representing the thickness of the domain of the variable λy, α_1, α_2, β_1, β_2 are positive constants and the functions ϕ_1, ϕ_2 are of class C^1 jointly in their arguments. The linear part of system (26) generates an exponentially stable C_0-semigroup in a suitable Banach space; see [43] as well as the related references therein. Under the assumption of the existence of a T_0-periodic solution $u_0 = (u_1^0, u_2^0)$ of the limit problem, obtained as $\lambda \to 0$, and suitable conditions on the growth of the derivatives of ϕ_1, ϕ_2 with respect to their arguments, it is shown in [44] that [43, Theorem 1] applies. This result guarantees the existence of $\lambda_0 > 0$ such that, for fixed $\hat{\lambda} \in (0,\lambda_0)$, system (26) has an isolated $T_{\hat{\lambda}}$-periodic solution $u_{\hat{\lambda}} = (u_1^{\hat{\lambda}}, u_2^{\hat{\lambda}})$. The crucial assumption of [43, Theorem 1] is that the zero eigenvalue of the linearized system around u_0 is simple. For a single damped wave equation of system (26) with the nonlinear term ϕ depending periodically on time t, the existence of periodic solutions was studied in [46].

Consider now a $T_{\hat{\lambda}}$-periodic perturbation of (26) of small amplitude $\varepsilon > 0$

$$
\begin{aligned}
&\frac{\partial^2 u_1}{\partial t^2} = \Delta_x u_1 + \frac{1}{\hat{\lambda}^2}\frac{\partial^2 u_1}{\partial y^2} - \beta_1\frac{\partial u_1}{\partial t} - \alpha_1 u_1 + \phi_1(x,\hat{\lambda} y,u_1,u_2) \\
&\qquad\quad + \varepsilon\psi_1(\varepsilon,t,x,\hat{\lambda} y,u_1,u_2), \\
&\frac{\partial^2 u_2}{\partial t^2} = \Delta_x u_2 + \frac{1}{\hat{\lambda}^2}\frac{\partial^2 u_2}{\partial y^2} - \beta_2\frac{\partial u_2}{\partial t} - \alpha_2 u_2 + \phi_2(x,\hat{\lambda} y,u_1,u_2) \\
&\qquad\quad + \varepsilon\psi_2(\varepsilon,t,x,\hat{\lambda} y,u_1,u_2), \\
&\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0.
\end{aligned}
\tag{27}
$$

If we assume that the superposition operators generated by the functions ϕ_1, ϕ_2, ψ_1, ψ_2 satisfy assumption (H2) of this paper, then (H1) and [47, Theorem 4.3.1] ensure that the C_0-semigroup generated by the linearization around $u_{\hat{\lambda}}$ of the unperturbed system, corresponding to $\varepsilon = 0$ in (27), is strongly contractive with respect to the Hausdorff measure

of noncompactness $\chi(\cdot)$, *i.e.*, χ-strongly contractive. Therefore, our abstract bifurcation result Theorem 2 applies to system (27).

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