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Existence of a solution for a three-point boundary value problem for a second-order differential equation at resonance

Juan J Nieto*

*Correspondence: juanjose.nieto.roig@usc.es
Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela, 15782, Spain
Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia

Abstract
We present a new existence result for a second-order nonlinear ordinary differential equation with a three-point boundary value problem when the linear part is noninvertible.
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Keywords: nonlinear ordinary differential equation; three-point boundary value problem; problem at resonance; existence of solution

1 Introduction

The study of multi-point boundary value problems for linear second-order ordinary differential equations goes back to the method of separation of variables [1]. Also, some questions in the theory of elastic stability are related to multi-point problems [2]. In 1987, Il'in and Moiseev [3, 4] studied some nonlocal boundary value problems. Then, for example, Gupta [5] considered a three-point nonlinear boundary value problem. For some recent works on nonlocal boundary value problems, we refer, for example, to [6–15] and references therein.

As indicated in [16], there has been enormous interest in nonlinear perturbations of linear equations at resonance since the seminal paper of Landesman and Lazer [17]; see [18] for further details.

Here we study the following nonlinear ordinary differential equation of second order subject to the three-point boundary condition:

$$\begin{aligned} &-u''(t) = f\bigl(t, u(t)\bigr), \quad t \in [0, T], \\ &u(0) = 0, \qquad \alpha u(\eta) = u(T), \end{aligned} \tag{1}$$

where $T > 0, f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function $\alpha \in \mathbb{R}$ and $\eta \in (0, T)$.

In this paper we consider the resonance case $\alpha\eta = T$ to obtain a new existence result. Although this situation has already been considered in the literature [19], we point out that our approach and methodology is different.

2 Linear problem

Consider the linear second-order three-point boundary value problem

$$\begin{aligned} &-u''(t) = \sigma(t), \quad t \in [0, T], \\ &u(0) = 0, \qquad \alpha u(\eta) = u(T) \end{aligned} \tag{2}$$

for a given function $\sigma \in C[0, T]$.

The general solution is

$$u(t) = c_1 + c_2 t - \int_0^t (t-s)\sigma(s)\,ds$$

with c_1, c_2 arbitrary constants.

From $u(0) = 0$, we get $c_1 = 0$. From the second boundary condition, we have

$$(T - \alpha\eta)c_2 = \int_0^T (T-s)\sigma(s)\,ds - \alpha \int_0^\eta (\eta - s)\sigma(s)\,ds. \tag{3}$$

2.1 Nonresonance case

If $\alpha\eta \neq T$, then

$$c_2 = \frac{1}{T - \alpha\eta}\left[\int_0^T (T-s)\sigma(s)\,ds - \alpha \int_0^\eta (\eta - s)\sigma(s)\,ds\right],$$

and the linear problem (2) has a unique solution for any $\sigma \in C[0, T]$. In this case, we say that (2) is a nonresonant problem since the homogeneous problem has only the trivial solution as a solution, *i.e.*, when $\sigma = 0$, $c_1 = c_2 = 0$ and $u = 0$. Note that the solution is given by

$$u(t) = \int_0^T g(t,s)\sigma(s)\,ds \tag{4}$$

with

$$g(t,s) = \begin{cases} \frac{t(T-s)}{T-\alpha\eta} - \frac{t\alpha(\eta-s)}{T-\alpha\eta} - (t-s), & 0 \le s < \min(\eta, t), \\ \frac{t(T-s)}{T-\alpha\eta} - \frac{t\alpha(\eta-s)}{T-\alpha\eta}, & 0 \le t < s < \eta < T, \\ \frac{t(T-s)}{T-\alpha\eta} - (t-s), & 0 \le \eta < s < t \le T, \\ \frac{t(T-s)}{T-\alpha\eta}, & \max(\eta, t) < s \le T. \end{cases}$$

For $T = 1$ this is precisely the function given in Lemma 2.3 of [20] or in Remark 12 of [21].

2.2 Resonance case

If $T = \alpha\eta$, then (3) is solvable if and only if

$$\int_0^T (T-s)\sigma(s)\,ds = \alpha \int_0^\eta (\eta - s)\sigma(s)\,ds, \tag{5}$$

and then (2) has a solution if and only if (5) holds. In such a case, (2) has an infinite number of solutions given by

$$u(t) = ct - \int_0^t (t-s)\sigma(s)\,ds, \quad c \in \mathbb{R}.$$

In particular ct, $c \in \mathbb{R}$ is a solution of the homogeneous linear equation

$$-u''(t) = 0, \quad t \in [0,T]$$

satisfying the boundary conditions

$$u(0) = 0, \qquad \alpha u(\eta) = u(T).$$

Note that

$$u(T) - u(\eta) = c_2 T - \int_0^T (T-s)\sigma(s)\,ds - c_2\eta + \int_0^\eta (\eta - s)\sigma(s)\,ds,$$

and then

$$c_2 = \frac{1}{T-\eta}\left[u(T) - u(\eta) + \int_0^T (T-s)\sigma(s)\,ds - \int_0^\eta (\eta-s)\sigma(s)\,ds\right].$$

We now use that $u(T) = \frac{T}{\eta}u(\eta)$ to get

$$\frac{1}{T-\eta}\left[u(T) - u(\eta)\right] = \frac{1}{T}u(T)$$

and

$$c_2 = \frac{1}{T-\eta}\left[\int_0^T (T-s)\sigma(s)\,ds - \int_0^\eta (\eta-s)\sigma(s)\,ds\right] + \frac{1}{T}u(T).$$

Hence the solution of (2) is given, implicitly, as

$$u(t) = \int_0^T \frac{t(T-s)}{T-\eta}\sigma(s)\,ds - \int_0^\eta \frac{t(\eta-s)}{T-\eta}\sigma(s)\,ds - \int_0^t (t-s)\sigma(s)\,ds + \frac{t}{T}u(T)$$

or, equivalently,

$$u(t) = \int_0^T k(t,s)\sigma(s)\,ds + \frac{t}{T}u(T), \tag{6}$$

where

$$k(t,s) = \begin{cases} s, & 0 \le s < \min(\eta, t), \\ t, & 0 \le t < s < \eta \le T, \\ \frac{t(T-s)}{T-\eta} - (t-s), & 0 \le \eta < s < t \le T, \\ \frac{t(T-s)}{T-\eta}, & \max(\eta, t) < s \le T. \end{cases}$$

We note that $k \in C([0,T] \times [0,T], \mathbb{R})$ and $k(t,s) \ge 0$ for every $(t,s) \in [0,T] \times [0,T]$.

3 Nonlinear problem

Defining the operators:

$$F : C[0,T] \to C[0,T],$$

$$[Fu](t) = f\bigl(t, u(t)\bigr), \quad u \in C[0,T], t \in [0,T],$$

$$K : C[0,T] \to C[0,T],$$

$$[K\sigma](t) = \int_0^T k(t,s)\sigma(s)\,ds, \quad \sigma \in C[0,T], t \in [0,T],$$

$$L : C[0,T] \to C[0,T],$$

$$[Lu](t) = \frac{t}{T}u(T), \quad u \in C[0,T], t \in [0,T],$$

the nonlinear problem is equivalent to

$$u = Nu,$$

where $N = K \circ F + L$.

We note that (6) can be written as

$$u(t) - \frac{t}{T}u(T) = \int_0^T k(t,s)\sigma(s)\,ds$$

and the nonlinear problem (1) as

$$u(t) - \frac{t}{T}u(T) = \int_0^T k(t,s)f\bigl(s, u(s)\bigr)\,ds.$$

This suggests to introduce the new function $v(t) = u(t) - \frac{t}{T}u(T)$. To find a solution u, we have to find v and $u(T)$.

For every constant $c \in \mathbb{R}$, we solve

$$v(t) = \int_0^T k(t,s)f\left(s, v(s) + \frac{s}{T}c\right)ds \tag{7}$$

and let $\varphi(c)$ be the set of solutions of (7). This set may be empty (no solution), a singleton (unique solution) or with more than one element (multiple solutions). For every $v_c \in \varphi(c)$, we consider

$$u_c(t) = v_c(t) + \frac{t}{T}c,$$

and hence

$$u_c(t) = \int_0^T k(t,s)f\bigl(s, u_c(s)\bigr)\,ds + \frac{t}{T}c.$$

If $c = u_c(T)$, then u_c is a solution of the nonlinear problem (1). We then look for fixed points of the map

$$c \in \mathbb{R} \longrightarrow u_c(T) \in \mathbb{R}.$$

For $c \in \mathbb{R}$ fixed, we try to solve the integral equation (7).

Assume that there exist $a, b \in C[0,T]$ and $\alpha \in [0,1)$ such that

$$\left|f(t,u)\right| \leq a(t) + b(t)|u|^{\alpha} \tag{8}$$

for every $t \in [0,T]$, $u \in \mathbb{R}$.

For $v \in C[0,T]$, define $F_c v \in C[0,T]$ as

$$[F_c v](t) = f\left(t, v(t) + \frac{t}{T}c\right).$$

Thus, a solution of (7) is precisely a fixed point of $K \circ F_c = K_c$. Note that K_c is a compact operator. For $v \in C[0,T]$, let $\|v\| = \sup_{t\in[0,T]} |v(t)|$.

For $\lambda \in (0,1)$, if $v = \lambda K_c(v)$ we have

$$v(t) = \lambda \int_0^T k(t,s) f\left(s, v(s) + \frac{s}{T}c\right) ds,$$

and

$$\left|v(t)\right| \leq \|k\| \int_0^T f\left(s, v(s) + \frac{s}{T}c\right) ds \leq \|k\| \cdot T\left[\|a\| + \|b\|\left(\|v\| + c\right)^{\alpha}\right].$$

Hence there exist constants a_0, b_0 such that

$$\|v\| \leq a_0 + b_0\left(\|v\| + c\right)^{\alpha} \tag{9}$$

for any $v \in C[0,T]$ and $\lambda \in (0,1)$ solution of $v = \lambda K_c(v)$. This implies that v is bounded independently of $\lambda \in (0,1)$, and hence by Schaefer's fixed point theorem (Theorem 4.3.2 of [22]), K_c has at least a fixed point, *i.e.*, for given c, equation (7) is solvable.

Now suppose f is Lipschitz continuous.

Then there exists $l > 0$ such that

$$\left|f(t,x) - f(t,y)\right| \leq l|x - y| \tag{10}$$

for every $t \in [0,T]$ and $x, y \in \mathbb{R}$.

Then, for $v, w \in C[0,T]$, we have

$$\left|[K_c v](t) - [K_c w](t)\right| \leq \int_0^T k(t,s) l \left|v(s) - w(s)\right| ds$$

and

$$\|K_c v - K_c w\| \leq \|k\| \cdot l \cdot T\|v - w\|.$$

Thus, for $l > 0$ small, equation (7) has a unique solution in view of the classical Banach contraction fixed point theorem.

Now, under conditions (8) and (10), set

$$c \in \mathbb{R} \longrightarrow v_c \in C[0,T],$$

where v_c is the unique solution of (7), and as a consequence of the contraction principle, this map is continuous.

Define the map

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R},$$
$$\varphi(c) = v_c(T).$$

If there exists $c \in \mathbb{R}$ such that $\varphi(c) = 0$, then for that c we have $v_c(T)$, and the function

$$u_c(t) = v_c(t) + \frac{t}{T}c$$

is such that $u_c(T) = c$, and therefore u_c is a solution of the original nonlinear problem (1).

Now, assume that

$$\lim_{u\to\pm\infty} f(t,u) = \pm\infty \tag{11}$$

uniformly on $t \in [0,T]$.

Then the growth of $\|v\|$ is sublinear in view of estimate (9). However, c growths linearly. Hence the norm of the function

$$v_c(s) + \frac{s}{T}c$$

growths asymptotically as c.

This implies that $\lim_{c\to\pm\infty} \varphi(c) = \pm\infty$, and there exists $c \in \mathbb{R}$ with $\varphi(c) = 0$.

We have the following result.

Theorem 3.1 *Suppose that f satisfies the growth conditions* (8) *and* (10). *If* (11) *holds, then* (1) *is solvable for l sufficiently small.*

Note that condition (11) is crucial since for $f(t,u) = \sigma(t)$ and, in view of (5), the problem (1) may have no solution.

Competing interests
The author declares that he has no competing interests.

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