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# Periodic solutions of radially symmetric systems with a singularity

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**Abstract**
In this paper, we study the existence of infinitely many periodic solutions to planar radially symmetric systems with certain strong repulsive singularities near the origin and with some semilinear growth near infinity. The proof of the main result relies on topological degree theory. Recent results in the literature are generalized and complemented.

**MSC:** 34C25

**Keywords:** periodic solution; singular systems; topological degree

## 1 Introduction

In this work, we are concerned with the existence of positive periodic solutions for the following radically symmetric system:

$$\ddot{x} + f\big(t, |x|\big)\frac{x}{|x|} = 0, \quad x \in \mathbb{R}^2 \setminus \{0\}, \tag{1.1}$$

where $f : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is $T$-periodic in the time variable $t$ for some $T > 0$ and satisfies the $L^1$-Carathéodory condition. Setting $r(t) = |x(t)|$, $f(t, r)$ may be singular at $r = 0$, we therefore look for non-collision solutions, *i.e.*, solutions which never attain the singularity.

Roughly speaking, system (1.1) is singular at 0 means that $f(t, r)$ becomes unbounded when $r \to 0^+$. We say that (1.1) is of repulsive type (attractive type) if $f(t, r) \to -\infty$ (respectively $f(t, r) \to +\infty$) when $r \to 0^+$.

Such a type of singular systems appears in many problems of applications. Such as, if we take $f(t, r) = c/r^2$ $(c > 0)$, it is the famous Newtonian equation

$$\ddot{x} + \frac{cx}{|x|^3} = 0, \quad x \in \mathbb{R}^2 \setminus \{0\},$$

which describes the motion of a particle subjected to the gravitational attraction of a sun that lies at the origin. If we take $f(t, r) = c/r^2$ $(c < 0)$, (1.1) may be used to model Rutherford's scattering of $\alpha$ particles by heavy atomic nuclei.

The question about the existence of non-collision periodic orbits for scalar equations and dynamical systems with singularities has attracted much attention of many researchers over many years [1–10]. There are two main lines of research in this area. The first one is the variational approach [11–13]. Usually, the proof requires some strong force condition, which was first introduced with this name by Gordon in [14], although the idea goes back at least to Poincaré [15]. Gordon's result, later improved by Capozzi, Greco and Salvatore [16], is stated as follows.

**Theorem 1.1** *Let $x(t) \in \mathbb{R}^2$ and the following assumptions hold.*

($A_1$) *The function $V$ is $T$-periodic in $t$, differentiable in $x \neq 0$ with continuous gradient, and such that*

$$\lim_{x\to 0} V(t,x) = -\infty.$$

($A_2$) *There exist $\nu \in [0,2)$ and positive constants $c_1$, $c_2$ such that*

$$V(t,x) \leq c_1|x|^{\nu} + c_2$$

*for every $t$ and $x \neq 0$.*

($A_3$) *There are a $C^1$-function $U : \mathbb{R}^2\backslash\{0\} \to \mathbb{R}$, a neighborhood $\mathcal{N}$ of $0$ and a positive constant $c_3$ such that*

$$\lim_{x\to 0} U(x) = -\infty \quad \text{and} \quad -V(t,x) \geq \left|\nabla U(x)\right|^2 - c_3$$

*for every $x \in \mathcal{N} \setminus \{0\}$, then, for every integer $k \geq 1$, the system*

$$\ddot{x} + \nabla V(t,x) = 0$$

*has a periodic solution with a minimal period $kT$.*

The strong force conditions ($A_2$), ($A_3$) guarantee that the minimization procedure does not lead to a collision orbit. This similar condition has been widely used for a voiding collisions in the singularity case. For example, if we consider the system

$$\ddot{x} = \frac{1}{|x|^{\alpha}} + f(t),$$

the strong force condition corresponds to the case $\alpha \geq 2$.

Besides the variational approach, topological methods have been widely applied, starting with the pioneering paper of Lazer and Solimini [17]. In particular, some classical tools have been used to study singular differential equations and dynamical systems in the literature, including the degree theory [18–23], the method of upper and lower solutions [24, 25], Schauder's fixed point theorem [26–28], some fixed point theorems in cones for completely continuous operators [29–32] and a nonlinear Leray-Schauder alternative principle [33–36]. Contrasting with the variational setting, the strong force condition plays here a different role linked to repulsive singularities. A counterexample in the paper of Lazer and Solimini [17] shows that a strong force assumption (unboundedness of the potential near the singularity) is necessary in some sense for the existence of positive periodic solutions in the scalar case.

However, compared with the case of strong singularities, the study of the existence of periodic solutions under the presence of weak singularities by topological methods is more recent and the number of references is much smaller. Several existence results can be found in [7, 26, 28].

As mentioned above, this paper is mainly motivated by the recent papers [19, 20]. The aim of this paper is to show that the topological degree theorem can be applied to the periodic problem. We prove the existence of large-amplitude periodic solutions whose minimal period is an integer multiple of $T$.

The rest of this paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, by the use of topological degree theory, we will state and prove the main results.

## 2 Preliminaries

In this section, we present some results which will be applied in Section 3. We may write the solutions of (1.1) in polar coordinates as follows:

$$x(t) = r(t)\big(\cos\varphi(t), \sin\varphi(t)\big). \tag{2.1}$$

Eq. (1.1) is then equivalent to the system

$$\begin{cases} \ddot{r} + f(t,r) - \frac{\mu^2}{r^3} = 0, \\ r^2\dot{\varphi} = \mu, \end{cases} \tag{2.2}$$

where $\mu$ is the (scalar) angular momentum of $x(t)$. Recall that $\mu$ is constant in time along any solution. In the following, when considering a solution of (2.2), we will always implicitly assume that $\mu \geq 0$ and $r > 0$.

If $x$ is a $T$-radially periodic, then $r$ must be $T$-periodic. We will prove the existence of a $T$-periodic solution $r$ of the first equation in (2.2). We thus consider the boundary value problem

$$\begin{cases} \ddot{r} + f(t,r) = \frac{\mu^2}{r^3}, \\ r(0) = r(T), \qquad \dot{r}(0) = \dot{r}(T). \end{cases} \tag{2.3}$$

Let $\mu = 0$, (2.3) can be written as the $T$-periodic problem

$$\ddot{r} + f(t,r) = 0. \tag{2.4}$$

Let $X$ be a Banach space of functions such that $C^1([0,T]) \subseteq X \subseteq C([0,T])$ with continuous immersions, and set $X_* = \{r \in X : \min r > 0\}$.

Define the following two operators:

$$D(L) = \big\{r \in W^{2,1}(0,T) : r(0) = r(T), \dot{r}(0) = \dot{r}(T)\big\},$$
$$L : D(L) \subset X \to L^1(0,T), \qquad Lr = \ddot{r}$$

and

$$N : X_* \to L^1(0,T), \qquad (Nr)(t) = -f\big(t, r(t)\big).$$

Taking $\sigma \in \mathbb{R}$ not belonging to the spectrum of $L$, (2.4) can be translated to the fixed problem

$$r = (L - \sigma I)^{-1}(N - \sigma I)r.$$

We will say that a set $\Omega \subseteq X$ is uniformly positively bounded below if there is a constant $\delta > 0$ such that $\min r \geq \delta$ for every $r \in \Omega$. In order to prove the main result of this paper, we need the following theorem, which has been proved in [18].

**Theorem 2.1** *Let $\Omega$ be an open bounded subset of $X$, uniformly positively bounded below. Assume that there is no solution of* (2.4) *on the boundary $\partial\Omega$, and that*

$$\deg\bigl(I - (L - \sigma I)^{-1}(N - \sigma I), \Omega, 0\bigr) \neq 0.$$

*Then, there exists a $k_1 \geq 1$ such that, for every integer $k \geq k_1$, system* (1.1) *has a periodic solution $x_k(t)$ with a minimal period $kT$, which makes exactly one revolution around the origin in the period time $kT$. The function $|x_k(t)|$ is $T$-periodic and, when restricted to* $[0, T]$, *it belongs to $\Omega$. Moreover, if $\mu_k$ denotes the angular momentum associated to $x_k(t)$, then*

$$\lim_{k \to \infty} \mu_k = 0.$$

## 3 Main results

First we introduce some known results on eigenvalues. Let $q(t)$ be a $T$-periodic potential such that $q \in L^1(\mathbb{R})$. Consider the eigenvalue problems of

$$x'' + \bigl(\lambda + q(t)\bigr)x = 0 \tag{3.1}$$

with the periodic boundary condition $(PC)$: $x(0) = x(T)$, $x'(0) = x'(T)$, or with the antiperiodic boundary condition $(APC)$: $x(0) = -x(T)$, $x'(0) = -x'(T)$. We use $\lambda_1^D(q) < \lambda_2^D(q) < \cdots < \lambda_n^D(q) < \cdots$ to denote all the eigenvalues of (3.1) with the Dirichlet boundary condition $(DC)$: $x(0) = x(T) = 0$.

The following are the standard results for eigenvalues. See, *e.g.*, reference [37].

($\mathrm{E}_1$) With respect to the periodic and anti-periodic eigenvalues, there exist sequences

$$-\infty < \overline{\lambda}_0(q) < \underline{\lambda}_1(q) \leq \overline{\lambda}_1(q) < \underline{\lambda}_2(q) \leq \overline{\lambda}_2(q) < \cdots < \underline{\lambda}_n(q) \leq \overline{\lambda}_n(q) < \cdots,$$

where $\underline{\lambda}_n(q), \overline{\lambda}_n(q) \to +\infty$ (as $n \to +\infty$), such that $\lambda$ is an eigenvalue of (3.1)-$(PC)$ if and only if $\lambda = \underline{\lambda}_n(q)$ or $\overline{\lambda}_n(q)$ with $n$ is even; and $\lambda$ is an eigenvalue of (3.1)-$(APC)$ if and only if $\lambda = \underline{\lambda}_n(q)$ or $\overline{\lambda}_n(q)$ with $n$ is odd.

($\mathrm{E}_2$) The comparison results hold for all of these eigenvalues. If $q_1 \leq q_2$, then

$$\underline{\lambda}_n(q_1) \geq \underline{\lambda}_n(q_2), \qquad \overline{\lambda}_n(q_1) \geq \overline{\lambda}_n(q_2), \qquad \lambda_n^D(q_1) \geq \lambda_n^D(q_2)$$

for any $n \geq 1$.

($E_3$) The eigenvalues $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$ can be recovered from the Dirichlet eigenvalues in the following way. For any $n \geq 1$,

$$\underline{\lambda}_n(q) = \min\{\lambda_n^D(q_{t_0}) : t_0 \in \mathbb{R}\}, \qquad \overline{\lambda}_n(q) = \max\{\lambda_n^D(q_{t_0}) : t_0 \in \mathbb{R}\},$$

where $q_{t_0}(t)$ denotes the translation of $q(t)$: $q_{t_0}(t) \equiv q(t + t_0)$.

Now we present our main result.

**Theorem 3.1** *Let the following assumptions hold.*

($H_1$) *There exist a constant $R_0 > 0$ and a function $f_0 \in C((0,\infty),\mathbb{R})$ such that*

$$f(t,r) \leq -f_0(r)$$

*for all $t$ and all $0 < r \leq R_0$, where $f_0$ satisfies*

$$\lim_{r\to 0^+} f_0(r) = +\infty$$

*and*

$$\lim_{r\to 0^+} \int_1^r f_0(r)\,dr = -\infty.$$

($H_2$) *There exist positive $T$-periodic continuous functions $\phi$, $\Phi$ such that*

$$\phi(t) \leq \liminf_{r\to +\infty} \frac{f(t,r)}{r} \leq \limsup_{r\to +\infty} \frac{f(t,r)}{r} \leq \Phi(t) \tag{3.2}$$

*uniformly in $t$. Moreover,*

$$\underline{\lambda}_1(\Phi) > 0. \tag{3.3}$$

*Then Eq.* (2.4) *has a $T$-periodic solution, and there exists a $k_1 \geq 1$ such that, for every integer $k \geq k_1$, Eq.* (1.1) *has a periodic solution with a minimal period $kT$, which makes exactly one revolution around the origin in the period time $kT$. Moreover, there exists a constant $C > 0$* (*independent of $\mu$ and $k$*) *such that*

$$\frac{1}{C} < |x_k(t)| < C \quad \textit{for every } t \in \mathbb{R} \textit{ and every } k \geq k_1,$$

*and if $\mu_k$ denotes the angular momentum associated to $x_k(t)$, then*

$$\lim_{k\to\infty} \mu_k = 0.$$

In order to apply Theorem 2.1, we consider the $T$-periodic problem (2.4).

**Lemma 3.2** *Suppose that $f(t,r)$ satisfies* ($H_1$) *and $\phi$, $\Phi$ satisfy* ($H_2$). *Then Eq.* (2.4) *has at least one positive $T$-periodic solution.*

Now we begin by showing that Lemma 3.2 holds, and use topological degree theory. To this end, we deform (2.4) to a simpler singular autonomous equation

$$r'' + ar = \frac{1}{r},$$

where $a$ for some positive constant satisfies $0 < a < (\pi/T)^2$ for all $t$. Consider the following homotopy equation:

$$r'' + f(t,r;\tau) = 0, \quad \tau \in [0,1], \tag{3.4}$$

where $f(t,r;\tau) = \tau f(t,r) + (1-\tau)(ar - \frac{1}{r})$. We need to find *a priori* estimates for the possible positive $T$-periodic solutions of (3.4).

Note that $f(t,r;\tau)$ satisfies the conditions $(\mathrm{H}_1)$ uniformly with respect to $\tau \in [0,1]$. Moreover, for each $\tau \in [0,1]$, $f(t,r;\tau)$ satisfies (3.2) with $\phi = \phi_\tau = \tau\phi(t) + (1-\tau)a$ and $\Phi = \Phi_\tau = \tau\Phi(t) + (1-\tau)a$. We will prove that $\Phi_\tau$ satisfy (3.3) uniformly in $\tau \in [0,1]$. The usual $L^p$-norm is denoted by $\|\cdot\|_p$, and the supremum norm of $\mathbb{C}[0,T]$ is denoted by $\|\cdot\|_\infty$.

This follows from the convexity of the first eigenvalues with respect to potentials.

**Lemma 3.3** *Given $q_0, q_1 \in L^1(0,T)$. Then, for all $\tau \in [0,1]$,*

$$\underline{\lambda}_1\bigl(\tau q_1 + (1-\tau)q_0\bigr) \geq \tau\underline{\lambda}_1(q_1) + (1-\tau)\underline{\lambda}_1(q_0). \tag{3.5}$$

*Proof* Put $q_\tau = \tau q_1 + (1-\tau)q_0$, $\tau \in [0,1]$. Then

$$\begin{aligned}
&\lambda_1^D(q_\tau)\\
&\quad= \inf_{\substack{\varphi \in H_0^1(0,T)\\ \|\varphi\|_2 = 1}} \int_0^T \bigl(\varphi'^2(t) - q_\tau(t)\varphi^2(t)\bigr)\,dt\\
&\quad= \inf_{\substack{\varphi \in H_0^1(0,T)\\ \|\varphi\|_2 = 1}} \left(\tau \int_0^T \bigl(\varphi'^2(t) - q_1(t)\varphi^2(t)\bigr)\,dt + (1-\tau)\int_0^T \bigl(\varphi'^2(t) - q_0(t)\varphi^2(t)\bigr)\,dt\right)\\
&\quad\geq \tau \inf_{\substack{\varphi \in H_0^1(0,T)\\ \|\varphi\|_2 = 1}} \int_0^T \bigl(\varphi'^2(t) - q_1(t)\varphi^2(t)\bigr)\,dt\\
&\qquad+ (1-\tau)\inf_{\substack{\varphi \in H_0^1(0,T)\\ \|\varphi\|_2 = 1}} \int_0^T \bigl(\varphi'^2(t) - q_0(t)\varphi^2(t)\bigr)\,dt\\
&\quad= \tau\lambda_1^D(q_1) + (1-\tau)\lambda_1^D(q_0).
\end{aligned}$$

For (3.5), applying $\lambda_1^D(q_\tau) \geq \tau\lambda_1^D(q_1) + (1-\tau)\lambda_1^D(q_0)$ to $q_i = q_{i,t_0}$, where $t_0 \in [0,T]$, we have

$$\lambda_1^D(q_{\tau,t_0}) \geq \tau\lambda_1^D(q_{1,t_0}) + (1-\tau)\lambda_1^D(q_{0,t_0})$$

for all $t_0$. Thus

$$\begin{aligned}\underline{\lambda}_1(q_\tau) &= \min_{t_0} \lambda_1^D(q_{\tau,t_0}) \geq \min_{t_0}\big(\tau\lambda_1^D(q_{1,t_0}) + (1-\tau)\lambda_1^D(q_{0,t_0})\big) \\ &\geq \min_{t_0} \tau\lambda_1^D(q_{1,t_0}) + (1-\tau)\min_{t_0}\lambda_1^D(q_{0,t_0}) \\ &= \tau\underline{\lambda}_1(q_1) + (1-\tau)\underline{\lambda}_1(q_0).\end{aligned}$$

Hence (3.5) holds. □

Applying Lemma 3.3 to $q_1 = \Phi$ and $q_0 = a$, we have

$$\underline{\lambda}_1(\Phi_\tau) \geq \tau\underline{\lambda}_1(\Phi) + (1-\tau)\underline{\lambda}_1(a) \geq \min\big(\underline{\lambda}_1(\Phi), \underline{\lambda}_1(a)\big) > 0.$$

Thus $\Phi_\tau$ defined above satisfy (3.3) uniformly in $\tau \in [0,1]$.

In the obtention of *a priori* estimates for all possible positive solutions to (3.4)-($PC$), we simply prove this for all possible positive solutions to (2.4)-($PC$), because $\phi_\tau$, $\Phi_\tau$ satisfy (3.3) and also (3.2) uniformly in $\tau \in [0,1]$.

**Lemma 3.4** *Assume that* $\underline{\lambda}_1(\Phi) > 0$ *of the equation* $y'' + (\lambda + \Phi(t))y = 0$, *then*

$$\|y'\|_2^2 \geq \int_0^T \Phi(t+t_0)y^2(t)\,dt + \lambda_1^D(\Phi_{t_0})\int_0^T y^2(t)\,dt.$$

*Proof* By the results for eigenvalues in ($E_3$), we have

$$\lambda_1^D(\Phi_{t_0}) \geq \underline{\lambda}_1(\Phi) > 0$$

for all $t_0 \in \mathbb{R}$.

Then, by the theory of linear second-order differential operators [38], the eigenvalues of $y'' + (\lambda + \Phi(t+t_0))y = 0$ with Dirichlet boundary conditions form a sequence $\lambda_1^D(\Phi_{t_0}) < \lambda_2^D(\Phi_{t_0}) < \cdots$ which tends to $+\infty$, and the corresponding eigenfunctions $\psi_1, \psi_2, \ldots$ are an orthonormal base of $L^2(0,T)$. Hence, given $c_i \in \mathbb{R}$ and $y \in H_0^1(0,T)$, we can write

$$y(t) = \sum_{i\geq 1} c_i\psi_i(t),$$

and

$$\begin{aligned}\int_0^T \big((y'(t))^2 - \Phi(t+t_0)y^2(t)\big)\,dt &= \sum_{i\geq 1} c_i^2 \int_0^T \big((\psi_i'(t))^2 - \Phi(t+t_0)\psi_i^2(t)\big)\,dt \\ &= \sum_{i\geq 1} c_i^2\lambda_i^D(\Phi_{t_0})\int_0^T \psi_i^2(t)\,dt \\ &\geq \lambda_1^D(\Phi_{t_0})\int_0^T y^2(t)\,dt.\end{aligned}$$

This completes the proof. □

**Lemma 3.5** *Under the assumptions as in Theorem* 3.1, *there exist* $C_2 > C_1 > 0$ *such that any positive* $T$*-periodic solution* $r(t)$ *of* (2.4)-($PC$) *satisfies*

$$C_1 < r(t_0) < C_2 \tag{3.6}$$

*for some* $t_0 \in [0, T]$.

*Proof* Let $r(t)$ be a positive $T$-periodic solution of (2.4)-($PC$). By ($H_1$), there is $C_1 > 0$ such that

$$f(t,s) < 0 \quad \text{for all } 0 < s < C_1.$$

Integrating (2.4) from 0 to $T$, we get

$$\int_0^T r''(t)\,dt + \int_0^T f\bigl(t, r(t)\bigr)\,dt = 0.$$

Thus $\int_0^T f(t, r(t))\,dt = 0$, there exist $t^* \in [0, T]$ such that $r(t^*) > C_1$.

Take some constant $\varepsilon_0 \in (0, \min\{\overline{\phi}, \underline{\lambda}_1(\Phi)\})$, where $\bar{\phi} = \frac{1}{T}\int_0^T \phi(t)\,dt$ is the average of $\phi(t)$. From ($H_2$) there is $C_2$ ($> C_1$) large enough such that

$$\phi(t) - \varepsilon_0 \le \frac{f(t,s)}{s} \le \Phi(t) + \varepsilon_0 \tag{3.7}$$

for all $t$ and $s \ge C_2$. We assert that $r(t_*) < C_2$ for some $t_*$. Otherwise, assume that $r(t) \ge C_2$ for all $t$.

Let

$$p(t) = \frac{f(t, r(t))}{r(t)} \in \bigl(\phi(t) - \varepsilon_0, \Phi(t) + \varepsilon_0\bigr).$$

Moreover, write $r$ as $r = \tilde{r} + \bar{r}$, then $\tilde{r}$ satisfies the following differential equation:

$$\tilde{r}'' + p(t)\tilde{r} + p(t)\bar{r} = 0. \tag{3.8}$$

Integrating (3.8) from 0 to $T$, we have

$$\int_0^T p(t)\tilde{r}(t)\,dt = -\bar{r}\int_0^T p(t)\,dt. \tag{3.9}$$

Multiplying (3.8) by $\tilde{r}$ and integrating, we get

$$\begin{aligned}
\|\tilde{r}'\|_2^2 &= \int_0^T p(t)\tilde{r}^2(t)\,dt + \bar{r}\int_0^T p(t)\tilde{r}(t)\,dt \\
&= \int_0^T p(t)\tilde{r}^2(t)\,dt - \bar{r}^2(t)\int_0^T p(t)\,dt \\
&\le \int_0^T p(t)\tilde{r}^2(t)\,dt,
\end{aligned} \tag{3.10}$$

where the fact $\frac{1}{T}\int_0^T p(t)\,dt > \bar{\phi} - \varepsilon_0 > 0$ is used.

Note that $\tilde{r}(t_0) = 0$ for some $t_0$, $\tilde{r}(t_0 + T) = 0$, so $\tilde{r}(t) \in H_0^1(t_0, t_0 + T)$. We assert that $\tilde{r} \equiv 0$. On the contrary, assume that $\tilde{r} \not\equiv 0$. Now, by (3.10), the first Dirichlet eigenvalue

$$\lambda_1^D(p|_{[t_0,t_0+T]}) = \inf_{\substack{\varphi \in H_0^1(t_0,t_0+T) \\ \varphi \neq 0}} \frac{\int_{t_0}^{t_0+T} (\varphi'^2(t) - p(t)\varphi^2(t))\,dt}{\int_{t_0}^{t_0+T} \varphi^2(t)\,dt} \le 0.$$

So,

$$\underline{\lambda}_1(p) = \min\{\lambda_1^D(p)\} \le 0.$$

On the other hand, $p(t) < \Phi(t) + \varepsilon_0$,

$$\underline{\lambda}_1(p) \ge \underline{\lambda}_1(\Phi + \varepsilon_0) = \underline{\lambda}_1(\Phi) - \varepsilon_0 > 0.$$

This is a contradiction.

Now it follows from (3.9) that $\bar{r} = 0$ and $r \equiv 0$, a contradiction to the positiveness of $r(t)$. We have proved that $r(t^*) > C_1$ for some $t^* \in [0, T]$ and $r(t_*) < C_2$ for some $t_* \in [0, T]$. Thus the intermediate value theorem implies that (3.6) holds. □

**Lemma 3.6** *There exist $C_3 > C_2 > 0$, $C_4 > 0$ such that any positive $T$-periodic solution $r(t)$ of* (2.4)-(*PC*) *satisfies*

$$\|r\|_\infty < C_3, \qquad \|r'\|_\infty < C_4.$$

*Proof* From $(H_2)$ and (3.7), we know that there is $h_0 > 0$ such that

$$f(t,s) \le (\Phi(t) + \varepsilon_0)s + h_0$$

for all $t$ and $s > 0$.

Multiplying (2.4) by $r$ and then integrating over $[0, T]$, we get

$$\begin{aligned}
\|r'\|_2^2 &= \int_0^T f(t, r(t))r(t)\,dt \\
&\le \int_0^T \big((\Phi(t) + \varepsilon_0)r(t) + h_0\big)r(t)\,dt \\
&= \int_0^T \Phi(t)r^2(t)\,dt + \varepsilon_0\|r\|_2^2 + h_0\|r\|_1.
\end{aligned} \tag{3.11}$$

Note from Lemma 3.5 that there exists $t_0$ satisfying $C_1 < r(t_0) < C_2$. Let $u(t) = r(t + t_0) - r(t_0)$, then $u \in H_0^1(0, T)$. Thus

$$\begin{aligned}
\int_0^T \Phi(t)r^2(t)\,dt &= \int_0^T \Phi(t + t_0)r^2(t + t_0)\,dt \\
&= \int_0^T \Phi(t + t_0)\big(r^2(t_0) + 2r(t_0)u(t) + u^2(t)\big)\,dt \\
&\le C_2^2\|\Phi\|_1 + 2C_2\|\Phi\|_2\|u\|_2 + \int_0^T \Phi(t + t_0)u^2(t)\,dt.
\end{aligned}$$

The other terms in (3.11) by the Hölder inequality can be estimated as follows:

$$\begin{aligned}
&\varepsilon_0\|r\|_2^2 \le \varepsilon_0\bigl(TC_2^2 + 2C_2T^{\frac{1}{2}}\|u\|_2 + \|u\|_2^2\bigr),\\
&h_0\|r\|_1 \le h_0\bigl(TC_2 + T^{\frac{1}{2}}\|u\|_2\bigr).
\end{aligned}$$

Thus (3.11) reads as

$$\|u'\|_2^2 \le A_0 + B_0\|u\|_2 + \varepsilon_0\|u\|_2^2 + \int_0^T \Phi(t+t_0)u^2(t)\,dt, \tag{3.12}$$

where $A_0 = \varepsilon_0TC_2^2 + h_0TC_2 + C_2^2\|\Phi\|_1$, $B_0 = 2\varepsilon_0C_2T^{\frac{1}{2}} + h_0T^{\frac{1}{2}} + 2C_2\|\Phi\|_2$ are positive constants.

On the other hand, using Lemma 3.4,

$$\underline{\lambda}_1\bigl(\Phi(t)\bigr)\|u\|_2^2 \le \lambda_1^D(\Phi_{t_0})\|u\|_2^2 \le \int_0^T \bigl(u'^2(t) - \Phi(t+t_0)u^2(t)\bigr)\,dt$$

we get from (3.12) that

$$\bigl(\varepsilon_0 - \underline{\lambda}_1\bigl(\Phi(t)\bigr)\bigr)\|u\|_2^2 + B_0\|u\|_2 + A_0 \ge 0.$$

Consequently, $\|u\|_2 < A_1$ for some $A_1 > 0$. By (3.12), one has $\|r'\|_2 = \|u'\|_2 < A_2$ for some $A_2 > 0$. From these, for any $t \in [t_0, t_0+T]$,

$$\begin{aligned}
|r(t)| &\le |r(t_0)| + \left|\int_{t_0}^t r'(t)\,dt\right|\\
&\le C_2 + T^{\frac{1}{2}}\|r'\|_2\\
&\le C_2 + T^{\frac{1}{2}}A_2 := C_3.
\end{aligned}$$

Thus $\|r\|_\infty < C_3$ is obtained.

As $\int_0^T f(t,r(t))\,dt = 0$, thus $\|f(t,r(t))\|_1 = 2\|f^+(t,r(t))\|_1$. Since $r(0) = r(T)$, there exists $t_1 \in [0,T]$ such that $r'(t_1) = 0$. Therefore

$$\begin{aligned}
\|r'\|_\infty &= \max_{0\le t\le T}|r'(t)| = \max_{0\le t\le T}\left|\int_{t_1}^t r''(s)\,ds\right|\\
&\le \int_0^T \bigl|f\bigl(s,r(s)\bigr)\bigr|\,ds = 2\int_0^T \bigl|f^+\bigl(s,r(s)\bigr)\bigr|\,ds\\
&\le 2\int_0^T \bigl|\bigl(\Phi^+(s) + \varepsilon_0\bigr)r(s) + h_0\bigr|\,ds\\
&\le 2\bigl(\bigl(\|\Phi^+\|_1 + T\varepsilon_0\bigr)C_3 + h_0T\bigr) := C_4,
\end{aligned}$$

where $\Phi^+(t) = \max\{\Phi(t), 0\}$, $f^+(t,r(t)) = \max\{f(t,r(t)), 0\}$. □

Next, the positive lower estimates for $m = \min_{t\in[0,T]} r(t)$ are obtained from the condition $(\mathrm{H}_1)$.

**Lemma 3.7** *There exists a constant $C_5 \in (0, C_1)$ such that any positive solution $r(t)$ of* (2.4)-*(PC) satisfies*

$$r(t) > C_5 \quad \text{for all } t.$$

*Proof* From ($H_1$), we fix some $B_1 \in (0, C_1)$ such that

$$f(t,s) < -C_4$$

for all $t$ and all $0 < s \le B_1$. Assume now that

$$m = \min_{t\in[0,T]} r(t) = r(t_2) < B_1.$$

By Lemma 3.5, $\max_t r(t) > C_1$. Let $t_3 > t_2$ be the first time instant such that $r(t) = B_1$. Then, for any $t \in [t_2, t_3]$, we have $r(t) \le B_1$. Hence, for $t \in [t_2, t_3]$,

$$r''(t) = -f\big(t, r(t)\big) > C_4 \ge 0.$$

As $r'(t_2) = 0$, $r'(t) > 0$ for $t \in (t_2, t_3]$. Therefore, the function $r : [t_2, t_3] \to \mathbb{R}$ has an inverse denoted by $\xi$.

Now multiplying (2.4) by $r'(t)$ and integrating over $[t_2, t_3]$, we get

$$\begin{aligned}\int_m^{B_1} -f\big(\xi(r), r\big)\, dr &= \int_{t_2}^{t_3} -f\big(t, r(t)\big) r'(t)\, dt \\ &= \int_{t_2}^{t_3} r''(t) r'(t)\, dt = \frac{1}{2}\big(r'(t_3)\big)^2 \le B_2\end{aligned}$$

for some $B_2 > 0$, where the results from Lemma 3.6 are used. By ($H_1$),

$$\int_m^{B_1} -f\big(\xi(r), r\big)\, dr \ge \int_m^{B_1} f_0(r)\, dr \to +\infty \tag{3.13}$$

if $m \to 0^+$. Thus we know from (3.13) that $m > C_5$ for some constant $C_5 > 0$. □

Now we give the proof of Lemma 3.2. Consider the homotopy equation (3.4), we can get *a priori* estimates as in Lemmas 3.5, 3.6, 3.7. That is, any positive $T$-periodic solution of (3.4) satisfies

$$C_5' < r(t) < C_3', \qquad \|r'\|_\infty < C_4'$$

for some positive constants $C_5'$, $C_3'$, $C_4'$. Define $C = \max\{1/C_5', C_3', C_4'\}$ and let the open bounded in $X$ be

$$\Omega = \left\{ r \in X : \frac{1}{C} < r(t) < C \text{ and } \big|r'(t)\big| < C \text{ for all } t \in [0, T] \right\}.$$

By the homotopy invariance of degree and the result of Capietto, Mawhin and Zanolin [39],

$$\deg\big(I - (L - \sigma I)^{-1}(N - \sigma I), \Omega, 0\big) = \deg(ar - 1/r, \Omega \cap \mathbb{R}, 0) = 1.$$

Thus (3.4), with $\tau = 1$, has at least one solution in $\Omega$, which is a positive $T$-periodic solution of (2.4). By Theorem 2.1, the proof of Theorem 3.1 is thus completed.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors read and approved the final manuscript.

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