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Well-posedness of a boundary value problem for a class of third-order operator-differential equations

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Abstract

This paper investigates the well-posedness of a boundary value problem on the semiaxis for a class of third-order operator-differential equations whose principal part has multiple real characteristics. We obtain sufficient conditions for the existence and uniqueness of the solution of a boundary value problem in the Sobolev-type space $W_2^3(R_+;H)$. These conditions are expressed in terms of the operator coefficients of the investigated equation. We find relations between the estimates of the norms of intermediate derivatives operators in the subspace $W_2^3(R_+;H)$ and the solvability conditions. Furthermore, we calculate the exact values of these norms. The results are illustrated with an example of the initial-boundary value problems for partial differential equations.

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1 Introduction

The paper is dedicated to the formulation and study of the well-posedness of a boundary value problem for a class of third-order operator-differential equations with a real and real multiple characteristic. Note that the differential equations whose characteristic equations have real different or real multiple roots find a wide application in modeling problems of mechanics and engineering, such as problems of heat mass transfer and filtration [1], dynamics of arches and rings [2], *etc.*

Suppose that H is a separable Hilbert space with a scalar product (x,y), $x,y \in H$, A is a self-adjoint positive definite operator on H ($A = A^* \geq cE$, $c > 0$, E is the identity operator), and H_γ ($\gamma \geq 0$) is the scale of Hilbert spaces generated by the operator A, *i.e.*, $H_\gamma = D(A^\gamma)$, $(x,y)_\gamma = (A^\gamma x, A^\gamma y)$, $x,y \in D(A^\gamma)$. For $\gamma = 0$ we consider that $H_0 = H$, $(x,y)_0 = (x,y)$, $x,y \in H$. Here the operator A^γ is determined from the spectral decomposition of the operator A, *i.e.*,

$$A^\gamma = \int_c^{+\infty} \sigma^\gamma \, dE_\sigma, \quad \gamma \geq 0,$$

where E_σ is the resolution of the identity for A.

By $L_2(R;H)$ we denote the Hilbert space of all vector-functions $u(t)$ defined on $R = (-\infty,+\infty)$ with values in H and the norm

$$\|u\|_{L_2(R;H)} = \left(\int_{-\infty}^{+\infty} \|u(t)\|^2\,dt\right)^{1/2}.$$

Similarly, we define the space $L_2(R_+;H)$, where $R_+ = [0,+\infty)$,

$$L_2(R_+;H) = \left\{u(t) : \|u\|_{L_2(R_+;H)} = \left(\int_0^{+\infty} \|u(t)\|^2\,dt\right)^{1/2} < +\infty\right\}.$$

Define the following spaces:

$$W_2^3(R;H) = \left\{u(t) : \frac{d^3u(t)}{dt^3} \in L_2(R;H), A^3u(t) \in L_2(R;H)\right\},$$
$$W_2^3(R_+;H) = \left\{u(t) : \frac{d^3u(t)}{dt^3} \in L_2(R_+;H), A^3u(t) \in L_2(R_+;H)\right\}$$

(for more details about these spaces, see [3, Ch.1]). Here and further, the derivatives are understood in the sense of distributions (see [3]).

The spaces $W_2^3(R;H)$ and $W_2^3(R_+;H)$ become Hilbert spaces with respect to the norms

$$\|u\|_{W_2^3(R;H)} = \left(\int_{-\infty}^{+\infty} \left(\left\|\frac{d^3u(t)}{dt^3}\right\|^2 + \|A^3u(t)\|^2\right)dt\right)^{1/2}$$

and

$$\|u\|_{W_2^3(R_+;H)} = \left(\int_0^{+\infty} \left(\left\|\frac{d^3u(t)}{dt^3}\right\|^2 + \|A^3u(t)\|^2\right)dt\right)^{1/2},$$

respectively.

Consider the following subspaces of $W_2^3(R_+;H)$:

$$\overset{\circ}{W}{}_2^3(R_+;H) = \left\{u(t) : u(t) \in W_2^3(R_+;H), \frac{d^nu(0)}{dt^n} = 0, n = 0,1,2\right\},$$
$$\overset{\circ}{W}{}_2^3(R_+;H;0,1) = \left\{u(t) : u(t) \in W_2^3(R_+;H), u(0) = \frac{du(0)}{dt} = 0\right\}.$$

By the theorem on intermediate derivatives, both of these spaces are complete [3].

Now let us state the boundary value problem under study.

In the Hilbert space H, we consider the following third-order operator-differential equation whose principal part has multiple characteristic:

$$\left(-\frac{d}{dt} + A\right)\left(\frac{d}{dt} + A\right)^2 u(t) + A_1\frac{d^2u(t)}{dt^2} + A_2\frac{du(t)}{dt} = f(t), \quad t \in R_+, \tag{1.1}$$

where $f(t) \in L_2(R_+;H)$, A is the self-adjoint positive definite operator defined above, and A_1, A_2 are linear, in general, unbounded operators on H. Assuming $u(t) \in W_2^3(R_+;H)$, we attach to equation (1.1) boundary conditions at zero of the form

$$\frac{d^i u(0)}{dt^i} = 0, \quad i = 0, 1. \tag{1.2}$$

Definition 1.1 If the vector function $u(t) \in W_2^3(R_+;H)$ satisfies equation (1.1) almost everywhere in R_+, then it is called a *regular solution* of equation (1.1).

Definition 1.2 If for any $f(t) \in L_2(R_+;H)$ there exists a regular solution of equation (1.1) satisfying the boundary conditions (1.2) in the sense of relation $\lim_{t\to 0} \|A^{5/2-i} \frac{d^i u(t)}{dt^i}\| = 0$, $i = 0, 1$ and the following inequality holds:

$$\|u\|_{W_2^3(R_+;H)} \le \text{const} \|f\|_{L_2(R_+;H)},$$

then we say that problem (1.1), (1.2) is *regularly solvable*.

The solvability of boundary value problems for operator-differential equations has been studied by many authors. Among such works, we should especially mention the papers by Gasymov, Kostyuchenko, Gorbachuk, Dubinskii, Shkalikov, Mirzoev, Jakubov, Aliev and their followers (see, *e.g.*, [4–11]) that are close to our paper. Allowing to treat both ordinary and partial differential operators from the same point of view, these equations are also interesting from the aspect that the well-posedness of boundary value problems for them is closely related to the spectral theory of polynomial operator pencils [4] (for comprehensive survey, see Shkalikov [12]). And, of course, well-posed solvability of the Cauchy problem and non-local boundary value problems for operator-differential equations as well as related spectral problems (see, *e.g.*, Shkalikov [13], Gorbachuk and Gorbachuk [14], Agarwal *et al.* [15]) are also of great interest.

In this paper, we obtain conditions for the regular solvability of boundary value problem (1.1) (1.2), which are expressed only in terms of the operator coefficients of equation (1.1). We also show the relationship between these conditions and the exact estimates for the norms of intermediate derivatives operators in the subspaces $\overset{\circ}{W}{}_2^3(R_+;H)$ and $\overset{\circ}{W}{}_2^3(R_+;H;0,1)$ with respect to the norm of the operator generated by the principal part of equation (1.1). Mirzoev [16] was the first who paid detailed attention to such relation (for more details about the calculation of the norms of intermediate derivatives operators, see [17]). To estimate these norms, he used the method of factorization of polynomial operator pencils which depend on a real parameter. Further these results have been developed in [18, 19].

It should be noted that all the above-mentioned works, unlike equation (1.1), consider the operator-differential equations with a simple characteristic. Although similar matters of solvability and related problems have already been studied for fourth-order operator-differential equations whose principal parts have multiple characteristic (see, for example, [20, 21], also [22] and some references therein), but they have not been studied for odd order operator-differential equations with multiple characteristic, including those of third-order. One of the goals of the present paper is to fill this gap.

2 Equivalent norms and conditional theorem on solvability of boundary value problem (1.1), (1.2)

We show that the norm of the operator generated by the principal part of equation (1.1) is equivalent to the initial norm $\|u\|_{W_2^3(R_+;H)}$ on the space $\overset{\circ}{W}{}_2^3(R_+;H;0,1)$.

Let P_0 denote the operator acting from the space $\overset{\circ}{W}{}_2^3(R_+;H;0,1)$ to the space $L_2(R_+;H)$ as follows:

$$P_0 u(t) \equiv \left(-\frac{d}{dt}+A\right)\left(\frac{d}{dt}+A\right)^2 u(t), \quad u(t) \in \overset{\circ}{W}{}_2^3(R_+;H;0,1).$$

Then the following theorem holds.

Theorem 2.1 *The operator P_0 is an isomorphism between the spaces $\overset{\circ}{W}{}_2^3(R_+;H;0,1)$ and $L_2(R_+;H)$.*

Proof First, we note that if $\xi \in H_{5/2}$, then $e^{-tA}\xi \in W_2^3(R_+;H)$ and if $\eta \in H_{3/2}$, then $tAe^{-tA}\eta \in W_2^3(R_+;H)$ (see, *e.g.*, [23]), where e^{-tA} is the strongly continuous semi-group of bounded operators generated by the operator $-A$.

Obviously, the homogeneous equation $P_0 u(t) = 0$ has only the trivial solution in $\overset{\circ}{W}{}_2^3(R_+;H;0,1)$. But the equation $P_0 u(t) = f(t)$ for any $f(t) \in L_2(R_+;H)$ has a solution $u(t) \in \overset{\circ}{W}{}_2^3(R_+;H;0,1)$ of the form

$$u(t) = \int_0^{+\infty} G(t-s)f(s)\,ds - \frac{1}{4}(E+2tA)\int_0^{+\infty} e^{-(t+s)A}\left(A^{-2}f(s)\right)ds,$$

where

$$G(t-s) = \frac{1}{4}\begin{cases}(E+2(t-s)A)e^{-(t-s)A}A^{-2} & \text{if } t-s>0,\\ e^{(t-s)A}A^{-2} & \text{if } t-s<0.\end{cases}$$

In fact, such a solution $u(t)$ satisfies the equation

$$\left(-\frac{d}{dt}+A\right)\left(\frac{d}{dt}+A\right)^2 u(t) = f(t)$$

and the conditions at zero $u(0) = \frac{du(0)}{dt} = 0$, therefore, it belongs to $W_2^3(R_+;H)$ (see, *e.g.*, [4, 5]).

Let us now show the boundedness of the operator P_0. We have

$$\begin{aligned}
\|P_0 u\|^2_{L_2(R_+;H)} &= \left\|\left(-\frac{d}{dt}+A\right)\left(\frac{d}{dt}+A\right)^2 u\right\|^2_{L_2(R_+;H)}\\
&= \left\|-\frac{d^3u}{dt^3} - A\frac{d^2u}{dt^2} + A^2\frac{du}{dt} + A^3u\right\|^2_{L_2(R_+;H)}\\
&= \left\|\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)} + \left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)} + \left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)} + \left\|A^3u\right\|^2_{L_2(R_+;H)}\\
&\quad + 2\operatorname{Re}\left(\frac{d^3u}{dt^3}, A\frac{d^2u}{dt^2}\right)_{L_2(R_+;H)} - 2\operatorname{Re}\left(\frac{d^3u}{dt^3}, A^2\frac{du}{dt}\right)_{L_2(R_+;H)}
\end{aligned}$$

$$-2\operatorname{Re}\left(\frac{d^3u}{dt^3}, A^3u\right)_{L_2(R_+;H)} - 2\operatorname{Re}\left(A\frac{d^2u}{dt^2}, A^2\frac{du}{dt}\right)_{L_2(R_+;H)}$$
$$-2\operatorname{Re}\left(A\frac{d^2u}{dt^2}, A^3u\right)_{L_2(R_+;H)} + 2\operatorname{Re}\left(A^2\frac{du}{dt}, A^3u\right)_{L_2(R_+;H)}. \qquad (2.1)$$

Since $u(t) \in \mathring{W}_2^3(R_+;H;0,1)$,

$$2\operatorname{Re}\left(\frac{d^3u}{dt^3}, A\frac{d^2u}{dt^2}\right)_{L_2(R_+;H)} = -\left\|A^{1/2}\frac{d^2u(0)}{dt^2}\right\|^2,$$
$$2\operatorname{Re}\left(\frac{d^3u}{dt^3}, A^2\frac{du}{dt}\right)_{L_2(R_+;H)} = -2\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)}, \qquad 2\operatorname{Re}\left(\frac{d^3u}{dt^3}, A^3u\right)_{L_2(R_+;H)} = 0,$$
$$2\operatorname{Re}\left(A\frac{d^2u}{dt^2}, A^2\frac{du}{dt}\right)_{L_2(R_+;H)} = 0,$$
$$2\operatorname{Re}\left(A\frac{d^2u}{dt^2}, A^3u\right)_{L_2(R_+;H)} = -2\left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)},$$
$$2\operatorname{Re}\left(A^2\frac{du}{dt}, A^3u\right)_{L_2(R_+;H)} = 0,$$

then (2.1) takes the form

$$\|P_0u\|^2_{L_2(R_+;H)} = \|u\|^2_{W_2^3(R_+;H)} + 3\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)} + 3\left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)} - \left\|A^{1/2}\frac{d^2u(0)}{dt^2}\right\|^2.$$

Further, by theorems on intermediate derivatives and traces [3, Ch.1], we obtain

$$\|P_0u\|^2_{L_2(R_+;H)} \le \text{const}\,\|u\|^2_{W_2^3(R_+;H)}.$$

Thus, the operator P_0 is bounded and bijective from the space $\mathring{W}_2^3(R_+;H;0,1)$ to the space $L_2(R_+;H)$. Therefore, due to the Banach inverse operator theorem, P_0 is an isomorphism between these spaces. The theorem is proved. □

Corollary 2.2 *It follows from Theorem* 2.1 *that the norm* $\|P_0u\|_{L_2(R_+;H)}$ *on the space* $\mathring{W}_2^3(R_+;H;0,1)$ *is equivalent to the initial norm* $\|u\|_{W_2^3(R_+;H)}$.

Before we state the conditional theorem on solvability of boundary value problem (1.1) (1.2), we prove the following lemma.

Lemma 2.3 *Let the operators* A_jA^{-j}, $j = 1,2$ *be bounded on* H *and let the operator* P_1 *act from the space* $\mathring{W}_2^3(R_+;H;0,1)$ *to the space* $L_2(R_+;H)$ *as follows*:

$$P_1u(t) \equiv A_1\frac{d^2u(t)}{dt^2} + A_2\frac{du(t)}{dt}, \quad u(t) \in \mathring{W}_2^3(R_+;H;0,1).$$

Then P_1 *is also bounded.*

Proof Since for $u(t) \in \mathring{W}_2^3(R_+;H;0,1)$, by virtue of the intermediate derivatives theorem [3, Ch.1], we obtain

$$\|P_1 u\|_{L_2(R_+;H)} \le \|A_1 A^{-1}\|_{H\to H} \left\| A \frac{d^2u}{dt^2} \right\|_{L_2(R_+;H)} + \|A_2 A^{-2}\|_{H\to H} \left\| A^2 \frac{du}{dt} \right\|_{L_2(R_+;H)} \le \text{const} \|u\|_{W_2^3(R_+;H)}.$$

The lemma is proved. □

Theorem 2.4 *Let the operators $A_j A^{-j}$, $j = 1,2$ be bounded on H and the following inequality hold*:

$$\sum_{j=1}^{2} n_{3-j} \|A_j A^{-j}\|_{H\to H} < 1,$$

where

$$n_j = \sup_{0\neq u \in \mathring{W}_2^3(R_+;H;0,1)} \frac{\|A^{3-j} \frac{d^j u}{dt^j}\|_{L_2(R_+;H)}}{\|P_0 u\|_{L_2(R_+;H)}}, \quad j = 1,2.$$

Then boundary value problem (1.1), (1.2) *is regularly solvable*.

Proof We represent boundary value problem (1.1), (1.2) in the form of the operator equation $P_0 u(t) + P_1 u(t) = f(t)$, where $f(t) \in L_2(R_+;H)$, $u(t) \in \mathring{W}_2^3(R_+;H;0,1)$. Since by Theorem 2.1 the operator P_0 has the bounded inverse P_0^{-1} which acts from $L_2(R_+;H)$ to $\mathring{W}_2^3(R_+;H;0,1)$, then, after the replacement $u(t) = P_0^{-1} v(t)$, we obtain the equation $(E + P_1 P_0^{-1}) v(t) = f(t)$ in the space $L_2(R_+;H)$.

Under the conditions of Theorem 2.4, we obtain

$$\begin{aligned} \|P_1 P_0^{-1} v\|_{L_2(R_+;H)} &= \|P_1 u\|_{L_2(R_+;H)} \\ &\le \|A_1 A^{-1}\|_{H\to H} \left\| A \frac{d^2u}{dt^2} \right\|_{L_2(R_+;H)} + \|A_2 A^{-2}\|_{H\to H} \left\| A^2 \frac{du}{dt} \right\|_{L_2(R_+;H)} \\ &\le \sum_{j=1}^{2} n_{3-j} \|A_j A^{-j}\|_{H\to H} \|P_0 u\|_{L_2(R_+;H)} \\ &= \sum_{j=1}^{2} n_{3-j} \|A_j A^{-j}\|_{H\to H} \|v\|_{L_2(R_+;H)}. \end{aligned}$$

Therefore, if the inequality $\sum_{j=1}^{2} n_{3-j} \|A_j A^{-j}\|_{H\to H} < 1$ holds, then the operator $E + P_1 P_0^{-1}$ is invertible and we can define $u(t)$ by the formula $u(t) = P_0^{-1}(E + P_1 P_0^{-1})^{-1} f(t)$. Moreover,

$$\|u\|_{W_2^3(R_+;H)} \le \|P_0^{-1}\|_{L_2(R_+;H)\to W_2^3(R_+;H)} \|(E + P_1 P_0^{-1})^{-1}\|_{L_2(R_+;H)\to L_2(R_+;H)} \|f\|_{L_2(R_+;H)} \le \text{const} \|f\|_{L_2(R_+;H)}.$$

The theorem is proved. □

Naturally, there arises a problem of finding exact values or estimates for the numbers n_j, $j = 1, 2$, and it is very important for extending the class of operator-differential equations of the form (1.1) for which our boundary value problem is solvable. We will make the calculations for n_j, $j = 1, 2$ in Section 4.

3 On spectral properties of some polynomial operator pencils and basic equalities for the functions in the space $W_2^3(R_+; H)$

Consider the following polynomial operator pencils depending on the real parameter β:

$$P_j(\lambda; \beta; A) = \left((i\lambda)^2 E + A^2\right)^3 - \beta (i\lambda)^{2j} A^{6-2j}, \quad j = 1, 2. \tag{3.1}$$

We need to clarify considering naturally arising pencils (3.1). Obviously, for $u(t) \in W_2^3(R; H)$, we obtain

$$\begin{aligned}
&\left\| \left(-\frac{d}{dt} + A\right)\left(\frac{d}{dt} + A\right)^2 u \right\|^2_{L_2(R;H)} \\
&\quad = \left\| -\frac{d^3 u}{dt^3} - A\frac{d^2 u}{dt^2} + A^2 \frac{du}{dt} + A^3 u \right\|^2_{L_2(R;H)} \\
&\quad = \left\| \frac{d^3 u}{dt^3} \right\|^2_{L_2(R;H)} + \left\| A\frac{d^2 u}{dt^2} \right\|^2_{L_2(R;H)} + \left\| A^2 \frac{du}{dt} \right\|^2_{L_2(R;H)} + \left\| A^3 u \right\|^2_{L_2(R;H)} \\
&\qquad + 2\operatorname{Re}\left(\frac{d^3 u}{dt^3}, A\frac{d^2 u}{dt^2}\right)_{L_2(R;H)} - 2\operatorname{Re}\left(\frac{d^3 u}{dt^3}, A^2\frac{du}{dt}\right)_{L_2(R;H)} \\
&\qquad - 2\operatorname{Re}\left(\frac{d^3 u}{dt^3}, A^3 u\right)_{L_2(R;H)} - 2\operatorname{Re}\left(A\frac{d^2 u}{dt^2}, A^2\frac{du}{dt}\right)_{L_2(R;H)} \\
&\qquad - 2\operatorname{Re}\left(A\frac{d^2 u}{dt^2}, A^3 u\right)_{L_2(R;H)} + 2\operatorname{Re}\left(A^2\frac{du}{dt}, A^3 u\right)_{L_2(R;H)}.
\end{aligned}$$

Since for $u(t) \in W_2^3(R; H)$,

$$\begin{aligned}
&2\operatorname{Re}\left(\frac{d^3 u}{dt^3}, A\frac{d^2 u}{dt^2}\right)_{L_2(R;H)} = 0, \qquad 2\operatorname{Re}\left(\frac{d^3 u}{dt^3}, A^2\frac{du}{dt}\right)_{L_2(R;H)} = -2\left\| A\frac{d^2 u}{dt^2} \right\|^2_{L_2(R;H)}, \\
&2\operatorname{Re}\left(\frac{d^3 u}{dt^3}, A^3 u\right)_{L_2(R;H)} = 0, \qquad 2\operatorname{Re}\left(A\frac{d^2 u}{dt^2}, A^2\frac{du}{dt}\right)_{L_2(R;H)} = 0, \\
&2\operatorname{Re}\left(A\frac{d^2 u}{dt^2}, A^3 u\right)_{L_2(R;H)} = -2\left\| A^2\frac{du}{dt} \right\|^2_{L_2(R;H)}, \qquad 2\operatorname{Re}\left(A^2\frac{du}{dt}, A^3 u\right)_{L_2(R;H)} = 0,
\end{aligned}$$

then, as a result, we obtain

$$\begin{aligned}
\left\| \left(-\frac{d}{dt} + A\right)\left(\frac{d}{dt} + A\right)^2 u \right\|^2_{L_2(R;H)} &= \left\| \frac{d^3 u}{dt^3} \right\|^2_{L_2(R;H)} + 3\left\| A\frac{d^2 u}{dt^2} \right\|^2_{L_2(R;H)} \\
&\quad + 3\left\| A^2\frac{du}{dt} \right\|^2_{L_2(R;H)} + \left\| A^3 u \right\|^2_{L_2(R;H)}.
\end{aligned} \tag{3.2}$$

If we use the Fourier transform in (3.2), then

$$\left\| \left(-\frac{d}{dt}+A\right)\left(\frac{d}{dt}+A\right)^2 u \right\|_{L_2(R;H)}^2 = \int_{-\infty}^{+\infty} \left(\left(\xi^2 E + A^2\right)^3 \tilde{u}(\xi), \tilde{u}(\xi)\right) d\xi,$$

where $\tilde{u}(\xi)$ is the Fourier transform of the function $u(t)$. Therefore, if $\beta \in R$, then

$$\left\| \left(-\frac{d}{dt}+A\right)\left(\frac{d}{dt}+A\right)^2 u \right\|_{L_2(R;H)}^2 - \beta \left\| A^{3-j} \frac{d^j u}{dt^j} \right\|_{L_2(R;H)}^2$$
$$= \int_{-\infty}^{+\infty} \left(\left(\left(\xi^2 E + A^2\right)^3 - \beta \xi^{2j} A^{6-2j}\right) \tilde{u}(\xi), \tilde{u}(\xi)\right) d\xi.$$

That is why to estimate n_j, $j = 1, 2$, it is necessary to study some properties of pencils (3.1).
The following theorem on factorization of pencils (3.1) holds.

Theorem 3.1 *Let* $\beta \in [0, \frac{27}{4})$. *Then polynomial operator pencils* (3.1) *are invertible on the imaginary axis and the following representations are true*:

$$P_j(\lambda; \beta; A) = F_j(\lambda; \beta; A) F_j(-\lambda; \beta; A), \quad j = 1, 2,$$

where

$$F_j(\lambda; \beta; A) = \prod_{s=1}^{3} \left(\lambda E - \omega_{j,s}(\beta) A\right) \equiv \lambda^3 E + \alpha_{1,j}(\beta)\lambda^2 A + \alpha_{2,j}(\beta)\lambda A^2 + A^3,$$

here $\operatorname{Re}\omega_{j,s}(\beta) < 0$, $s = 1, 2, 3$, *the numbers* $\alpha_{1,j}(\beta)$, $\alpha_{2,j}(\beta)$ *are positive and satisfy the following systems of equations*:

$$\begin{aligned} &(1)\ \textit{for } j = 1, \quad \begin{cases} -2\alpha_{2,1}(\beta) + \alpha_{1,1}^2(\beta) - 3 = 0, \\ 2\alpha_{1,1}(\beta) - \alpha_{2,1}^2(\beta) + 3 = \beta; \end{cases} \\ &(2)\ \textit{for } j = 2, \quad \begin{cases} -2\alpha_{2,2}(\beta) + \alpha_{1,2}^2(\beta) - 3 = -\beta, \\ 2\alpha_{1,2}(\beta) - \alpha_{2,2}^2(\beta) + 3 = 0. \end{cases} \end{aligned} \tag{3.3}$$

Proof It is clear that

$$P_j(\lambda; \beta; \sigma) = \left((i\lambda)^2 + \sigma^2\right)^3 - \beta (i\lambda)^{2j} \sigma^{6-2j}, \quad j = 1, 2, \tag{3.4}$$

are the characteristic polynomial operator pencils (3.1), where $\sigma \in \sigma(A)$ ($\sigma(A)$ denotes the spectrum of the operator A). Let $\lambda = i\xi$, $\xi \in R$. Then characteristic polynomials (3.4) satisfy the following relations:

$$P_j(\lambda; \beta; \sigma) = P_j(i\xi; \beta; \sigma) = \sigma^6 \left(\frac{\xi^2}{\sigma^2} + 1\right)^3 \left[1 - \beta \frac{(\frac{\xi^2}{\sigma^2})^j}{(\frac{\xi^2}{\sigma^2} + 1)^3}\right]$$
$$\geq \sigma^6 \left(\frac{\xi^2}{\sigma^2} + 1\right)^3 \left[1 - \beta \sup_{\frac{\xi^2}{\sigma^2} \geq 0} \frac{(\frac{\xi^2}{\sigma^2})^j}{(\frac{\xi^2}{\sigma^2} + 1)^3}\right], \quad j = 1, 2.$$

Since

$$\sup_{\frac{\xi^2}{\sigma^2}\geq 0} \frac{(\frac{\xi^2}{\sigma^2})^j}{(\frac{\xi^2}{\sigma^2}+1)^3} = \frac{4}{27}, \quad j=1,2,$$

we obtain

$$P_j(i\xi;\beta;\sigma) > 0, \quad j=1,2 \tag{3.5}$$

for $\beta \in [0, \frac{27}{4})$. Inequalities (3.5) imply that polynomials (3.4) have no roots on the imaginary axis for $\beta \in [0, \frac{27}{4})$. Besides, it can be seen from (3.4) that each of the characteristic polynomial $P_j(\lambda;\beta;\sigma)$ has exactly three roots in the left half-plane for $\sigma \in \sigma(A)$. Since polynomials (3.4) are homogeneous with respect to the arguments λ and σ, then the following factorization is true for them:

$$P_j(\lambda;\beta;\sigma) = F_j(\lambda;\beta;\sigma)F_j(-\lambda;\beta;\sigma), \quad j=1,2, \tag{3.6}$$

where

$$F_j(\lambda;\beta;\sigma) = \prod_{s=1}^{3}\bigl(\lambda - \omega_{j,s}(\beta)\sigma\bigr) \equiv \lambda^3 + \alpha_{1,j}(\beta)\lambda^2\sigma + \alpha_{2,j}(\beta)\lambda\sigma^2 + \sigma^3,$$

$\operatorname{Re}\omega_{j,s}(\beta) < 0$, $s=1,2,3$, and the numbers $\alpha_{1,j}(\beta)$, $\alpha_{2,j}(\beta)$ are positive according to Vieta's formulas and satisfy systems of equations (3.3) derived from (3.6) during the comparison of same degree coefficients. Further, using the spectral decomposition of the operator A, from equalities (3.6) we obtain the assertions of the theorem. The theorem is proved. □

Now, we state a theorem playing a significant role in the subsequent study. Let us introduce another notation, which will be used in the proof of that theorem: $D(R_+;H_3)$ will denote the linear set of infinitely differentiable functions with values in H_3 and compact support in R_+. As is well known, the space $D(R_+;H_3)$ is everywhere dense in $W_2^3(R_+;H)$ (see [3, Ch.1]).

Theorem 3.2 *Let $\beta \in [0, \frac{27}{4})$. Then, for any $u(t) \in W_2^3(R_+;H)$, the following relation holds*:

$$\begin{aligned} &\left\| \left(-\frac{d}{dt}+A\right)\left(\frac{d}{dt}+A\right)^2 u\right\|^2_{L_2(R_+;H)} - \beta\left\| A^{3-j}\frac{d^j u}{dt^j}\right\|^2_{L_2(R_+;H)} \\ &\quad = \left\| F_j\left(\frac{d}{dt};\beta;A\right)u\right\|^2_{L_2(R_+;H)} + \bigl(S_j(\beta)\varphi,\varphi\bigr)_{H^3}, \end{aligned} \tag{3.7}$$

where

$$H^3 = \bigoplus_{p=1}^{3} H, \qquad \varphi = \left(\varphi_k = A^{3-k-1/2}\frac{d^k u(0)}{dt^k}\right)_{k=0}^{2},$$

$$S_j(\beta) = \begin{pmatrix} \alpha_{2,j}(\beta)-1 & \alpha_{1,j}(\beta)+1 & 2 \\ \alpha_{1,j}(\beta)+1 & \alpha_{1,j}(\beta)\alpha_{2,j}(\beta)-1 & \alpha_{2,j}(\beta)+1 \\ 2 & \alpha_{2,j}(\beta)+1 & \alpha_{1,j}(\beta)-1 \end{pmatrix}.$$

Proof It suffices to prove the theorem for functions $u(t) \in D(R_+; H_3)$. We have

$$\begin{aligned}
&\left\| F_j\left(\frac{d}{dt};\beta;A\right)u\right\|^2_{L_2(R_+;H)} \\
&\quad = \left\|\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)} + \alpha_{1,j}^2(\beta)\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)} \\
&\qquad + \alpha_{2,j}^2(\beta)\left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)} + \left\|A^3u\right\|^2_{L_2(R_+;H)} + 2\alpha_{1,j}(\beta)\operatorname{Re}\left(\frac{d^3u}{dt^3}, A\frac{d^2u}{dt^2}\right)_{L_2(R_+;H)} \\
&\qquad + 2\alpha_{2,j}(\beta)\operatorname{Re}\left(\frac{d^3u}{dt^3}, A^2\frac{du}{dt}\right)_{L_2(R_+;H)} + 2\operatorname{Re}\left(\frac{d^3u}{dt^3}, A^3u\right)_{L_2(R_+;H)} \\
&\qquad + 2\alpha_{1,j}(\beta)\alpha_{2,j}(\beta)\operatorname{Re}\left(A\frac{d^2u}{dt^2}, A^2\frac{du}{dt}\right)_{L_2(R_+;H)} \\
&\qquad + 2\alpha_{1,j}(\beta)\operatorname{Re}\left(A\frac{d^2u}{dt^2}, A^3u\right)_{L_2(R_+;H)} \\
&\qquad + 2\alpha_{2,j}(\beta)\operatorname{Re}\left(A^2\frac{du}{dt}, A^3u\right)_{L_2(R_+;H)}.
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
&\left\| F_j\left(\frac{d}{dt};\beta;A\right)u\right\|^2_{L_2(R_+;H)} \\
&\quad = \left\|\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)} + \left(\alpha_{1,j}^2(\beta) - 2\alpha_{2,j}(\beta)\right)\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)} \\
&\qquad + \left(\alpha_{2,j}^2(\beta) - 2\alpha_{1,j}(\beta)\right)\left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)} + \left\|A^3u\right\|^2_{L_2(R_+;H)} - \alpha_{1,j}(\beta)\|\varphi_2\|^2 \\
&\qquad - 2\alpha_{2,j}(\beta)\operatorname{Re}(\varphi_2,\varphi_1) - 2\operatorname{Re}(\varphi_2,\varphi_0) + \left(1 - \alpha_{1,j}(\beta)\alpha_{2,j}(\beta)\right)\|\varphi_1\|^2 \\
&\qquad - 2\alpha_{1,j}(\beta)\operatorname{Re}(\varphi_1,\varphi_0) - \alpha_{2,j}(\beta)\|\varphi_0\|^2.
\end{aligned} \tag{3.8}$$

Making similar calculations for $\|(-\frac{d}{dt}+A)(\frac{d}{dt}+A)^2u\|^2_{L_2(R_+;H)}$, we obtain

$$\begin{aligned}
&\left\|\left(-\frac{d}{dt}+A\right)\left(\frac{d}{dt}+A\right)^2u\right\|^2_{L_2(R_+;H)} \\
&\quad = \left\|\frac{d^3u}{dt^3}\right\|^2_{L_2(R_+;H)} + 3\left\|A\frac{d^2u}{dt^2}\right\|^2_{L_2(R_+;H)} \\
&\qquad + 3\left\|A^2\frac{du}{dt}\right\|^2_{L_2(R_+;H)} + \left\|A^3u\right\|^2_{L_2(R_+;H)} - \|\varphi_2\|^2 + 2\operatorname{Re}(\varphi_2,\varphi_1) \\
&\qquad + 2\operatorname{Re}(\varphi_2,\varphi_0) + 2\operatorname{Re}(\varphi_1,\varphi_0) - \|\varphi_0\|^2.
\end{aligned} \tag{3.9}$$

From (3.8), taking into account (3.9) and applying Theorem 3.1, we get the validity of (3.7). The theorem is proved. □

Corollary 3.3 *If* $u(t) \in \overset{\circ}{W}{}_2^3(R_+;H)$ *and* $\beta \in [0, \frac{27}{4})$, *then*

$$\left\| \left(-\frac{d}{dt}+A\right)\left(\frac{d}{dt}+A\right)^2 u \right\|_{L_2(R_+;H)}^2 - \beta \left\| A^{3-j}\frac{d^j u}{dt^j} \right\|_{L_2(R_+;H)}^2 = \left\| F_j\left(\frac{d}{dt};\beta;A\right)u \right\|_{L_2(R_+;H)}^2. \tag{3.10}$$

Corollary 3.4 *If* $u(t) \in \overset{\circ}{W}{}_2^3(R_+;H;0,1)$ *and* $\beta \in [0, \frac{27}{4})$, *then*

$$\left\| \left(-\frac{d}{dt}+A\right)\left(\frac{d}{dt}+A\right)^2 u \right\|_{L_2(R_+;H)}^2 - \beta \left\| A^{3-j}\frac{d^j u}{dt^j} \right\|_{L_2(R_+;H)}^2 = \left\| F_j\left(\frac{d}{dt};\beta;A\right)u \right\|_{L_2(R_+;H)}^2 + \left(S_j(\beta;0,1)\tilde{\varphi},\tilde{\varphi}\right)_H, \tag{3.11}$$

where $S_j(\beta;0,1) = \alpha_{1,j}(\beta) - 1$ *is obtained from* $S_j(\beta)$ *by discarding the first two rows and columns, here* $\tilde{\varphi} = A^{1/2}\frac{d^2u(0)}{dt^2}$.

4 On the values of the numbers n_j, $j = 1, 2$

It is easy to check that the norms $\|u\|_{W_2^3(R_+;H)}$ and $\|(-\frac{d}{dt}+A)(\frac{d}{dt}+A)^2u\|_{L_2(R_+;H)}$ are equivalent on $\overset{\circ}{W}{}_2^3(R_+;H)$. Then it follows from the theorem on intermediate derivatives [3, Ch.1] that the following numbers are finite:

$$n_{0,j} = \sup_{0\neq u\in \overset{\circ}{W}{}_2^3(R_+;H)} \frac{\|A^{3-j}\frac{d^ju}{dt^j}\|_{L_2(R_+;H)}}{\|(-\frac{d}{dt}+A)(\frac{d}{dt}+A)^2u\|_{L_2(R_+;H)}}, \quad j = 1, 2.$$

Lemma 4.1 $n_{0,j} = \frac{2}{3\sqrt{3}}$, $j = 1, 2$.

Proof Passing to the limit as $\beta \to \frac{27}{4}$ in (3.10), we see that, for any function $u(t) \in \overset{\circ}{W}{}_2^3(R_+;H)$, the following inequality holds:

$$\frac{27}{4}\left\| A^{3-j}\frac{d^j u}{dt^j} \right\|_{L_2(R_+;H)}^2 \leq \left\| \left(-\frac{d}{dt}+A\right)\left(\frac{d}{dt}+A\right)^2 u \right\|_{L_2(R_+;H)}^2,$$

i.e., $n_{0,j} \leq \frac{2}{3\sqrt{3}}$. We need to show that here we have the equality. To do this, it suffices to show that for any $\varepsilon > 0$, there exists a function $u_\varepsilon(t) \in \overset{\circ}{W}{}_2^3(R_+;H)$ such that

$$\left\| \left(-\frac{d}{dt}+A\right)\left(\frac{d}{dt}+A\right)^2 u_\varepsilon \right\|_{L_2(R_+;H)}^2 - \left(\frac{27}{4}+\varepsilon\right)\left\| A^{3-j}\frac{d^j u_\varepsilon}{dt^j} \right\|_{L_2(R_+;H)}^2 < 0.$$

Note that the procedure of constructing such functions $u_\varepsilon(t)$ is thoroughly described in [17] (in addition, the one for fourth-order equations with multiple characteristic is available in [20]). This method is applicable to our case, too. Therefore, we omit the respective part of the proof. So lemma is proved. □

Remark 4.2 Since $\overset{\circ}{W}{}_2^3(R_+;H) \subset \overset{\circ}{W}{}_2^3(R_+;H;0,1)$, then $n_j \geq n_{0,j} = \frac{2}{3\sqrt{3}}$, $j = 1, 2$. Therefore, there arises the question: When do we have $n_j = \frac{2}{3\sqrt{3}}$, $j = 1, 2$?

Denote by μ_j the root of the equation $S_j(\beta;0,1) = 0$ in the interval $(0, \frac{27}{4})$ if such exists.

Theorem 4.3 *The following relation holds*:

$$n_j = \begin{cases} \frac{2}{3\sqrt{3}} & \text{for } S_j(\beta;0,1) \neq 0, \beta \in (0, \frac{27}{4}), \\ \mu_j^{-1/2} & \text{otherwise.} \end{cases}$$

Proof If $n_j = \frac{2}{3\sqrt{3}}$, then it follows from equation (3.11) that for any function $u(t) \in \overset{\circ}{W}{}_2^3(R_+;H;0,1)$ and for all $\beta \in [0, \frac{27}{4})$, the following inequality holds:

$$\left\| F_j\left(\frac{d}{dt};\beta;A\right)u\right\|^2_{L_2(R_+;H)} + \left(S_j(\beta;0,1)\tilde{\varphi},\tilde{\varphi}\right)_H \geq \|P_0u\|^2_{L_2(R_+;H)}\left(1-\beta n_j^2\right) > 0. \tag{4.1}$$

Now let us consider the Cauchy problem

$$F_j\left(\frac{d}{dt};\beta;A\right)u(t) = 0, \tag{4.2}$$

$$\frac{d^k u(0)}{dt^k} = 0, \quad k = 0,1, \tag{4.3}$$

$$\frac{d^2 u(0)}{dt^2} = A^{-1/2}\tilde{\varphi}, \quad \tilde{\varphi} \in H. \tag{4.4}$$

Since by Theorem 3.1, for $\beta \in [0, \frac{27}{4})$, the polynomial operator pencil $F_j(\lambda;\beta;A)$ is of the form $F_j(\lambda;\beta;A) = \prod_{s=1}^{3}(\lambda E - \omega_{j,s}(\beta)A)$, where $\operatorname{Re}\omega_{j,s}(\beta) < 0$, $s = 1,2,3$, then Cauchy problem (4.2)-(4.4) has a unique solution $u_\beta(t) \in W_2^3(R_+;H)$, which can be expressed as:

$$u_\beta(t) = e^{\omega_{j,1}(\beta)tA}\eta_1 + e^{\omega_{j,2}(\beta)tA}\eta_2 + e^{\omega_{j,3}(\beta)tA}\eta_3,$$

where $\eta_1,\eta_2,\eta_3 \in H_{5/2}$ are uniquely determined by the conditions at zero (4.3), (4.4). Therefore, if we rewrite the inequality (4.1) for function $u_\beta(t)$, then for $\beta \in [0, \frac{27}{4})$ we will have

$$\left\| F_j\left(\frac{d}{dt};\beta;A\right)u_\beta\right\|^2_{L_2(R_+;H)} + \left(S_j(\beta;0,1)\tilde{\varphi},\tilde{\varphi}\right)_H > 0.$$

This means that $(S_j(\beta;0,1)\tilde{\varphi},\tilde{\varphi})_H > 0$, and therefore, $S_j(\beta;0,1) > 0$ for all $\beta \in [0, \frac{27}{4})$. Now let $S_j(\beta;0,1) > 0$ for all $\beta \in [0, \frac{27}{4})$. Then it follows from (3.11) that for any function $u(t) \in \overset{\circ}{W}{}_2^3(R_+;H;0,1)$ and for all $\beta \in [0, \frac{27}{4})$,

$$\|P_0u\|^2_{L_2(R_+;H)} - \beta\left\|A^{3-j}\frac{d^ju}{dt^j}\right\|^2_{L_2(R_+;H)} > 0.$$

Passing here to the limit as $\beta \to \frac{27}{4}$, we obtain that $n_j \leq \frac{2}{3\sqrt{3}}$, and hence, $n_j = \frac{2}{3\sqrt{3}}$.

Continuing the proof of the theorem, we suppose that $n_j > \frac{2}{3\sqrt{3}}$. Then $n_j^{-2} \in (0, \frac{27}{4})$. Note that for $\beta \in (0, n_j^{-2})$, we have

$$\|P_0u\|^2_{L_2(R_+;H)} - \beta\left\|A^{3-j}\frac{d^ju}{dt^j}\right\|^2_{L_2(R_+;H)} \geq \|P_0u\|^2_{L_2(R_+;H)}\left(1-\beta n_j^2\right) > 0.$$

Therefore, from (3.11) we find that for $\beta \in (0, n_j^{-2})$, the following inequality holds:

$$\left\| F_j\left(\frac{d}{dt};\beta;A\right)u\right\|^2_{L_2(R_+;H)} + \left(S_j(\beta;0,1)\tilde{\varphi},\tilde{\varphi}\right)_H > 0.$$

Applying this inequality again to the solution $u(t)$ of Cauchy problem (4.2)-(4.4), we obtain $S_j(\beta;0,1) > 0$ for all $\beta \in [0, n_j^{-2})$. On the other hand, it follows from the definition of n_j that, for any $\beta \in (n_j^{-2}, \frac{27}{4})$, there exists a function $v_\beta(t) \in \overset{\circ}{W}{}_2^3(R_+;H;0,1)$ such that

$$\|P_0 v_\beta\|^2_{L_2(R_+;H)} < \beta \left\| A^{3-j}\frac{d^j v_\beta}{dt^j}\right\|^2_{L_2(R_+;H)}.$$

Taking into account this inequality in (3.11), we obtain

$$\left\| F_j\left(\frac{d}{dt};\beta;A\right)v_\beta\right\|^2_{L_2(R_+;H)} + \left(S_j(\beta;0,1)\tilde{\varphi}_\beta,\tilde{\varphi}_\beta\right)_H < 0,$$

where $\tilde{\varphi}_\beta = A^{1/2}\frac{d^2 v_\beta(0)}{dt^2}$. Therefore, there exists a vector $\tilde{\varphi}_\beta \in H$ such that

$$\left(S_j(\beta;0,1)\tilde{\varphi}_\beta,\tilde{\varphi}_\beta\right)_H < 0.$$

Thus, $S_j(\beta;0,1) < 0$ for $\beta \in (n_j^{-2}, \frac{27}{4})$. And since $S_j(\beta;0,1)$ is a continuous function of the argument β in the interval $[0, \frac{27}{4})$, then $S_j(n_j^{-2};0,1) = 0$. It follows from these arguments that the equation $S_j(\beta;0,1) = 0$ has a root in the interval $(0, \frac{27}{4})$. Now let $S_j(\beta;0,1) = 0$ have a root in the interval $(0, \frac{27}{4})$. This means that the inequality $S_j(\beta;0,1) > 0$ cannot be satisfied for any $\beta \in [0, \frac{27}{4})$. Therefore, according to our earlier reasonings in the proof of this theorem, we have $n_j > \frac{2}{3\sqrt{3}}$. Obviously, for the root μ_j of the equation $S_j(\beta;0,1) = 0$, we have that $\mu_j \geq n_j^{-2}$, because the proof of the theorem for $\beta \in [0, n_j^{-2})$ implies that $S_j(\beta;0,1) > 0$. And since $S_j(n_j^{-2};0,1) = 0$, we obtain $n_j^{-2} = \mu_j$. The theorem is proved. □

Remark 4.4 From Theorem 4.3, it becomes clear that to find the numbers n_j, $j = 1,2$, we must solve the equations $S_j(\beta;0,1) = 0$, $j = 1,2$, together with systems (3.3) respectively. In this case, it is necessary to take into account the properties of the numbers $\alpha_{1,j}(\beta)$, $\alpha_{2,j}(\beta)$, $j = 1,2$.

The following theorem holds.

Theorem 4.5 $n_1 = \frac{2}{3\sqrt{3}}$, $n_2 = \frac{1}{\sqrt{2}(\sqrt{5}+1)^{1/2}}$.

Proof In view of Remark 4.4, in the case $j = 1$, we have $n_1 = \frac{2}{3\sqrt{3}}$ due to the negativity of $\alpha_{2,1}(\beta)$, despite $\alpha_{1,1}(\beta) = 1 \Rightarrow \alpha_{2,1}(\beta) = -1 \Rightarrow \beta = 4 \in (0, \frac{27}{4})$. In the case $j = 2$, we have $\alpha_{1,2}(\beta) = 1 \Rightarrow \alpha_{2,2}(\beta) = -\sqrt{5}$ or $\alpha_{2,2}(\beta) = \sqrt{5}$. Then $\beta = -2(\sqrt{5}-1) \notin (0, \frac{27}{4})$ or $\beta = 2(\sqrt{5}+1) \in (0, \frac{27}{4})$, respectively. Therefore, $n_2 = \frac{1}{\sqrt{2}(\sqrt{5}+1)^{1/2}}$. The theorem is proved. □

5 Solvability of boundary value problem (1.1), (1.2). Example

The results obtained above allow us to establish exact conditions for regular solvability of boundary value problem (1.1), (1.2). These conditions are expressed in terms of the operator coefficients of equation (1.1).

Theorem 5.1 *Let the operators A_jA^{-j}, $j = 1, 2$ be bounded on H and the following inequality hold*:

$$\frac{1}{\sqrt{2}(\sqrt{5}+1)^{1/2}}\left\|A_1A^{-1}\right\|_{H\to H}+\frac{2}{3\sqrt{3}}\left\|A_2A^{-2}\right\|_{H\to H}<1.$$

Then boundary value problem (1.1), (1.2) *is regularly solvable*.

Note that the above conditions for regular solvability of boundary value problem (1.1), (1.2) are easily verified in applications because they are expressed in terms of the operator coefficients of equation (1.1).

Let us illustrate our solvability results with an example of an initial-boundary value problem for a partial differential equation.

Example 5.2 On the half-strip $R_+ \times [0, \pi]$, consider the problem

$$\left(-\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)^2 u(t,x)+p(x)\frac{\partial^4 u(t,x)}{\partial x^2\,\partial t^2}+q(x)\frac{\partial^5 u(t,x)}{\partial x^4\,\partial t}=f(t,x), \tag{5.1}$$

$$\frac{\partial^i u(0,x)}{\partial t^i}=0, \quad i=0,1, \tag{5.2}$$

$$\frac{\partial^{2k} u(t,0)}{\partial x^{2k}}=\frac{\partial^{2k} u(t,\pi)}{\partial x^{2k}}=0, \quad k=0,1,2, \tag{5.3}$$

where $p(x)$, $q(x)$ are bounded functions on $[0, \pi]$ and $f(t, x) \in L_2(R_+; L_2[0, \pi])$. Note that problem (5.1)-(5.3) is a special case of boundary value problem (1.1), (1.2). In fact, here we have $H = L_2[0, \pi]$, $A_1 = p(x)\frac{\partial^2}{\partial x^2}$ and $A_2 = q(x)\frac{\partial^4}{\partial x^4}$. The operator A is defined on $L_2[0, \pi]$ by the relation $Au = -\frac{d^2u}{dx^2}$ and the conditions $u(0) = u(\pi) = 0$.

Applying Theorem 5.1, we obtain that under the condition

$$\frac{1}{\sqrt{2}(\sqrt{5}+1)^{1/2}}\sup_{0\le x\le\pi}\left|p(x)\right|+\frac{2}{3\sqrt{3}}\sup_{0\le x\le\pi}\left|q(x)\right|<1,$$

problem (5.1)-(5.3) has a unique solution in the space $W^{3,6}_{t,x,2}(R_+; L_2[0, \pi])$.

Remark 5.3 Using the same procedure, we can obtain similar results for equation (1.1) on the semiaxis R_+ with boundary conditions $u(0) = \frac{d^2u(0)}{dt^2} = 0$ or $\frac{du(0)}{dt} = \frac{d^2u(0)}{dt^2} = 0$.

Competing interests
The authors declare that they have no competing interests.

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