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# Symmetric positive solutions to a second-order boundary value problem with integral boundary conditions

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**Abstract**

This paper investigates the existence of concave symmetric positive solutions and establishes corresponding iterative schemes for a second-order boundary value problem with integral boundary conditions. The main tool is a monotone iterative technique. Meanwhile, an example is worked out to demonstrate the main results.

**Keywords:** integral boundary conditions; iterative; monotone positive solution; symmetric; completely continuous

## 1 Introduction

The theory of boundary value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlinear problems with integral boundary conditions; we refer readers to [1–3] for examples and references.

At the same time, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point and nonlocal boundary value problems as special cases.

Hence, increasing attention is paid to boundary value problems with integral boundary conditions [4–8]. Generally, the Guo-Krasnosel' skii fixed point theorem in a cone, the Leggett-Williams fixed point theorem, the method of upper and lower solutions and the monotone iterative technique play extremely important roles in proving the existence of solutions to boundary value problems. In particular, we would like to mention some excellent results.

In [4], Ma studied the following problem:

$$\begin{cases} u^{(4)}(t) = h(t)f(t,u), & 0 < t < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s)\,ds, \\ u''(0) = u''(1) = \int_0^1 q(s)u(s)\,ds, \end{cases}$$

where $p, q \in L^1[0,1]$, $h$ and $f$ are continuous. The existence of at least one symmetric positive solution is obtained by the application of the fixed point index in cones.

In 2010, Wang *et al.* [7] considered the second-order boundary value problem with the integral boundary conditions

$$\begin{cases} (\phi(u'(t)))' + f(t, u(t), u'(t)) = 0, & 0 < t < 1, \\ u(0) - k_1 u'(0) = \int_0^1 h_1(u(s))\,ds, \\ u(1) + k_2 u'(1) = \int_0^1 h_2(u(s))\,ds, \end{cases}$$

where $\phi, f, h_1$ and $h_2$ are continuous, $k_1$ and $k_2$ are nonnegative constants. The existence result was obtained by applying the method of upper and lower solutions and Leray-Schauder degree theory. Theorem 1 (see [7]) supposed that the upper and lower solutions exist, and then, the theory of differential inequalities was used to prove that there is a solution to the boundary value problem between the upper and lower solutions.

Different from [7], [9] is not based on the assumption that the upper and lower solutions to the boundary value problem should exist, but constructs the specific form of the symmetric upper and lower solutions. The author in [9] investigated a second-order Sturm-Liouville boundary value problem

$$\begin{cases} w''(t) + h(t)f(w(t)) = 0, & 0 < t < 1, \\ \alpha w(0) - \beta w'(0) = 0, & \alpha w(1) + \beta w'(1) = 0. \end{cases}$$

And by applying monotone iterative techniques, author proved the existence of n symmetric positive solutions.

To the best of our knowledge, no contribution exists concerning the existence of solutions for a boundary value problem with integral boundary conditions by applying monotone iterative techniques. Inspired by the work mentioned above, we concentrate on the following problem:

$$u''(x) + f\big(x, u(x), u'(x)\big) = 0, \quad 0 < x < 1, \tag{1.1}$$

$$u(0) = u(1) = \int_0^1 p(s)u(s)\,ds, \tag{1.2}$$

where $p \in L^1[0,1]$. We construct a specific form of the symmetric upper and lower solutions, and by applying monotone iterative techniques, we construct successive iterative schemes for approximating solutions.

The difficulty of this paper is that the nonlinear term $f$ depends on $u'$, which leads to complexities to prove the properties of the operator $T$, especially the monotonicity of the operator $T$. In Lemma 2.2, we skillfully use the cone's character to overcome the mentioned obstacle. In addition, it is worth stating that the first term of our iterative scheme is a simple function or a constant function. Therefore, the iterative scheme is feasible. Under the appropriate assumptions on nonlinear term, this paper aims to establish a new and general result on the existence of a symmetric positive solution to BVP (1.1) and (1.2).

## 2 Preliminaries

**Definition 2.1** Let $E$ be a Banach space, a nonempty convex closed set $P \subset E$ is said to be a cone provided the following hypotheses are satisfied:

(i)  if $u \in P$, $\lambda \geq 0$, then $\lambda u \in P$;

(ii)  if $u \in P$ and $-u \in P$, then $u = 0$.

Every cone $P \subset E$ induces a partial ordering '$\leq$' on E defined by

$$u \leq v \quad \text{if and only if} \quad v - u \in P.$$

**Definition 2.2**  Let $(E, \leq)$ be an ordered Banach space. An operator $\varphi : E \to E$ is said to be nondecreasing (nonincreasing) provided that $\varphi(u) \leq \varphi(v)$ ($\varphi(u) \geq \varphi(v)$) for all $u, v \in E$ with $u \leq v$. If the inequality is strict, then $\varphi$ is said to be strictly nondecreasing (nonincreasing).

**Definition 2.3**  Let $E = C^1[0,1]$, $u \in E$ is said to be concave on $[0,1]$ if

$$u\big(\lambda x_1 + (1 - \lambda)x_2\big) \geq \lambda u(x_1) + (1 - \lambda)u(x_2)$$

for any $x_1, x_2 \in [0,1]$ and $\lambda \in [0,1]$.

We consider the Banach space $E = C^1[0,1]$ equipped with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$, where $\|u\|_\infty = \max_{x \in [0,1]} |u(x)|$. In this paper, a symmetric positive solution $u^*$ of (1.1) means a function which is symmetric and positive on $(0,1)$ and satisfies equation (1.1) as well as the boundary conditions (1.2).

In this paper, we always suppose that the following assumptions hold:

($H_1$) $f \in C([0,1] \times [0,+\infty) \times R, [0,+\infty))$, $f(x,u,v) = f(1 - x,u,-v)$ for $x \in [\frac{1}{2},1]$, and $f(x,u,v) > 0$ for all $(x,u,v) \in [0,1] \times [0,+\infty) \times R$;

($H_2$) $f(x,\cdot,v)$ is nondecreasing for each $(x,v) \in [0,\frac{1}{2}] \times R$, $f(x,u,\cdot)$ is nondecreasing for $(x,u) \in [0,\frac{1}{2}] \times [0,+\infty)$;

($H_3$) $p \in L^1[0,1]$ is nonnegative and $0 \leq \mu < 1$, where $\mu = \int_0^1 p(s)\, ds$.

Denote

$$C^+[0,1] = \big\{u \in E : u(x) \geq 0, x \in [0,1]\big\},$$

$$P = \big\{u \in E : u(x) \geq 0 \text{ is concave and } u(x) = u(1 - x), x \in [0,1]\big\}.$$

It is easy to see that $P$ is a cone in $E$.

For any $y \in C^+[0,1]$, suppose that $u$ is a solution of the following BVP:

$$\begin{cases} u''(x) + f(x,y(x),y'(x)) = 0, & 0 < x < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s)\, ds. \end{cases}$$

Then we can easily get the solution:

$$u(x) = \int_0^1 H(x,t)f\big(t,y(t),y'(t)\big)\, dt, \tag{2.1}$$

where

$$H(x,t) = G(x,t) + \frac{1}{1 - \mu}\int_0^1 G(t,s)p(s)\, ds, \qquad G(t,s) = \begin{cases} t(1 - s), & 0 \leq t \leq s \leq 1, \\ s(1 - t), & 0 \leq s \leq t \leq 1 \end{cases}$$

and

$$\mu = \int_0^1 p(s)\,ds.$$

During the process of getting the above solution, we can also know

$$u'(x) = \int_0^1 (1-t)f\big(t,y(t),y'(t)\big)\,dt - \int_0^x f\big(t,y(t),y'(t)\big)\,dt \tag{2.2}$$

for $x \in [0,1]$.

**Lemma 2.1** *If* $(H_3)$ *is satisfied, the following results are true:*
1.  $H(x,t) \ge 0, \text{for } x,t \in [0,1]; H(x,t) > 0 \text{ for } x,t \in (0,1).$
2.  $G(1-x,1-t) = G(x,t), G(x,t) \le G(x,x) \text{ for } x,t \in [0,1].$

For any $y \in C^+[0,1]$, $T : P \to E$ is defined

$$(Ty)(x) = \int_0^1 H(x,t)f\big(t,y(t),y'(t)\big)\,dt \quad \text{for } x \in [0,1]. \tag{2.3}$$

**Lemma 2.2** *If* $(H_3)$ *is satisfied, $T : P \to P$ is completely continuous, i.e., $T$ is continuous and compact. Moreover, $T$ is nondecreasing provided that* $(H_2)$ *holds.*

*Proof* For any $y \in P$, from the definition of $Ty$, we know

$$\begin{cases} (Ty)''(x) + f(x,y(x),y'(x)) = 0, & 0 < x < 1, \\ (Ty)(0) = (Ty)(1) = \int_0^1 p(s)(Ty)(s)\,ds. \end{cases}$$

Obviously, $Ty$ is concave. From the expression of $Ty$, combining with Lemma 2.1, we know that $Ty$ is nonnegative on $[0,1]$. We now prove that $Ty$ is symmetric about $\frac{1}{2}$.

For $x \in [0,\frac{1}{2}]$, then $(1-x) \in [\frac{1}{2},1]$, and

$$\begin{aligned} (Ty)(1-x) &= \int_0^1 \left( G(1-x,t) + \frac{1}{1-\mu}\int_0^1 G(t,s)p(s)\,ds \right)f\big(t,y(t),y'(t)\big)\,dt \\ &= \int_0^1 G(1-x,t)f\big(t,y(t),y'(t)\big)\,dt \\ &\quad + \frac{1}{1-\mu}\int_0^1\int_0^1 G(t,s)p(s)f\big(t,y(t),y'(t)\big)\,ds\,dt \\ &= \int_1^0 G(1-x,1-t)f\big(1-t,y(1-t),y'(1-t)\big)\,d(1-t) \\ &\quad + \frac{1}{1-\mu}\int_0^1\int_0^1 G(t,s)p(s)f\big(t,y(t),y'(t)\big)\,ds\,dt \\ &= \int_0^1 G(x,t)f\big(1-t,y(t),-y'(t)\big)\,dt \\ &\quad + \frac{1}{1-\mu}\int_0^1\int_0^1 G(t,s)p(s)f\big(t,y(t),y'(t)\big)\,ds\,dt \end{aligned}$$

$$= \int_0^1 G(x,t)f\big(t,y(t),y'(t)\big)\,dt + \frac{1}{1-\mu}\int_0^1\int_0^1 G(t,s)p(s)f\big(t,y(t),y'(t)\big)\,ds\,dt$$

$$= (Ty)(x).$$

Similarly, we have

$$(Ty)(1-x) = (Ty)(x) \quad \text{for } x \in \left[\frac{1}{2},1\right].$$

So, $TP \subset P$. The continuity of $T$ is obvious. We now prove that $T$ is compact. Let $\Omega \subset P$ be a bounded set. Then there exists $R$ such that

$$\Omega = \big\{y \in P \mid \|y\| \le R\big\}.$$

For any $y \in \Omega$, we have

$$0 \le f\big(s,y(s),y'(s)\big) \le \max\big\{f\big(s,y,y'\big) \mid s \in [0,1], y \in [0,R], y' \in [-R,R]\big\} =: M.$$

Therefore, from (2.3), we have

$$\big\|(Ty)\big\|_\infty \le M + \frac{\mu}{1-\mu}M = \frac{M}{1-\mu}, \qquad \big\|(Ty)'\big\|_\infty \le M.$$

So, $\|(Ty)\|$ is uniformly bounded. Now we prove $Ty$ is equi-continuous. For $0 \le x_1 \le x_2 \le \frac{1}{2}$, we have

$$\big|(Ty)(x_2) - (Ty)(x_1)\big|$$

$$= \left|\int_0^1 \big(G(x_2,t) - G(x_1,t)\big)f\big(t,y(t),y'(t)\big)\,dt\right|$$

$$\le \begin{cases} \int_0^1 |(x_2-x_1)(1-t)|f(t,y(t),y'(t))\,dt, & 0 \le x_1 \le x_2 \le t \le 1, \\ \int_0^1 |t(x_1-x_2)|f(t,y(t),y'(t))\,dt, & 0 \le t \le x_1 \le x_2 \le 1, \\ \int_0^1 |t(1-x_2)-x_1(1-t)|f(t,y(t),y'(t))\,dt, & 0 \le x_1 \le t \le x_2 \le 1. \end{cases}$$

$$\le M|x_2-x_1|.$$

Moreover,

$$\big|(Ty)'(x_2) - (Ty)'(x_1)\big| = \left|\int_{x_1}^{x_2} f\big(t,y(t),y'(t)\big)\,dt\right| \le M|x_2-x_1|.$$

And the similar results can be obtained for $\frac{1}{2} \le x_1 \le x_2 \le 1$ and $0 \le x_1 \le \frac{1}{2} \le x_2 \le 1$.

The Arzelà-Ascoli theorem guarantees that $T\Omega$ is relatively compact, which means $T$ is compact.

Finally, we show that $Ty$ is nondecreasing about $y$.

For any $y_i(x) \in P$ ($i = 1, 2$) with $y_1(x) \le y_2(x)$. By the properties of a cone, we have $y_2(x) - y_1(x) \in P$ for $x \in [0,1]$. Then $y_2(x) - y_1(x) \ge 0$ is concave and symmetric about $\frac{1}{2}$. Therefore,

$$\begin{cases} y_2'(x) \ge y_1'(x) & \text{for } x \in [0,\frac{1}{2}], \\ y_2'(x) \le y_1'(x) & \text{for } x \in [\frac{1}{2},1]. \end{cases}$$

Hence, for $x \in [0, \frac{1}{2}]$, by (H$_2$) and the definition of $Ty$, we have

$$(Ty_1)(x) - (Ty_2)(x) = \int_0^1 H(x,t)f(t, y_1(t), y_1'(t)) \, dt$$

$$- \int_0^1 H(x,t)f(t, y_2(t), y_2'(t)) \, dt \leq 0.$$

Furthermore, we have

$$(Ty_2)'(x) - (Ty_1)'(x) = \int_0^1 (1-s)f(s, y_2(s), y_2'(s)) \, ds - \int_0^1 (1-s)f(s, y_1(s), y_1'(s)) \, ds$$

$$+ \int_0^x f(s, y_1(s), y_1'(s)) \, ds - \int_0^x f(s, y_2(s), y_2'(s)) \, ds.$$

In order to prove $(Ty_2)(x) - (Ty_1)(x)$ is concave, we need to prove $(Ty_2)'(x) - (Ty_1)'(x)$ is nonincreasing. Let $0 \leq x_1 \leq x_2 \leq \frac{1}{2}$, then

$$(Ty_2)'(x_2) - (Ty_1)'(x_2) - (Ty_2)'(x_1) + (Ty_1)'(x_1)$$

$$= \int_{x_1}^{x_2} f(s, y_1(s), y_1'(s)) \, ds - \int_{x_1}^{x_2} f(s, y_2(s), y_2'(s)) \, ds \leq 0.$$

A similar result can be obtained for $x \in [\frac{1}{2}, 1]$. And it is easy to see that $(Ty_2)(x) - (Ty_1)(x)$ is symmetric about $\frac{1}{2}$. So, $(Ty_2 - Ty_1) \in P$ and thus $T$ is nondecreasing. □

## 3 Existence and iterative of solutions for BVP (1.1) and (1.2)

**Theorem 3.1** *Assume that* (H$_1$)-(H$_3$) *hold. If there exist two positive numbers $a_1 < a$ such that*

$$\sup_{x \in [0, \frac{1}{2}]} f(x, a, a) \leq a_1, \tag{3.1}$$

*where $a$ and $a_1$ satisfy*

$$a \geq \max \left\{ \frac{2\mu}{9(1-\mu)}, \frac{1}{8} + \frac{\mu}{6(1-\mu)}, \frac{1}{2} \right\} a_1, \tag{3.2}$$

*then problem* (1.1) *and* (1.2) *has a concave symmetric positive solution $w^*, v^* \in P$ with*

$$\|w^*\|_\infty \leq a \quad and \quad \lim_{n \to \infty} T^n w_0 = w^*, \quad where \ w_0(x) = ax(1-x) + \frac{3}{4}a,$$

$$\|v^*\|_\infty \leq a \quad and \quad \lim_{n \to \infty} T^n v_0 = v^*, \quad where \ v_0(x) = 0.$$

*Proof* We denote $P_a = \{w \in P : \|w\| \leq a\}$. In what follows, we first prove $TP_a \subset P_a$.

Let $w \in P_a$, then $0 \leq w(x) \leq \max_{x \in [0,1]} w(x) = \|w\|_\infty \leq a$, $\max_{x \in [0,1]} |w'(x)| = w'(0) \leq a$. By assumption (H$_2$) and (3.1), for $x \in [0, \frac{1}{2}]$, we have

$$0 \leq f(x, w(x), w'(x)) \leq f(x, a, a) \leq \sup_{x \in [0, \frac{1}{2}]} f(x, a, a) \leq a_1. \tag{3.3}$$

For any $w(x) \in P_a$, by Lemma 2.2, we know $Tw \in P$ and, as a result,

$$
\begin{aligned}
\|Tw\|_\infty &= (Tw)\left(\frac{1}{2}\right) \\
&= \int_0^1 G\left(\frac{1}{2}, t\right) f\left(t, w(t), w'(t)\right) dt + \frac{1}{1-\mu} \int_0^1 \int_0^1 G(t,s) p(s) f\left(t, w(t), w'(t)\right) ds\, dt \\
&\le \frac{1}{8} a_1 + \frac{a_1}{1-\mu} \int_0^1 \int_0^1 t(1-t) p(s)\, ds\, dt \\
&= \frac{1}{8} a_1 + \frac{a_1 \mu}{6(1-\mu)} \le a
\end{aligned}
$$

and

$$
\left\|(Tw)'\right\|_\infty = (Tw)'(0) = \int_0^1 (1-s) f\left(s, w(s), w'(s)\right) ds \le \frac{1}{2} a_1 \le a.
$$

Hence, $\|Tw\| \le a$. Thus, we get $TP_a \subset P_a$. Let $w_0(x) = ax(1-x) + \frac{3}{4} a$ for $x \in [0,1]$, then $\|w_0\| = a$ and $w_0(x) \in P_a$. Let $w_1(x) = Tw_0$, then $w_1 \in P_a$. We denote

$$
w_{n+1} = Tw_n = T^{n+1} w_0 \quad (n = 0, 1, 2, \ldots). \tag{3.4}
$$

Since $TP_a \subset P_a$, we have $w_n \in P_a$ $(n = 0, 1, 2, \ldots)$. From Lemma 2.2, $T$ is compact, we assert that $\{w_n\}_{n=1}^\infty$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^\infty$ and there exists $w^* \in P_a$ such that $w_{n_k} \to w^*$. From the definition of $T$, (3.1) and (3.2), we have

$$
\begin{aligned}
w_1(x) &= (Tw_0)(x) \\
&= \int_0^1 \left( G(x,t) + \frac{1}{1-\mu} \int_0^1 G(t,s) p(s)\, ds \right) f\left(t, w_0(t), w_0'(t)\right) dt \\
&= \int_0^1 G(x,t) f\left(t, w_0(t), w_0'(t)\right) dt + \frac{1}{1-\mu} \int_0^1 \int_0^1 G(t,s) p(s) f\left(t, w_0(t), w_0'(t)\right) ds\, dt \\
&\le \frac{1}{2} a_1 \left(x - x^2\right) + \frac{\mu a_1}{6(1-\mu)} \\
&\le ax(1-x) + \frac{3}{4} a = w_0.
\end{aligned}
$$

On the other hand, we notice that

$$
w_0''(x) - w_1''(x) \le a_1 - 2a \le 0.
$$

So, $w_0(x) - w_1(x) \in P_a$. By Lemma 2.2, we know $Tw_1 \le Tw_0$, which means $w_2 \le w_1$, $0 < x < 1$. By induction, $w_{n+1} \le w_n$, $0 < x < 1$ $(n = 0, 1, 2, \ldots)$. Hence, we assert that $w_n \to w^*$. Let $n \to \infty$ in (3.4) to obtain $Tw^* = w^*$ since T is continuous. It is well known that the fixed point of the operator $T$ is the solution of BVP (1.1) and (1.2). Therefore, $w^*$ is a concave symmetric positive solution of BVP (1.1) and (1.2).

Let $v_0 = 0$, $x \in [0,1]$, then $v_0 \in P_a$. Let $v_1 = Tv_0$, then $v_1 \in P_a$, we denote

$$
v_{n+1} = Tv_n = T^{n+1} v_0, \quad n = 0, 1, 2, \ldots. \tag{3.5}
$$

Similarly to $\{v_n\}_{n=1}^{\infty}$, we assert that $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and there exists $v^* \in P_a$ such that $v_{n_k} \to v^*$. Now, since $v_1 \geq v_0$, by Lemma 2.2, we know $Tv_1 \geq Tv_0$, which means $v_2 \geq v_1$, $0 < x < 1$. By induction, $v_{n+1} \geq v_n$, $0 < x < 1$ ($n = 0, 1, 2, \ldots$). Hence, we assert that $v_n \to v^*$, $Tv^* = v^*$, and $v^*(x) > 0$, $x \in (0, 1)$. Therefore, $v^*$ is a concave symmetric positive solution of BVP (1.1) and (1.2). □

**Remark** The existence of a solution under the assumptions of Theorem 3.1 is just a consequence of Schauder's fixed point theorem. The monotone iterative technique adds the information about the approximation sequences.

**Example** Consider the following second-order boundary value problem with integral boundary conditions:

$$
\begin{cases}
u''(x) + \left[\frac{1}{80} + \frac{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}{20}\right][(u')^2 + \ln(u(x)+1) + 1] = 0, & 0 < x < 1, \\
u(0) = u(1) = \int_0^1 s u(s)\, ds.
\end{cases}
\tag{3.6}
$$

And we have

$$
f(x, u, v) = \left[\frac{1}{80} + \frac{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}{20}\right]\left[(v)^2 + \ln(u+1) + 1\right].
$$

It is easy to check that the assumptions (H$_1$)-(H$_3$) hold and $\mu = 1/2$. Set $a = 10$, $a_1 = 15$. Then we can verify that condition (3.1) is satisfied. Then applying Theorem 3.1, BVP (3.6) has a concave symmetric positive solution $w^*, v^* \in P$ with

$$
\|w^*\|_\infty \leq 10 \quad and \quad \lim_{n\to\infty} T^n w_0 = w^*, \quad where \ w_0(x) = 10x(1-x) + \frac{15}{2},
$$

$$
\|v^*\|_\infty \leq 10 \quad and \quad \lim_{n\to\infty} T^n v_0 = v^*, \quad where \ v_0(x) = 0.
$$

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