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Nonlocal Navier-Stokes problem with a small parameter

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Abstract

Initial nonlocal boundary value problems for a Navier-Stokes equation with a small parameter is considered. The uniform maximal regularity properties of the corresponding stationary Stokes operator, well-posedness of a nonstationary Stokes problem and the existence, uniqueness and uniformly L^p estimates for the solution of the Navier-Stokes problem are established.

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1 Introduction

Consider the following Navier-Stokes problem with a parameter:

$$\frac{\partial u}{\partial t} - \Delta_\varepsilon u + (u \cdot \nabla)u + \nabla \varphi = f(x, t), \quad \operatorname{div} u = 0, x \in G, t \in (0, T), \quad (1.1)$$

$$L_{kj\varepsilon} u = \sum_{i=0}^{m_{kj}} \varepsilon^{\sigma_i} \left[\alpha_{kji} \frac{\partial^i u}{\partial x_k^i}(G_{k0}) + \beta_{kji} \frac{\partial^i u}{\partial x_k^i}(G_{kb}) \right] = 0, \quad (1.2)$$

$$k = 1, 2, \dots, n, j = 1, 2, u(x, 0) = a(x), \quad (1.3)$$

where

$$\Delta_\varepsilon u = \varepsilon \sum_{k=1}^n \frac{\partial^2 u}{\partial x_k^2}, \quad \sigma_i = \frac{1}{2} \left(i + \frac{1}{q} \right), \quad q \in (1, \infty),$$

$$G = \prod_{k=1}^n (0, b_k), \quad G_{k0} = (x_1, x_2, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_n),$$

$$G_{kb} = (x_1, x_2, \dots, x_{k-1}, b_k, x_{k+1}, \dots, x_n), \quad 0 \leq m_{kj} \leq 1,$$

$\alpha_{kji}, \beta_{kji}$ are complex numbers, ε is a small positive parameter,

$$u = (u_1(x, t), u_2(x, t), \dots, u_n(x, t)), \quad \varphi = \varphi(x, t)$$

represent the unknown velocity and pressure, respectively,

$$f = (f_1(x, t), f_2(x, t), \dots, f_n(x, t))$$

represents a given external force and a denotes the initial velocity. This problem is characterized by nonlocality of boundary conditions and by presence of a small term ε which corresponds to the inverse of Reynolds number Re very large for the Navier-Stokes equations. From both the theoretical and computational points of view, singularly perturbed problems and asymptotic behavior of the Navier-Stokes equations with small viscosity when the boundary is either characteristic or non-characteristic have been well studied; see, e.g., [1–6]. In the present work, we established a uniform time of existence and estimates for solutions of problem (1.1)-(1.3). It is clear that for $\varepsilon = 1$, choosing the boundary conditions locally and $m_{kj} = 0$, problem (1.1)-(1.3) is reduced to the classical Navier-Stokes problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla)u + \nabla \varphi &= f(x, t), \quad \operatorname{div} u = 0, \quad x \in \Omega, \quad t \in (0, T), \\ u|_{\partial\Omega} &= 0, \quad u(x, 0) = a(x). \end{aligned} \tag{1.4}$$

Note that the existence of weak or strong solutions and regularity properties of classical Navier-Stokes problems were extensively studied, e.g., in [1–3, 5, 7–33]. There is extensive literature on the solvability of the initial value problem for the Navier-Stokes equation (see, e.g., [25] for further papers cited there). Hopf [20] proved the existence of a global weak solution of (1.4) using the Faedo-Galerkin approximation and an energy inequality. Another approach to problem (1.4) is to use semigroup theory. Kato and Fujita [18, 22, 34] and Sobolevskii [27] transformed equation (1.4) into an evolution equation in the Hilbert space L^2 . They proved the existence of a unique global strong solution for any square-summable initial velocity when $n = 2$. On the other hand, when $n = 3$ they proved the existence of a unique local strong solution if the initial velocity has some regularity. Other contributions in this field have also assumed some regularity of the initial velocity corresponding to the Stokes problem; see, for example, Solonnikov [26] and Heywood [21]. Afterward, Giga and Sohr [13] improved this result in two directions. First, they generalized the result of Solonnikov for spaces with different exponents in space and time, and the estimate obtained was global in time. Here, first at all, we consider the nonlocal (boundary value problem) BVP for the following differential operator equation (DOE) with small parameters:

$$\begin{aligned} -\Delta_\varepsilon u + (A + \lambda)u &= f(x), \quad x \in G, \\ L_{kj\varepsilon} u &= \sum_{i=0}^{m_{kj}} \varepsilon_k^{\sigma_i} \left[\alpha_{kji} \frac{\partial^i u}{\partial x_k^i}(G_{k0}) + \beta_{kji} \frac{\partial^i u}{\partial x_k^i}(G_{kb}) \right] = 0, \\ k &= 1, 2, \dots, n, \quad j = 1, 2, \end{aligned} \tag{1.5}$$

where A is a linear operator in a Banach space E , α_{kji} , β_{kji} are complex numbers, ε_k are positive and λ is a complex parameter. We show that problem (1.5) has a unique solution $u \in W^{2,q}(G; E(A), E)$ for $f \in W^{m,q}(G; E)$ and $\lambda \in S_{\psi, \varkappa}$ with sufficiently large $\varkappa > 0$, and the

following coercive uniform estimate holds:

$$\sum_{k=1}^n \sum_{i=0}^{m+2} \varepsilon_k^{\frac{i}{m+2}} |\lambda|^{1-\frac{i}{2}} \left\| \frac{\partial^i u}{\partial x_k^i} \right\|_{L^q(G;E)} + \|Au\|_{L^q(G;E)} \leq C \|f\|_{W^{m,q}(G;E)}, \quad m \geq 0,$$

with $C(q)$ independent of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \lambda$ and f .

Further, we consider the nonlocal BVP for the stationary Stokes system with small parameters

$$\begin{aligned} \Delta_\varepsilon u + \nabla \varphi &= f(x, t), \quad \operatorname{div} u = 0, \quad x \in G, t \in (0, T), \\ L_{kj\varepsilon} u &= \sum_{i=0}^{m_{kj}} \varepsilon_k^{\sigma_i} \left[\alpha_{kji} \frac{\partial^i u}{\partial x_k^i}(G_{k0}) + \beta_{kji} \frac{\partial^i u}{\partial x_k^i}(G_{kb}) \right] = 0, \\ k &= 1, 2, \dots, n, j = 1, 2, u(x, 0) = 0, \end{aligned} \tag{1.6}$$

where

$$u = (u_1(x, t), u_2(x, t), \dots, u_n(x, t)), \quad f = (f_1(x, t), f_2(x, t), \dots, f_n(x, t)).$$

Then we consider the initial nonlocal BVP for the following nonstationary Stokes equation with small parameters:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta_\varepsilon u + \nabla \varphi &= f(x, t), \quad \operatorname{div} u = 0, \quad x \in G, t \in (0, T), \\ L_{kj\varepsilon} u &= \sum_{i=0}^{m_{kj}} \varepsilon_k^{\sigma_i} \left[\alpha_{kji} \frac{\partial^i u}{\partial x_k^i}(G_{k0}) + \beta_{kji} \frac{\partial^i u}{\partial x_k^i}(G_{kb}) \right] = 0, \\ k &= 1, 2, \dots, n, j = 1, 2, u(x, 0) = 0. \end{aligned} \tag{1.7}$$

Problem (1.7) can be expressed as the abstract parabolic problem with a parameter

$$\frac{du}{dt} + O_{\varepsilon,q} u = f(t), \quad u(0) = 0, \tag{1.8}$$

where $O_{\varepsilon,q}$ is a stationary parameter depending on the Stokes operator in a solenoidal space $L^q_\sigma(G; R^n)$ defined by

$$\begin{aligned} D(O_{\varepsilon,q}) &= W^{2,q}_\sigma(G, L_{kj\varepsilon}) = \{u \in W^{2,q}(G), \operatorname{div} u = 0, L_{kj\varepsilon} u = 0, k = 1, 2, \dots, n, j = 1, 2\}, \\ O_{\varepsilon,q} u &= -P \Delta_\varepsilon u. \end{aligned}$$

We prove that the operator $O_{\varepsilon,q}$ is positive in $L^q(G; R^n)$ uniformly with respect to parameters $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ and also is a generator of a holomorphic semigroup. Then, by using L^p -maximal regularity theorems (see, e.g., [33, 35]) for abstract parabolic equations (1.8), we obtain that for every $f \in L^p(0, T; L^q(\Omega; R^n)) = B(p, q)$, $p, q \in (1, \infty)$, there is a unique solution $(u, \nabla \varphi)$ of problem (1.8) and the following uniform estimate holds:

$$\left\| \frac{\partial u}{\partial t} \right\|_{B(p,q)} + \sum_{k=1}^n \left\| \varepsilon_k \frac{\partial^2 u}{\partial x_k^2} \right\|_{B(p,q)} + \|\nabla \varphi\|_{B(p,q)} \leq C \|f\|_{B(p,q)}$$

with $C = C(T, \Omega, p, q)$ independent of f and ε . Afterwards, by using the above uniform coercive estimate, we derive local uniform existence and uniform *a priori* estimates of a solution of problem (1.1)-(1.3).

Modern analysis methods, particularly abstract harmonic analysis, the operator theory, the interpolation of Banach spaces, the theory of semigroups of linear operators, embedding and trace theorems in vector-valued Sobolev-Lions spaces are the main tools implemented to carry out the analysis.

2 Notations, definitions and background

Let E be a Banach space and $L^p(\Omega; E)$ denotes the space of strongly measurable E -valued functions that are defined on the measurable subset $\Omega \subset R^n$ with the norm

$$\|f\|_{L^p} = \|f\|_{L^p(\Omega; E)} = \left(\int_{\Omega} \|f(x)\|_E^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

The Banach space E is called a UMD-space if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

is bounded in $L_p(R, E)$, $p \in (1, \infty)$ (see, e.g., [36]). UMD spaces include, e.g., L_p , l_p spaces and Lorentz spaces L_{pq} , $p, q \in (1, \infty)$.

Let \mathbb{C} be the set of complex numbers and

$$S_\psi = \{ \lambda \in \mathbb{C}, |\arg \lambda| \leq \psi \cup \{0\}, 0 \leq \psi < \pi \},$$

$$S_{\psi, \varkappa} = \{ \lambda \in S_\psi, |\lambda| > \varkappa > 0 \}.$$

A linear operator A is said to be ψ -positive in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and $\|(A + \lambda I)^{-1}\|_{B(E)} \leq M(1 + |\lambda|)^{-1}$ for any $\lambda \in S_\psi$, $0 \leq \psi < \pi$, where I is the identity operator in E , $B(E)$ is the space of bounded linear operators in E . It is known [30, §1.15.1] that there exist the fractional powers A^θ of a positive operator A . Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \|A^\theta u\|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, 0 < \theta < \infty.$$

Let \mathbb{N} denote the set of natural numbers. A set $G \subset B(E_1, E_2)$ is called R -bounded (see, e.g., [36]) if there is a positive constant C such that for all $T_1, T_2, \dots, T_m \in G$ and $u_1, u_2, \dots, u_m \in E_1$, $m \in \mathbb{N}$,

$$\int_{\Omega} \left\| \sum_{j=1}^m r_j(y) T_j u_j \right\|_{E_2} dy \leq C \int_{\Omega} \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$ -valued random variables on Ω . The smallest C for which the above estimate holds is called an R -bound of the collection G and denoted by $R(G)$.

A set $G_h \subset B(E_1, E_2)$ is called uniform R -bounded if there is a constant C independent of $h \in Q$ such that for all $T_1(h), T_2(h), \dots, T_m(h) \in G_h$ and $u_1, u_2, \dots, u_m \in E_1, m \in \mathbb{N}$,

$$\int_{\Omega} \left\| \sum_{j=1}^m r_j(y) T_j(h) u_j \right\|_{E_2} dy \leq C \int_{\Omega} \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy,$$

which implies that $\sup_{h \in Q} R(G_h) \leq C$.

The ψ -positive operator A is said to be R -positive in a Banach space E if the set $L_A = \{\xi(A + \xi)^{-1} : \xi \in S_{\psi}, 0 \leq \psi < \pi\}$, is R -bounded.

The operator $A(t)$ is said to be ψ -positive in E uniformly in $t \in \sigma$ with bound $M > 0$ if $D(A(t))$ is independent of t , $D(A(t))$ is dense in E and $\|(A(t) + \lambda)^{-1}\| \leq M(1 + |\lambda|)^{-1}$ for all $\lambda \in S_{\psi}, 0 \leq \psi < \pi$, where M does not depend on t and λ .

Let E_0 and E be two Banach spaces, and let E_0 be continuously and densely embedded into E . Let Ω be a measurable set in R^n and m be a positive integer. Let $W^{m,p}(\Omega; E_0, E)$ denote the space consisting of all functions $u \in L^p(\Omega; E_0)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k^m} \in L^p(\Omega; E)$, with the norm

$$\|u\|_{W^{m,p}(\Omega; E_0, E)} = \|u\|_{L^p(\Omega; E_0)} + \sum_{k=1}^n \left\| \frac{\partial^m u}{\partial x_k^m} \right\|_{L^p(\Omega; E)} < \infty.$$

For $n = 1, \Omega = (a, b), a, b \in \mathbb{N}$, the space $W^{m,p}(\Omega; E_0, E)$ will be denoted by $W^{m,p}(a, b; E_0, E)$.

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α , we write C_{α} .

3 Boundary value problems for abstract elliptic equations

In this section, we consider problem (1.5). We derive the maximal regularity properties of this problem.

It should be noted that BVPs for DOEs were studied, e.g., in [35–38] and [6, 26, 27, 39–43]. For references, see [35, 43]. Let $\alpha_{kj} = \alpha_{kjm_k}$ and $\beta_{kj} = \beta_{kjm_k}$. First, we prove the following theorem.

Theorem 3.1 *Let the following conditions be satisfied:*

- (1) E is a UMD space and A is an R -positive operator in E for $0 \leq \psi < \pi$;
- (2) $q \in (1, \infty), \eta_k = (-1)^{m_1} \alpha_{k1} \beta_{k2} - (-1)^{m_2} \alpha_{k2} \beta_{k1} \neq 0, 0 < t_k \leq 1, k = 1, 2, \dots, n$.

Then problem (1.5) has a unique solution $u \in W^{2,q}(G; E(A), E)$ for $f \in L^q(G; E)$ and $\lambda \in S_{\psi, \varkappa}$ with sufficiently large $\varkappa > 0$. Moreover, the following coercive uniform estimate holds:

$$\sum_{k=1}^n \sum_{j=0}^2 \varepsilon_k^{\frac{j}{2}} |\lambda|^{1-\frac{j}{2}} \left\| \frac{\partial^j u}{\partial x_k^j} \right\|_{L^q(G; E)} + \|Au\|_{L^q(G; E)} \leq C \|f\|_{L^q(G; E)} \tag{3.1}$$

with $C(q)$ independent of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \lambda$ and f .

Proof Let us consider the BVP

$$-\varepsilon_1 \frac{\partial^2 u}{\partial x_1^2} - \varepsilon_2 \frac{\partial^2 u}{\partial x_2^2} + (A + \lambda)u(x_1, x_2) = f(x_1, x_2), \quad L_{kj\varepsilon} u = 0, \quad k, j = 1, 2, \tag{3.2}$$

where $x_1, x_2 \in G_2 = (0, b_1) \times (0, b_2)$, $L_{kj\epsilon}$ are defined by equalities (3.1)-(3.2). For the investigation (3.4), we consider the following BVP for ordinary DOE:

$$Lu = -\epsilon_2 \frac{d^2 u}{dx_2^2} + (A + \lambda)u(x_2) = f(x_2), \quad y \in (0, b_2), \quad L_{2j\epsilon_2} u = 0, \quad (3.3)$$

where L_{2k} are boundary conditions of type (1.5) on $(0, b_2)$. By virtue of [40, Theorem 3.2], we obtain that problem (3.3) has a unique solution $u \in W^{2,q}(0, b_2; E(A), E)$ for $f \in L^q(0, b_2; E)$, $\lambda \in S_{\psi, \varkappa}$, with sufficiently large $\varkappa > 0$, and the following coercive uniform estimate holds:

$$\sum_{j=0}^2 \epsilon_2^{\frac{j}{2}} |\lambda|^{1-\frac{j}{2}} \|u^{(j)}\|_{L^q(0, b_2; E)} + \|Au\|_{L^q(0, b_2; E)} \leq C \|f\|_{L^q(0, b_2; E)}. \quad (3.4)$$

Since $L^q(0, b_1; L^q(0, b_2; E)) = L^q(G_2; E)$, problem (3.2) can be expressed as the following problem:

$$-\epsilon_1 \frac{du}{dx_1^2} + (B_{\epsilon_2} + \lambda)u(x_1) = f(x_1), \quad L_{k1\epsilon_1} u = 0, \quad (3.5)$$

where B_{ϵ_2} is the differential operator in $L^q(0, b_1; E)$ generated by problem (3.3), i.e.,

$$D(B_{\epsilon_2}) = W^{2,q}(0, b_2; E(A), E, L_{2k\epsilon_2}), \quad B_{\epsilon_2} u = -\epsilon_2 \frac{d^2 u}{dx_2^2} + Au(x_2).$$

By virtue of [35, Theorem 4.5.2], $F = L^q(0, b_2; E) \in \text{UMD}$ provided $E \in \text{UMD}$, $q \in (1, \infty)$. Hence, by virtue of [40, Theorem 3.2] and [41, Theorem 3.1], the operator B_{ϵ_2} is uniformly R -positive in F . Then, by applying again [40, Theorem 3.2], we get that for $f \in L^q(0, b_1; F)$, $\lambda \in S_{\psi, \varkappa}$ and sufficiently large $\varkappa > 0$, problem (3.5) has a unique solution $u \in W^{2,q}(0, b_1; D(B), F)$, and the following coercive uniform estimate holds:

$$\sum_{j=0}^2 \epsilon_1^{\frac{j}{2}} |\lambda|^{1-\frac{j}{2}} \|u^{(j)}\|_{L^q(0, b_1; F)} + \|B_{\epsilon_2} u\|_{L^q(0, b_1; F)} \leq C \|f\|_{L^q(0, b_1; F)}. \quad (3.6)$$

The estimate (3.4) implies the uniform estimate

$$\sum_{j=0}^2 \epsilon_2^{\frac{j}{2}} |\lambda|^{1-\frac{j}{2}} \|u^{(j)}\|_{L^q(0, b_1; F)} \leq C \|B_{\epsilon_2} u\|_{L^q(0, b_1; F)}. \quad (3.7)$$

By using (3.4) and (3.7), we obtain that problem (3.7) has a unique solution $u \in W^{2,q}(G_2; E(A), E)$ for $f \in L^q(G_2; E)$, $\lambda \in S_{\psi, \varkappa}$ with sufficiently large $\varkappa > 0$, and the coercive uniform estimate holds

$$\sum_{k=1}^2 \sum_{j=0}^2 \epsilon_k^{\frac{j}{2}} |\lambda|^{1-\frac{j}{2}} \left\| \frac{\partial^j u}{\partial x_k^j} \right\|_{L^q(G_2; E)} + \|Au\|_{L^q(G_2; E)} \leq C \|f\|_{L^q(G_2; E)}.$$

Further, by continuing this process n -times, we obtain the assertion.

From Theorem 3.1 we obtain the following. □

Corollary 3.1 Let $0 < \varepsilon_k \leq 1$, $(-1)^{m_{k1}}\alpha_{k1}\beta_{k2} - (-1)^{m_{k2}}\alpha_{k2}\beta_{k1} \neq 0$. For $f \in L^q(G; R^n)$, $q \in (1, \infty)$ and for $\lambda \in S_{\psi, \varkappa}$ with sufficiently large $\varkappa > 0$, there is a unique solution u of problem (1.5) and the following uniform coercive estimate holds:

$$\sum_{k=1}^n \sum_{i=0}^2 |\lambda|^{1-\frac{i}{2}} \varepsilon_k^{\frac{i}{2}} \left\| \frac{\partial^i u}{\partial x_k^i} \right\|_q \leq C \|f\|_q,$$

with $C = C(q)$ independent of f , ε_k and λ .

Proof Let us put $E = R^n$ and $A = \varkappa > 0$ in Theorem 3.1. It is known that the operator $A = \varkappa > 0$ is R -positive in R^n (see, e.g., [36]). So, the estimate (3.1) implies Corollary 3.1.

Consider the differential operator Q_ε generated by problem (1.5), i.e.,

$$D(Q_\varepsilon) = W^{2,q}(G; L_{kj\varepsilon}) = \{u \in W^{2,q}(G), L_{kj\varepsilon} u = 0, k = 1, 2, \dots, n, j = 1, 2\},$$

$$Q_\varepsilon u = -\Delta_\varepsilon u + Au.$$

From Theorem 3.1 we obtain the following. □

Result 3.1 For $\lambda \in S_{\psi, \varkappa}$, there is a resolvent $(Q_\varepsilon + \lambda)^{-1}$ of the operator Q_ε satisfying the following uniform estimate:

$$\sum_{k=1}^n \sum_{i=0}^2 |\lambda|^{1-\frac{i}{2}} \varepsilon_k^{\frac{i}{2}} \left\| \frac{\partial^i}{\partial x_k^i} (Q_\varepsilon + \lambda)^{-1} \right\|_{B(L^q(G; E))} \leq C.$$

It is clear that the solution u of problem (1.5) depends on parameters $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, i.e., $u = u(x, \varepsilon)$. In view of Theorem 3.1, we established estimates for the solution of (1.5) uniformly in $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$.

4 Regularity properties of solutions for DOEs with parameters

In this section, we show the separability properties of problem (1.5) in Sobolev spaces $W^{m,q}(G; E)$. The main result is the following theorem.

Theorem 4.1 Let the following conditions be satisfied:

- (1) E is a UMD space and A is an R -positive operator in E ;
- (2) m is a positive integer $q \in (1, \infty)$, $0 < t_k \leq 1$, and

$$\eta_k = (-1)^{m_1}\alpha_{k1}\beta_{k2} - (-1)^{m_2}\alpha_{k2}\beta_{k1} \neq 0, \quad k = 1, 2, \dots, n.$$

Then problem (3.1)-(3.2) has a unique solution $u \in W^{2+m,q}(G; E(A), E)$ for $f \in W^{m,q}(G; E)$, $\lambda \in S_{\psi, \varkappa}$, with sufficiently large $\varkappa > 0$, and the following coercive uniform estimate holds:

$$\sum_{k=1}^n \sum_{i=0}^{m+2} \varepsilon_k^{\frac{i}{m+2}} |\lambda|^{1-\frac{i}{m+2}} \left\| \frac{\partial^i u}{\partial x_k^i} \right\|_{W^{m,q}(G; E)} + \|Au\|_{L^q(G; E)} \leq C \|f\|_{W^{m,q}(G; E)} \tag{4.1}$$

with $C = C(q, A)$ independent of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \lambda$ and f .

Consider first the following nonlocal BVP for an ordinary DOE with a small parameter:

$$\begin{aligned} (L_t + \lambda)u &= -tu^{(2)}(x) + (A + \lambda)u(x) = f(x), \quad x \in (0, 1), \\ L_{kt}u &= \sum_{i=0}^{m_k} t^{\sigma_i} [\alpha_{ki}u^{(i)}(0) + \beta_{ki}u^{(i)}(1)] = f_k, \quad k = 1, 2, \end{aligned} \tag{4.2}$$

where $\sigma_i = \frac{i}{2} + \frac{1}{2p}$, $m_k \in \{0, m+1\}$, α_{ki}, β_{ki} are complex numbers, t is positive, λ is a complex parameter and A is a linear operator in E . Let $A_\lambda = A + \lambda I$.

To prove the main result, we need the following result in [37, Theorem 2.1].

Theorem A *Let E be a UMD space, A be a ψ -positive operator in E with bound M , $0 \leq \psi < \pi$. Let m be a positive integer, $1 < p < \infty$, and $\alpha \in (\frac{1}{2p}, \frac{1}{2p} + m)$. Then, for $\lambda \in S_\psi$, an operator $-A_\lambda^{\frac{1}{2}}$ generates a semigroup $e^{-xA_\lambda^{\frac{1}{2}}}$ which is holomorphic for $x > 0$. Moreover, there exists a positive constant C (depending only on M, ψ, m, α and p) such that for every $u \in (E, E(A^m))^{\frac{\alpha}{m} - \frac{1}{2mp}, p}$ and $\lambda \in S_\psi$,*

$$\int_0^\infty \|A_\lambda^\alpha e^{-xA_\lambda^{\frac{1}{2}}} u\|^p dx \leq C [\|u\|_{(E, E(A^m))^{\frac{\alpha}{m} - \frac{1}{2mp}, p}}^p + |\lambda|^{\alpha p - \frac{1}{2}} \|u\|_E^p].$$

In a similar way as in [43, §1.8.2, Theorem 2], we obtain the following lemma.

Lemma 4.1 *Let m and j be integer numbers, $0 \leq j \leq m-1$, $\theta_j = \frac{pj+1}{pm}$, $0 < t \leq 1$, $x_0 \in [0, b]$. Then, for $u \in W_p^m(0, b; E_0, E)$, the transformation $u \rightarrow u^{(j)}(x_0)$ is bounded linear from $W_p^m(0, b; E_0, E)$ onto $(E_0, E)_{\theta_j, p}$ and the following inequality holds:*

$$t^{\theta_j} \|u^{(j)}(x_0)\|_{(E_0, E)_{\theta_j, p}} \leq C (\|tu^{(m)}\|_{L_p(0, b; E)} + \|u\|_{L_p(0, b; E_0)}).$$

Consider at first the homogeneous problem of (4.2)

$$\begin{aligned} (L_t + \lambda)u &= -tu^{(2)}(x) + (A + \lambda)u(x) = 0, \\ L_{kt}u &= \sum_{i=0}^{m_k} t^{\sigma_i} [\alpha_{ki}u^{(i)}(0) + \beta_{ki}u^{(i)}(1)] = f_k, \quad k = 1, 2. \end{aligned} \tag{4.3}$$

Let

$$E_k = (E(A), E)_{\theta_k, p}.$$

Lemma 4.2 *Let A be an R -positive operator in a UMD space E and*

$$0 < t \leq 1, \quad \eta = (-1)^{m_1} \alpha_1 \beta_2 - (-1)^{m_2} \alpha_2 \beta_1 \neq 0.$$

Then problem (4.3) has a unique solution $u \in W^{m+2, p}(0, 1; E(A), E)$ for $f_k \in E_k$, $p \in (1, \infty)$, $\lambda \in S_\psi$, and the coercive uniform estimate holds

$$\sum_{i=0}^{m+2} t^{\frac{i}{m+2}} |\lambda|^{1 - \frac{i}{m+2}} \|u^{(i)}\|_{L^p(0, 1; E)} + \|Au\|_{L^p(0, 1; E)} \leq M \sum_{k=1}^2 (\|f_k\|_{E_k} + |\lambda|^{1 - \theta_k} \|f_k\|_E). \tag{4.4}$$

Proof In a similar way as in [40, Theorem 3.1], we obtain the representation of the solution of (4.3)

$$\begin{aligned}
 u(x) = & t^{-\frac{1}{2p}} \left\{ e^{-xt} A_\lambda^{-\frac{1}{2}} \left[C_{11} + d_{11}(\lambda, t) \right] + e^{-(1-x)t} A_\lambda^{-\frac{1}{2}} \left[C_{12} + d_{12}(\lambda, t) \right] \right\} A_\lambda^{-\frac{m_1}{2}} f_1 \\
 & + t^{-\frac{1}{2p}} \left\{ e^{-xt} A_\lambda^{-\frac{1}{2}} \left[C_{21} + d_{21}(\lambda, t) \right] + e^{-(1-x)t} A_\lambda^{-\frac{1}{2}} \left[C_{22} + d_{22}(\lambda, t) \right] \right\} A_\lambda^{-\frac{m_2}{2}} f_2, \quad (4.5)
 \end{aligned}$$

where C_{ij} and d_{ij} are uniformly bounded operators. Then, in view of positivity of A , we obtain from (4.5)

$$\begin{aligned}
 & \sum_{i=0}^{m+2} t^{\frac{i}{m+2}} |\lambda|^{1-\frac{i}{m+2}} \|u^{(i)}\|_{L^p(0,1;E)} + \|Au\|_{L^p(0,1;E)} \\
 & \leq Ct^{-\frac{1}{2p}} \sum_{k=1}^2 \left[\sum_{i=0}^{m+2} |\lambda|^{\mu_i} \|A_\lambda^{-\frac{(m_k-i)}{2}} \left[e^{-(1-x)t} A_\lambda^{-\frac{1}{2}} + e^{-xt} A_\lambda^{-\frac{1}{2}} \right] f_k\|_{L^p(0,1;E)} \right. \\
 & \quad \left. + \|AA_\lambda^{-\frac{m_k}{2}} e^{-xt} A_\lambda^{-\frac{1}{2}} f_k\|_{L^p(0,1;E)} \right] \leq Ct^{-\frac{1}{2p}}, \quad (4.6)
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{k=1}^2 \left[\sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{2}} \|A_\lambda^{-(1-\frac{mm_k}{2(m+2)} + \frac{i}{2})} \|A_\lambda^{1-\frac{m_k}{m+2}} e^{-xt} A_\lambda^{-\frac{1}{2}} f_k\|_{L^p(0,1;E)} \right. \\
 & \quad \left. + \|AA_\lambda^{-\frac{m_k}{m+2}} \left[e^{-xt} A_\lambda^{-\frac{1}{2}} + e^{-(1-x)t} A_\lambda^{-\frac{1}{2}} \right] f_k\|_{L^p(0,1;E)} \right] \\
 & \leq Ct^{-\frac{1}{2p}} \sum_{k=1}^2 \left[\left(\int_0^1 \|A_\lambda^{1-\frac{m_k}{m+2}} \left[e^{-xt} A_\lambda^{-\frac{1}{2}} + e^{-(1-x)t} A_\lambda^{-\frac{1}{2}} \right] f_k\|^p dx \right)^{\frac{1}{p}} \right. \\
 & \quad \left. + \left(\int_0^1 \|AA_\lambda^{-\frac{m_k}{m+2}} \left[e^{-xt} A_\lambda^{-\frac{1}{2}} + e^{-(1-x)t} A_\lambda^{-\frac{1}{2}} \right] f_k\|^p dx \right)^{\frac{1}{p}} \right]. \quad (4.7)
 \end{aligned}$$

By changing the variable $xt^{-\frac{1}{2}} = y$ and in view of Theorem A, we obtain

$$\begin{aligned}
 & t^{-\frac{1}{2p}} \left(\int_0^1 \|A_\lambda^{-\frac{m_k}{m+2}} \left[e^{-xt} A_\lambda^{-\frac{1}{2}} + e^{-(1-x)t} A_\lambda^{-\frac{1}{2}} \right] f_k\|^p dx \right)^{\frac{1}{p}} \\
 & \leq M_1 \sum_{k=1}^2 [\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|]. \quad (4.8)
 \end{aligned}$$

By using the estimate (4.8), by virtue of Theorem A, we get the uniform estimate

$$\begin{aligned}
 & \sum_{k=1}^2 t^{-\frac{1}{2p}} \left(\int_0^1 \|AA_\lambda^{-\frac{m_k}{m+2}} \left[e^{-xt} A_\lambda^{-\frac{1}{2}} + e^{-(1-x)t} A_\lambda^{-\frac{1}{2}} \right] f_k\|^p dx \right)^{\frac{1}{p}} \\
 & \leq M_2 \sum_{k=1}^2 [\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|]. \quad (4.9)
 \end{aligned}$$

Then from (4.6)-(4.9) we obtain (4.4). □

Now we can represent a more general result for nonhomogeneous problem (4.2).

Theorem 4.2 *Assume that the following conditions are satisfied:*

- (1) E is a UMD space and A is an R -positive operator in E ;
- (2) $\eta = (-1)^{m_1} \alpha_1 \beta_2 - (-1)^{m_2} \alpha_2 \beta_1 \neq 0$, $\theta_k = \frac{mk}{m+2} + \frac{1}{2p}$, $k = 1, 2$, $0 < t \leq 1$, $p \in (1, \infty)$.

Then the operator $u \rightarrow \{(L_t + \lambda)u, L_{1t}u, L_{2t}u\}$ is an isomorphism from $W^{m+2,p}(0, 1; E(A), E)$ onto $W^{m,p}(0, 1; E) \times E_1 \times E_2$ for $\lambda \in S_{\psi, \varkappa}$ with large enough $\varkappa > 0$. Moreover, the uniform coercive estimate holds

$$\begin{aligned} & \sum_{j=0}^{m+2} t^{\frac{j}{m+2}} |\lambda|^{1-\frac{j}{m+2}} \|u^{(j)}\|_{L^p(0,1;E)} + \|Au\|_{L^p(0,1;E)} \\ & \leq C \left[\|f\|_{W^{m,p}(0,1;E)} + \sum_{k=1}^2 (\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|_E) \right]. \end{aligned} \tag{4.10}$$

Proof The uniqueness of a solution of problem (4.2) is obtained from Lemma 4.2. Let us define

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \in [0, 1], \\ 0, & \text{if } x \notin [0, 1]. \end{cases}$$

We will show that problem (4.2) has a solution $u \in W^{m+2,p}(0, 1; E(A), E)$ for $f \in W^{m,p}(0, 1; E)$, $f_k \in E_k$ and $u = u_1 + u_2$, where u_1 is the restriction on $[0, 1]$ of the solution of the equation

$$(L_t + \lambda)u = \bar{f}(x), \quad x \in R = (-\infty, \infty), \tag{4.11}$$

and u_2 is a solution of the problem

$$(L_t + \lambda)u = 0, \quad L_{kt}u = f_k - L_{kt}u_1. \tag{4.12}$$

A solution of equation (4.11) is given by

$$u(x) = F^{-1}L^{-1}(\lambda, t, \xi)F\bar{f} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} L^{-1}(\lambda, t, \xi)(F\bar{f})(\xi) d\xi,$$

where $L(\lambda, t, \xi) = A + t\xi^2 + \lambda$. It follows from the above expression that

$$\begin{aligned} & \sum_{j=0}^{m+2} t^{\frac{j}{m+2}} |\lambda|^{1-\frac{j}{m+2}} \|u^{(j)}\|_{L^p(R;E)} + \|Au\|_{L^p(R;E)} \\ & = \sum_{j=0}^{m+2} t^{\frac{j}{m+2}} |\lambda|^{1-\frac{j}{m+2}} \|F^{-1}\xi^j L^{-1}(\lambda, t, \xi)F\bar{f}\|_{L^p(R;E)} + \|F^{-1}AL^{-1}(\lambda, t, \xi)F\bar{f}\|_{L^p(R;E)}. \end{aligned} \tag{4.13}$$

It is sufficient to show that the operator-functions

$$\begin{aligned} \Psi_{t\lambda}(\xi) &= AL^{-1}(\lambda, t, \xi)(1 + \xi^m)^{-1}, \\ \sigma_{t\lambda}(\xi) &= \sum_{j=0}^{m+2} t^{\frac{j}{m+2}} |\lambda|^{1-\frac{j}{m+2}} \xi^j L^{-1}(\lambda, t, \xi)(1 + \xi^m)^{-1} \end{aligned}$$

are uniform Fourier multipliers in $L^p(R; E)$. Actually, due to the positivity of A , we have

$$\begin{aligned} \|L^{-1}(\lambda, t, i\xi)\| &\leq M(1 + t\xi^2 + |\lambda|)^{-1}, \\ \|\Psi_{t,\lambda}(\xi)\| &= \|A[A + \lambda + t\xi^2]^{-1}\| \leq C_1. \end{aligned} \tag{4.14}$$

It is clear to observe that

$$\xi \frac{d}{d\xi} \Psi_{t,\lambda}(\xi) = -2t\xi^2 AL^{-2}(\lambda, t, \xi) = [-2t\xi^2 L^{-1}(\lambda, t, \xi)] AL^{-1}(\lambda, t, \xi).$$

Due to R -positivity of the operator A , the sets

$$\{-2t\xi^2[A + t\xi^2 + \lambda]^{-1}\}, \quad \{A[A + t\xi^2 + \lambda]^{-1}\}, \quad \xi \in \mathbb{R} \setminus \{0\}$$

are R -bounded. Then, in view of the Kahane contraction principle, from the product properties of the collection of R -bounded operators (see, e.g., [36] Lemma 3.5, Proposition 3.4), we obtain

$$\begin{aligned} R\left\{\xi^i \frac{d^i}{d\xi^i} \Psi_{t,\lambda}(\xi) : \xi \in \mathbb{R} \setminus \{0\}\right\} &\leq C_1, \\ R\left\{\xi^i \frac{d^i}{d\xi^i} \sigma_{t,\lambda}(\xi) : \xi \in \mathbb{R} \setminus \{0\}\right\} &\leq C_2, \quad i = 0, 1. \end{aligned} \tag{4.15}$$

By [33, Theorem 3.4] it follows that $\Psi_{t,\lambda}(\xi)$ and $\sigma_{t,\lambda}(\xi)$ are the uniform collection of multipliers in $L^p(R; E)$. Then in view of (4.13) we obtain that problem (4.11) has a solution $u \in W^{m+2,p}(R; E(A), E)$ and the uniform coercive estimate holds

$$\sum_{j=0}^{m+2} t^{\frac{j}{m+2}} |\lambda|^{1-\frac{j}{m+2}} \|u^{(j)}\|_{L^p(R; E)} + \|Au\|_{L^p(R; E)} \leq C \|\bar{f}\|_{L^p(R; E)}. \tag{4.16}$$

Let u_1 be the restriction of u on $(0, 1)$. The estimate (4.16) implies that $u_1 \in W^{m+2,p}(0, 1; E(A), E)$. By virtue of Lemma 4.1, we get

$$u_1^{(m_k)}(\cdot) \in (E(A); E)_{\theta_k, p}, \quad k = 1, 2.$$

Hence, $L_k u_1 \in E_k$. Thus, by virtue of Lemma 4.2, problem (4.12) has a unique solution $u_2(x)$ that belongs to the space $W^{m+2,p}(0, 1; E(A), E)$ and

$$\begin{aligned} &\sum_{j=0}^{m+2} t^{\frac{j}{m+2}} |\lambda|^{1-\frac{j}{m+2}} \|u_2^{(j)}\|_{L^p(0,1; E)} + \|Au_2\|_{L^p(0,1; E)} \\ &\leq C \sum_{k=1}^2 [\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|_E + t^{\theta_k} \|u_1^{(m_k)}\|_{C([0,1]; E_k)} + t^{\theta_k} |\lambda|^{1-\theta_k} \|u_1\|_{C([0,1]; E)}]. \end{aligned} \tag{4.17}$$

Moreover, from (4.16) we obtain

$$\sum_{j=0}^{m+2} t^{\frac{j}{m+2}} |\lambda|^{1-\frac{j}{m+2}} \|u_1^{(j)}\|_{L^p(0,1; E)} + \|Au_1\|_{L^p(0,1; E)} \leq C \|f\|_{W^{m,p}(0,1; E)}. \tag{4.18}$$

Therefore, by Lemma 4.1 and by estimate (4.17), we obtain

$$t^{\theta_k} \|u_1^{(m_k)}(\cdot)\|_{E_k} \leq C \|u_1\|_{W_t^{m+2,p}(0,1;E(A),E)} \leq C \|f\|_{W^{m,p}(0,1;E)}. \tag{4.19}$$

So, in view of Lemma 4.1 and estimates (4.17)-(4.19), we get

$$\begin{aligned} & \sum_{j=0}^{m+2} t^{\frac{j}{m+2}} |\lambda|^{1-\frac{j}{m+2}} \|u_2^{(j)}\|_{L^p(0,1;E)} + \|Au_2\|_{L^p(0,1;E)} \\ & \leq C \left(\|f\|_{L^p(0,1;E)} + \sum_{k=1}^2 (\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|_E) \right). \end{aligned} \tag{4.20}$$

Finally, from (4.18) and (4.20) we obtain (4.10). □

Now, we can prove the main result of this section.

Proof of Theorem 4.1 Let $G_2 = (0, b_1) \times (0, b_2)$. It is clear to see that

$$W^{m,q}(G_2; E) = W^{m,q}(0, b_1; X_0, X) = W^{m,q}(0, b_1; X) \cap L^q(0, b_1; X_0),$$

where $X_0 = W^{m,q}(0, b_2; E)$ and $X = L^q(0, b_2; E)$.

Let us consider the BVP

$$-\varepsilon_1 \frac{\partial^2 u}{\partial x_1^2} - \varepsilon_2 \frac{\partial^2 u}{\partial x_2^2} + (A + \lambda)u(x_1, x_2) = f(x_1, x_2), \quad L_{kj\varepsilon} u = 0, \quad k, j = 1, 2, \tag{4.21}$$

where $L_{kj\varepsilon}$ are defined by equalities (1.5). Problem (4.21) can be expressed as the following BVP for an ordinary DOE:

$$Lu = -\varepsilon_1 \frac{d^2 u}{dx_1^2} + (B_{\varepsilon_2} + \lambda)u(x_2) = f(x_2), \quad x_1 \in (0, b_1), \quad L_{k1\varepsilon_1} u = 0, \tag{4.22}$$

where $L_{k1\varepsilon_1}$ are boundary conditions of type (3.2), B_{ε_2} is the operator acting in X_0 and X defined by

$$\begin{aligned} B_{\varepsilon_2} u &= -\varepsilon_2 \frac{d^2 u}{dx_2^2} + Au(x_2), \quad D(B_{\varepsilon_2}) = W^{m+2,q}(0, b_2; E(A), E, L_{2k\varepsilon_2}), \\ & W^{2,q}(0, b_2; E(A), E, L_{2k\varepsilon_2}). \end{aligned}$$

Since X_0 and X are UMD spaces, (see, e.g., [35, Theorem 4.5.2]) by virtue of Theorem 4.2, we obtain that problem (4.22) has a unique solution $u \in W^{m+2,q}(0, b_1; D(B_{\varepsilon_2}), X)$ for $f \in W^{m,q}(0, b_1; X)$ and $\lambda \in S_{\psi, \varkappa}$ with sufficiently large $\varkappa > 0$. Moreover, the coercive uniform estimates holds

$$\begin{aligned} & \sum_{i=0}^{m+2} \varepsilon_1^{\frac{i}{m+2}} |\lambda|^{1-\frac{i}{m+2}} \|u^{(i)}\|_{L^q(0,b_1;X)} + \|B_{\varepsilon_2} u\|_{L^q(0,b_1;X)} \leq C \|f\|_{W^{m,q}(0,b_1;X)}, \\ & \sum_{i=0}^2 \varepsilon_1^{\frac{i}{2}} |\lambda|^{1-\frac{i}{2}} \|u^{(i)}\|_{L^q(0,b_1;X_0)} + \|B_{\varepsilon_2} u\|_{L^q(0,b_1;X_0)} \leq C \|f\|_{L^q(0,b_1;X_0)}. \end{aligned} \tag{4.23}$$

From (4.23) we obtain that problem (4.22) has a unique solution

$$u \in W^{m+2,q}(G_2; E(A), E) \quad \text{for } W^{m,q}(G_2; E).$$

Moreover, the uniform coercive estimates hold

$$\sum_{i=0}^{m+2} \varepsilon_1^{\frac{i}{m+2}} |\lambda|^{1-\frac{i}{m+2}} \|u^{(i)}\|_{L^q(0,b_1;X)} + \|B_{\varepsilon_2} u\|_{L^q(0,b_1;X)} \leq C \|f\|_{W^{m,q}(G_2;E)}. \tag{4.24}$$

By applying Theorem 4.2 for $f_k = 0$ and $E = X$, we get the following uniform estimate:

$$\sum_{j=0}^{m+2} \varepsilon_2^{\frac{j}{m+2}} |\lambda|^{1-\frac{j}{m+2}} \|u^{(j)}\|_X + \|Au\|_X \leq C \|B_{\varepsilon_2} u\|_{W^{m,q}(0,b_2;E)}. \tag{4.25}$$

From estimates (4.24)-(4.25) we conclude the corresponding claim for problem (4.21). Then, by continuing this process n -times, we obtain the assertion. \square

5 Nonlocal initial-boundary value problems for the Stokes system with small parameters

In this section, we show the uniform maximal regularity properties of the nonlocal initial value problem for nonstationary Stokes equations (1.6).

The function $u \in W_{\sigma}^{2,q}(G, L_{kj\varepsilon}) = \{u \in W^{2,q}(G; R^n), L_{kj\varepsilon} u = 0, \operatorname{div} u = 0\}$ satisfying equation (1.6) a.e. on G is called the stronger solution of problem (1.6).

Let $W^{s,q}(G)$, $0 < s < \infty$ be the Sobolev space of order s such that $W^{0,q}(G) = L^q(G)$. For $q \in (1, \infty)$, let $X_q = L_{\sigma}^q(G)$ denote the closure of $C_{0\sigma}^{\infty}(G)$ in $L^p(G; R^n)$, where

$$C_{0\sigma}^{\infty}(G) = \{u \in C_0^{\infty}(G), \operatorname{div} u = 0\}.$$

It is known that (see, e.g., Fujiwara and Morimoto [17]) a vector field $u \in L^q(G; R^n)$ has the Helmholtz decomposition, i.e., all $u \in L^q(G; R^n)$ can be uniquely decomposed as $u = u_0 + \nabla \varphi$ with $u_0 \in L_{\sigma}^q(G)$, $u_0 = P_q u$, where $P_q = P$ is a projection operator from $L^q(G; R^n)$ to $L_{\sigma}^q(G)$ and $\varphi \in L_{\operatorname{loc}}^q(\bar{G})$, $\nabla \varphi \in L^q(G; R^n)$, so that

$$\|\nabla \varphi\|_q \leq C \|u\|_q, \quad \|\varphi\|_{L^q(G \cap B)} \leq C \|u\|_q$$

with C independent of u , where B is an open ball in R^n and $\|u\|_p$ denotes the norm of u in $L^q(G; R^n)$ or $L^q(G)$.

Then problem (1.6) can be reduced to the following BVP:

$$\begin{aligned} -P \Delta_{\varepsilon} u + \lambda u &= f(x), \quad x \in G, \\ L_{kj\varepsilon} u &= \sum_{i=0}^{m_{kj}} \varepsilon_k^{\sigma_{ik}} \left[\alpha_{kji} \frac{\partial^i u}{\partial x_k^i}(G_{k0}) + \beta_{kji} \frac{\partial^i u}{\partial x_k^i}(G_{kb}) \right] = 0, \\ k &= 1, 2, \dots, n, j = 1, 2. \end{aligned} \tag{5.1}$$

Consider the parameter-dependent Stokes operator $O_\varepsilon = O_{\varepsilon,q}$ generated by problem (5.1), i.e.,

$$D(O_\varepsilon) = (W_\sigma^{2,q}(G; L_{kj}))^n, \quad O_\varepsilon u = -P \Delta_\varepsilon u.$$

From Corollary 3.1 we get that the operator O_ε is positive and also is a generator of a bounded holomorphic semigroup $S_\varepsilon(t) = \exp(-O_\varepsilon t)$ for $t > 0$.

In a similar way as in [18], we show the following.

Proposition 5.1 *The following estimate holds:*

$$\|O_\varepsilon^\alpha S_\varepsilon(t)\| \leq Ct^{-\alpha}$$

uniformly in $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ for $\alpha \geq 0$ and $t > 0$.

Proof From Result 3.1 we obtain that the operator O_ε is uniformly positive in $L_q(G; R^n)$, i.e., for $\lambda \in S_{\psi, \varkappa}$, $0 < \psi < \pi$, the following estimate holds:

$$\|(O_\varepsilon + \lambda)^{-1}\| \leq \frac{M}{|\lambda|},$$

where the constant M is independent of λ and ε . Then, by using the Danford integral and operator calculus as in [18], we obtain the assertion. \square

Now consider problem (1.7). The main theorem in this section is the following.

Theorem 5.1 *Let $0 < \varepsilon_k \leq 1$, $(-1)^{mk_1} \alpha_{k1} \beta_{k2} - (-1)^{mk_2} \alpha_{k2} \beta_{k1} \neq 0$ and $p, q \in (1, \infty)$. Then there is a unique solution $(u, \nabla \varphi)$ of problem (1.7) for $f \in L^p(0, T; L^q(G; R^n)) = B(p, q)$ and $a \in B_{p,q}^{\frac{2-\frac{2}{p}}{p}}$. Moreover, the following uniform estimate holds:*

$$\left\| \frac{\partial u}{\partial t} \right\|_{B(p,q)} + \sum_{k=1}^n \left\| \varepsilon_k \frac{\partial^2 u}{\partial x_k^2} \right\|_{B(p,q)} + \|\nabla \varphi\|_{B(p,q)} \leq C \|f\|_{B(p,q)} + \|a\|_{B_{p,q}^{\frac{2-\frac{2}{p}}{p}}(G)} \quad (5.2)$$

with $C = C(T, G, p, q)$ independent of f and ε .

Proof Problem (1.7) can be expressed as the following abstract parabolic problem with a small parameter:

$$\frac{du}{dt} + O_\varepsilon u = f(t), \quad u(0) = a. \quad (5.3)$$

If we put $E = L^q(G; R^n)$ and $A = \varkappa > 0$ in Theorem 3.1, then the Result 3.1 implies that the operator O_ε is uniformly positive and generates bounded holomorphic semigroup in $L^q(G; R^n)$ uniformly in ε_k . Moreover, by using [41, Theorem 3.1] we get that operator O_ε is R -positive in E . Since E is a UMD space, in a similar way as in [33, Theorem 4.2], we obtain that for $f \in L^p(0, T; E)$ and $a \in (D(O_\varepsilon), E)_{\frac{1}{p}, p}$, there is a unique solution $u \in W^{1,p}(0, T, D(O_\varepsilon), E)$ of problem (5.3) so that the following uniform estimate holds:

$$\left\| \frac{du}{dt} \right\|_{L^p(0,T;E)} + \|O_\varepsilon u\|_{L^p(0,T;E)} \leq C (\|f\|_{L^p(0,T;E)} + \|a\|_{(D(O_\varepsilon), E)_{\frac{1}{p}, p}}). \quad (5.4)$$

From (5.4) for all $u \in W_{\sigma}^{2,q}(G, L_{k_j \varepsilon})$, we get the following estimate:

$$\sum_{k=1}^n \varepsilon_k \left\| \frac{\partial^2 u}{\partial x_k^2} \right\|_q \leq C \|A_{\varepsilon} u\|_q$$

uniformly in $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$. □

6 Existence and uniqueness for the Navier-Stokes equation with parameters

In this section, we study the Navier-Stokes problem (1.1)-(1.3) in the space X_q . Problem (1.1)-(1.3) can be expressed as

$$\frac{du}{dt} + O_{\varepsilon} u = Fu + Pf, \quad u(0) = 0, \quad t > 0, \tag{6.1}$$

where

$$Fu = -P(u, \nabla)u.$$

We consider equation (6.1) in an integral form

$$u(t) = S_{\varepsilon}(t)a + \int_0^t S_{\varepsilon}(t-s)[Fu(s) + Pf(s)] ds, \quad t > 0. \tag{6.2}$$

To prove the main result, we need the following result which are obtained in a similar way as in [11, Theorem 2].

Lemma 6.1 *For any $0 \leq \alpha \leq 1$, the domain $D(O_{\varepsilon}^{\alpha})$ is the complex interpolation space $[X_q, D(O_{\varepsilon})]_{\alpha}$.*

Lemma 6.2 *For each $k = 1, 2, \dots, n$, the operator $u \rightarrow O_{\varepsilon}^{-\frac{1}{2}} P(\frac{\partial}{\partial x_k})u$ extends uniquely to a uniformly bounded linear operator from $L^q(G; R^n)$ to X_q .*

Proof Since O_{ε} is a positive operator, it has fractional powers O_{ε}^{α} . From Lemma 6.1, it follows that the domain $D(O_{\varepsilon}^{\alpha})$ is continuously embedded in $X_q \cap H_q^{2\alpha}(G; R^n)$ for any $\alpha > 0$, where $H_q^{2\alpha}(G; R^n)$ is the vector-valued Bessel space. Then, by using the duality argument and due to uniform positivity of $O_{\varepsilon}^{\frac{1}{2}}$, we obtain the following uniformly in ε estimate:

$$\left\| O_{\varepsilon}^{-\frac{1}{2}} P\left(\frac{\partial}{\partial x_k}\right)u \right\|_{L^q(G; R^n)} \leq C \|u\|_{X_q}. \tag{6.3}$$

By reasoning as in [12], we obtain the following. □

Lemma 6.3 *Let $0 \leq \delta < \frac{1}{2} + \frac{n}{2}(1 - \frac{1}{q})$. Then the following estimate holds:*

$$\|\varepsilon O_{\varepsilon}^{-\delta} P(u, \nabla)v\|_q \leq M \|O_{\varepsilon}^{\theta} u\|_q \|O_{\varepsilon}^{\sigma} u\|_q$$

uniformly in ε with some constant $M = M(\delta, \theta, q, \sigma)$ provided that $\theta > 0, \sigma > 0, \sigma + \delta > \frac{1}{2}$ and

$$\theta + \sigma + \delta > \frac{n}{2q} + \frac{1}{2}.$$

Proof Assume that $0 < \nu < \frac{n}{2}(1 - \frac{1}{q})$. Since $D(O_\varepsilon^\alpha)$ is continuously embedded in $X_q \cap H_q^{2\alpha}(G; R^n)$, and since $L^{q'}(G; R^n) \cap X_{q'}$ is the same as $X_{s'}$, by the Sobolev embedding theorem, we obtain that the operator

$$O_{\varepsilon, q'}^{-\nu} : X_{q'} \rightarrow D(O_{\varepsilon, q'}^\nu) \rightarrow X_{s'}$$

is bounded, where

$$\frac{1}{s'} = \frac{1}{q'} - \frac{2\nu}{n}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

By the duality argument then, we get that the operator $u \rightarrow O_{\varepsilon, q}^{-\nu}$ is bounded from X_s to X_q , where

$$\frac{1}{s} = 1 - \frac{1}{s'} = \frac{1}{q} + \frac{2\nu}{n}.$$

Consider first the case $\delta > \frac{1}{2}$. Since $P(u, \nabla)v$ is bilinear in u, v , it suffices to prove the estimate on a dense subspace. Therefore, assume that u and v are smooth. Since $\operatorname{div} u = 0$, we get

$$(u, \nabla)v = \sum_{k=1}^n \frac{\partial}{\partial x_k}(u_k v).$$

Taking $\nu = \delta - \frac{1}{2}$, using the uniform boundedness of $O_{\varepsilon, q}^{-\nu}$ from X_s to X_q and Lemma 6.2 for all $\varepsilon > 0$, we obtain

$$\|\varepsilon O_\varepsilon^{-\delta} P(u, \nabla)v\|_q = \left\| \varepsilon O_{\varepsilon, q}^{\frac{1}{2}-\nu} \sum_{k=1}^n P \frac{\partial}{\partial x_k}(u_k v) \right\|_q \leq \| |u| |v| \|_s.$$

By assumption we can take r and η such that

$$\frac{1}{r} \geq \frac{1}{q} - \frac{2\theta}{n}, \quad \frac{1}{\eta} \geq \frac{1}{q} - \frac{2\sigma}{n}, \quad \frac{1}{r} + \frac{1}{\eta} = \frac{1}{s}, \quad r > 1, \eta < \infty.$$

Since $D(O_{\varepsilon, q}^\alpha)$ is continuously embedded in $X_q \cap H_q^{2\alpha}(G; R^n)$, by the Sobolev embedding, we get

$$\| |u| |v| \|_s \leq \|u\|_r \|v\|_\eta \leq M \|O_{\varepsilon, q}^\theta u\|_r \|O_{\varepsilon, q}^\sigma v\|_\eta,$$

i.e., we have the required result for $\delta > \frac{1}{2}$. In particular, we get the following uniform estimate:

$$\|\varepsilon O_\varepsilon^{-\frac{1}{2}} P(u, \nabla)v\|_q \leq M \|O_{\varepsilon, q}^\theta u\|_r \|O_{\varepsilon, q}^\sigma v\|_\eta, \quad \theta + \beta \geq \frac{n}{2q}, \quad \beta > 0.$$

Similarly, we obtain

$$\|\varepsilon P(u, \nabla)v\|_q \leq C \|u\|_r \|v\|_\eta \leq C \|O_{\varepsilon, q}^\theta u\|_r \|O_{\varepsilon, q}^{\beta+\frac{1}{2}} v\|_\eta$$

for $\frac{1}{r} + \frac{1}{\eta} = \frac{1}{q}$ and $\delta = 0$. The above two estimates show that the map $v \rightarrow P(u, \nabla)v$ is a uniform bounded operator from $D(O_\varepsilon^\beta)$ to $D(O_\varepsilon^{-\frac{1}{2}})$ and from $D(O_\varepsilon^{\beta+\frac{1}{2}})$ to X_q . By using Lemma 6.1 and the interpolation of Banach spaces [30, §1.3.2] for $0 \leq \delta \leq \frac{1}{2}$, we obtain

$$\|\varepsilon P(u, \nabla)v\|_q \leq C \|O_{\varepsilon,q}^\theta u\|_r \|O_{\varepsilon,q}^\sigma v\|_\eta.$$

By using Lemma 6.3 and the iteration argument, by reasoning as in Fujita and Kato [18], we obtain the following. □

Theorem 6.1 *Let $0 < \varepsilon_k \leq 1$, $(-1)^{m_{k1}} \alpha_{k1} \beta_{k2} - (-1)^{m_{k2}} \alpha_{k2} \beta_{k1} \neq 0$. Let $\gamma < 1$ be a real number and $\delta \geq 0$ such that*

$$\frac{n}{2q} - \frac{1}{2} \leq \gamma, \quad -\gamma < \delta < 1 - |\gamma|.$$

Suppose that $a \in D(O_\varepsilon^\gamma)$, and that $\|O_\varepsilon^{-\delta} Pf(t)\|$ is continuous on $(0, T)$ and satisfies

$$\|O_\varepsilon^{-\delta} Pf(t)\| = o(t^{\gamma+\delta-1}) \quad \text{as } t \rightarrow 0.$$

Then there is $T_ \in (0, T)$ independent of ε and a local solution of (6.2) such that*

- (1) $u \in C([0, T_*]; D(O_\varepsilon^\gamma))$, $u(0) = a$;
- (2) $u \in C((0, T_*]; D(O_\varepsilon^\alpha))$ for some $T_* > 0$;
- (3) $\|O_\varepsilon^\alpha u(t)\| = o(t^{\gamma-\alpha})$ as $t \rightarrow 0$ for all α with $\gamma < \alpha < 1 - \delta$ uniformly with respect to the parameter ε .

Moreover, the solution of (5.2) is unique if

- (4) $u \in C((0, T_*]; D(O_\varepsilon^\beta))$;
- (5) $\|O_\varepsilon^\alpha u(t)\| = o(t^{\gamma-\beta})$ as $t \rightarrow 0$ for some β with $\beta > |\gamma|$ uniformly in ε .

Proof We introduce the following iteration scheme:

$$u_0(t) = S_\varepsilon(t)a + \int_0^t S_\varepsilon(t-s)Pf(s) ds, \tag{6.4}$$

$$u_{m+1}(t) = u_0(t) + \int_0^t S_\varepsilon(t-s)Fu_m(s) ds, \quad m \geq 0.$$

By estimating the term $u_0(t)$ in (6.4) and by using Proposition 5.1 for $\gamma \leq \alpha < 1 - \delta$, we get

$$\begin{aligned} \|O_\varepsilon^\alpha u_0(t)\| &\leq \|O_\varepsilon^\alpha S_\varepsilon(t)a\| + \int_0^t \|O_\varepsilon^{\alpha+\delta} S_\varepsilon(t-s)\| \|O_\varepsilon^{-\delta} Pf(s)\| ds \\ &\leq \|O_\varepsilon^\alpha S_\varepsilon(t)a\| + C_{\alpha+\delta} \int_0^t \|(t-s)\|^{-(\alpha+\delta)} \|O_\varepsilon^{-\delta} Pf(s)\| ds \leq M_\alpha t^{\gamma-\alpha} \end{aligned}$$

uniformly in ε with

$$M_\alpha = \sup_{0 < t \leq T, \varepsilon > 0} t^{\alpha-\gamma} \|O_\varepsilon^{\alpha+\delta} S_\varepsilon(t)a\| + C_{\alpha+\delta} NB(1 - \delta - \alpha, \gamma + \alpha),$$

where $N = \sup_{0 < t \leq T} t^{1-\gamma-\delta} \|O_\varepsilon^{-\delta} Pf(t)\|$ and $B(a, b)$ is the beta function. Here we suppose $\gamma + \delta > 0$. By induction assume that $u_m(t)$ satisfies the following estimate:

$$\|O_\varepsilon^\alpha u_m(t)\| \leq M_{\alpha m} t^{\gamma-\alpha}, \quad \gamma \leq \alpha < 1 - \delta. \quad (6.5)$$

We will estimate $O_\varepsilon^\alpha u_{m+1}(t)$ by using (6.2). To estimate the term $\|O_\varepsilon^{-\delta} Fu_m(s)\|$, we suppose

$$\theta + \sigma + \delta = 1 + \gamma, \quad \gamma < \theta < 1 - \delta, \gamma < \sigma < 1 - \delta, \theta > 0, \sigma > 0, \delta + \sigma > \frac{1}{2},$$

so that the numbers θ, σ, δ satisfy the assumptions of Lemma 6.3. Using Lemma 6.3 and (6.5), we get the following uniform estimate:

$$\|O_\varepsilon^{-\delta} Fu_m(s)\| \leq CM_{\theta m} M_{\sigma m} s^{\gamma+\delta-1}.$$

Therefore, we obtain

$$\begin{aligned} \|O_\varepsilon^\alpha u_m(t)\| &\leq M_\alpha t^{\gamma-\alpha} + M_{\alpha+\delta} \int_0^t \|(t-s)\|^{-(\alpha+\delta)} \|O_\varepsilon^{-\delta} Fu_m(s)\| ds \\ &\leq M_{\alpha m+1} t^{\gamma-\alpha} \end{aligned}$$

with

$$M_{\alpha m+1} = M_\alpha + M_{\alpha+\delta} MB(1 - \delta - \alpha, \gamma + \delta) M_{\theta m} M_{\sigma m}.$$

Since we get the uniform estimates with respect to the parameter ε , the remaining part of the proof is the same as in [12, Theorem 2.3], so this part is omitted. \square

Competing interests

The author declares that they have no competing interests.

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