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Blow-up phenomena and global existence for the periodic two-component Dullin-Gottwald-Holm system

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Abstract

This paper is concerned with blow-up phenomena and global existence for the periodic two-component Dullin-Gottwald-Holm system. We first obtain several blow-up results and the blow-up rate of strong solutions to the system. We then present a global existence result for strong solutions to the system.

MSC: 35G25; 35L05

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1 Introduction

In this paper, we consider the following periodic two-component Dullin-Gottwald-Holm (DGH) system:

$$\begin{cases} m_t - Au_x + um_x + 2u_x m + \gamma u_{xxx} + \rho\rho_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t \geq 0, x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $m = u - u_{xx}$, $A > 0$ and γ are constants.

System (1.1) has been recently derived by Zhu *et al.* in [1] by following Ivanov's approach [2]. It was shown in [1] that the DGH system is completely integrable and can be written as a compatibility condition of two linear systems

$$\Psi_{xx} = \left(-\xi^2\rho^2 + \xi\left(m - \frac{A}{2} + \frac{\gamma}{2}\right) + \frac{1}{4}\right)\Psi$$

and

$$\Psi_t = \left(\frac{1}{2\xi} - u + \gamma\right)\Psi_x + \frac{1}{2}u_x\Psi,$$

where ξ is a spectral parameter. Moreover, this system has the following two Hamiltonians:

$$E(u,\rho) = \frac{1}{2}\int \left(u^2 + u_x^2 + (\rho - 1)^2\right) dx$$

and

$$F(u,\rho) = \frac{1}{2}\int \left(u^3 + uu_x^2 - Au^2 - \gamma u_x^2 + 2u(\rho - 1) + u(\rho - 1)^2\right) dx.$$

For $\rho = 0$ and $m = u - \alpha^2 u_{xx}$, (1.1) becomes the DGH equation [3]

$$u_t - \alpha^2 u_{txx} - Au_x + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}),$$

where A and α are two positive constants, modeling unidirectional propagation of surface waves on a shallow layer of water which is at rest at infinity, $u(t,x)$ stands for fluid velocity. It is completely integrable with a bi-Hamiltonian and a Lax pair. Moreover, its traveling wave solutions include both the KdV solitons and the CH peakons as limiting cases [3]. The Cauchy problem of the DGH equation has been extensively studied, *cf.* [4–13].

For $\rho \not\equiv 0$, $\gamma = 0$, system (1.1) becomes the two-component Camassa-Holm system [2]

$$\begin{cases} m_t - Au_x + um_x + 2u_x m + \rho\rho_x = 0, \\ \rho_t + (u\rho)_x = 0, \end{cases} \tag{1.2}$$

where $\rho(t,x)$ is in connection with the free surface elevation from scalar density (or equilibrium), and the parameter A characterizes a linear underlying shear flow. System (1.2) describes water waves in the shallow water regime with nonzero constant vorticity, where the nonzero vorticity case indicates the presence of an underlying current. A large amount of literature was devoted to the Cauchy problem (1.2); see [14–22].

The Cauchy problem (1.1) has been discussed in [1]. Therein Zhu and Xu established the local well-posedness to system (1.1), derived the precise blow-up scenario and investigated the wave breaking for it. The aim of this paper is to further study the blow-up phenomena for strong solutions to (1.1) and to present a global existence result.

Our paper is organized as follows. In Section 2, we briefly state some needed results including the local well posedness of system (1.1), the precise blow-up scenario and some useful lemmas to study blow-up phenomena and global existence. In Section 3, we give several new blow-up results and the precise blow-up rate. In Section 4, we present a new global existence result of strong solutions to (1.1).

Notation Given a Banach space Z, we denote its norm by $\|\cdot\|_Z$. Since all space of functions is over $\mathbb{S}$, for simplicity, we drop $\mathbb{S}$ in our notations if there is no ambiguity.

2 Preliminaries

In this section, we will briefly give some needed results in order to pursue our goal.

With $m = u - u_{xx}$, we can rewrite system (1.1) as follows:

$$\begin{cases} u_t - u_{txx} - Au_x + \gamma u_{xxx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \rho\rho_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t \geq 0, x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Note that if $G(x) := \frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}$, $x \in \mathbb{R}$ is the kernel of $(1-\partial_x^2)^{-1}$, then $(1-\partial_x^2)^{-1}f = G * f$ for all $f \in L^2(\mathbb{S})$, $G * m = u$. Here we denote by $*$ the convolution. Using this identity, we can rewrite system (2.1) as follows:

$$\begin{cases} u_t + (u-\gamma)u_x = -\partial_x G * (u^2 + \frac{1}{2}u_x^2 + (\gamma - A)u + \frac{1}{2}\rho^2), & t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t \geq 0, x \in \mathbb{R}. \end{cases} \tag{2.2}$$

The local well-posedness of the Cauchy problem (2.1) can be obtained by applying Kato's theorem. As a result, we have the following well-posedness result.

Lemma 2.1 [1] *Given the initial data* $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, *there exists a maximal* $T = T(\|(u_0, \rho_0)\|_{H^s \times H^{s-1}}) > 0$ *and a unique solution*

$$(u, \rho) \in C\big([0,T); H^s \times H^{s-1}\big) \cap C^1\big([0,T); H^{s-1} \times H^{s-2}\big)$$

of (2.1). *Moreover, the solution* (u, ρ) *depends continuously on the initial data* (u_0, ρ_0) *and the maximal time of existence* $T > 0$ *is independent of* s.

Consider now the following initial value problem:

$$\begin{cases} q_t = u(t,q), & t \in [0,T), \\ q(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{2.3}$$

where u *denotes the first component of the solution* (u, ρ) *to* (2.1).

Lemma 2.2 [1] *Let* (u, ρ) *be the solution of* (2.1) *with the initial data* $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$. *Then Eq.* (2.3) *has a unique solution* $q \in C^1([0,T) \times \mathbb{R}; \mathbb{R})$. *Moreover, the map* $q(t, \cdot)$ *is an increasing diffeomorphism of* $\mathbb{R}$ *with*

$$q_x(t,x) = \exp\left(\int_0^t u_x\big(s, q(s,x)\big)\,ds\right) > 0, \quad (t,x) \in [0,T) \times \mathbb{R}.$$

Lemma 2.3 [1] *Let* (u,ρ) *be the solution of* (2.1) *with the initial data* $(u_0,\rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, *and* $T > 0$ *be the maximal time of existence. Then we have*

$$\rho\big(t,q(t,x)\big)q_x(t,x) = \rho_0(x), \quad (t,x) \in [0,T) \times \mathbb{S}.$$

Moreover, if there exists an $x_0 \in \mathbb{S}$ *such that* $\rho_0(x_0) = 0$, *then* $\rho(t,q(t,x_0)) = 0$ *for all* $t \in [0,T)$.

Next, we give two useful conservation laws of strong solutions to (2.1).

Lemma 2.4 [1] *Let* (u,ρ) *be the solution of* (2.1) *with the initial data* $(u_0,\rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, *and let* $T > 0$ *be the maximal time of existence. Then, for all* $t \in [0,T)$, *we have*

$$\int_{\mathbb{S}} \big(u^2 + u_x^2 + \rho^2\big)\,dx = \int_{\mathbb{S}} \big(u_0^2 + u_{0,x}^2 + \rho_0^2\big)\,dx := E_0.$$

Lemma 2.5 *Let* (u,ρ) *be the solution of* (2.1) *with the initial data* $(u_0,\rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, *and let* $T > 0$ *be the maximal time of existence. Then, for all* $t \in [0,T)$, *we have*

$$\int_{\mathbb{S}} u(t,x)\,dx = \int_{\mathbb{S}} u_0(x)\,dx.$$

Proof By the first equation in (2.1), we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} u(t,x)\,dx &= \int_{\mathbb{S}} u_t\,dx \\ &= \int_{\mathbb{S}} (u_{txx} + Au_x - \gamma u_{xxx} - 3uu_x + 2u_x u_{xx} + uu_{xxx} - \rho\rho_x)\,dx = 0.\end{aligned}$$

This completes the proof of the lemma. □

Then we state the following precise blow-up mechanism of (2.1).

Lemma 2.6 [1] *Let* (u,ρ) *be the solution of* (2.1) *with the initial data* $(u_0,\rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, *and let* $T > 0$ *be the maximal time of existence. Then the solution blows up in finite time if and only if*

$$\liminf_{t\to T^-}\Big\{\inf_{x\in\mathbb{S}} u_x(t,x)\Big\} = -\infty.$$

Lemma 2.7 [23] *Let* $t_0 > 0$ *and* $v \in C^1([0,t_0);H^2(\mathbb{R}))$. *Then, for every* $t \in [0,t_0)$, *there exists at least one point* $\xi(t) \in \mathbb{R}$ *with*

$$m(t) := \inf_{x\in\mathbb{R}}\big\{v_x(t,x)\big\} = v_x\big(t,\xi(t)\big),$$

and the function m *is almost everywhere differentiable on* $(0,t_0)$ *with*

$$\frac{d}{dt}m(t) = v_{tx}\big(t,\xi(t)\big) \quad \text{a.e. on } (0,t_0).$$

Lemma 2.8 [24] (i) *For every* $f \in H^1(\mathbb{S})$, *we have*

$$\max_{x\in[0,1]} f^2(x) \le \frac{e+1}{2(e-1)} \|f\|_{H^1}^2,$$

where the constant $\frac{e+1}{2(e-1)}$ *is sharp.*

(ii) *For every* $f \in H^3(\mathbb{S})$, *we have*

$$\max_{x\in[0,1]} f^2(x) \le c\|f\|_{H^1}^2,$$

with the best possible constant c lying within the range $(1, \frac{13}{12}]$. *Moreover, the best constant c is* $\frac{e+1}{2(e-1)}$.

Lemma 2.9 [25] *If* $f \in H^3(\mathbb{S})$ *is such that* $\int_{\mathbb{S}} f(x)\,dx = \frac{a_0}{2}$, *then, for every* $\epsilon > 0$, *we have*

$$\max_{x\in[0,1]} f^2(x) \le \frac{\epsilon+2}{24}\int_{\mathbb{S}} f_x^2\,dx + \frac{\epsilon+2}{4\epsilon}a_0^2.$$

Moreover,

$$\max_{x\in[0,1]} f^2(x) \le \frac{\epsilon+2}{24}\|f\|_{H^1(\mathbb{S})}^2 + \frac{\epsilon+2}{4\epsilon}a_0^2.$$

Lemma 2.10 [26] *Assume that a differentiable function* $y(t)$ *satisfies*

$$y'(t) \le -Cy^2(t) + K, \tag{2.4}$$

where C, K *are positive constants. If the initial datum* $y(0) = y_0 < -\sqrt{\frac{K}{C}}$, *then the solution to* (2.4) *goes to* $-\infty$ *before t tends to* $\frac{1}{-Cy_0+\frac{K}{y_0}}$.

3 Blow-up phenomena

In this section, we discuss the blow-up phenomena of system (2.1). Firstly, we prove that there exist strong solutions to (2.1) which do not exist globally in time.

Theorem 3.1 *Let* $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \ge 2$, *and* T *be the maximal time of the solution* (u, ρ) *to* (2.1) *with the initial data* (u_0, ρ_0). *If there is some* $x_0 \in \mathbb{S}$ *such that* $\rho_0(x_0) = 0$ *and*

$$u_0'(x_0) = \inf_{x\in\mathbb{S}} u_0'(x) < -\sqrt{\frac{e+1}{2(e-1)}E_0 + |\gamma - A|\sqrt{\frac{8(e+1)}{e-1}}E_0^{\frac{1}{2}}},$$

then the corresponding solution to (2.1) *blows up in finite time.*

Proof Applying Lemma 2.1 and a simple density argument, we only need to show that the above theorem holds for some $s \ge 2$. Here we assume $s = 3$ to prove the above theorem.

Define now

$$m(t) := \inf_{x\in\mathbb{S}}\bigl[u_x(t,x)\bigr], \quad t \in [0, T).$$

By Lemma 2.7, we let $\xi(t) \in \mathbb{S}$ be a point where this infimum is attained. It follows that

$$m(t) = u_x\bigl(t, \xi(t)\bigr) \quad \text{and} \quad u_{xx}\bigl(t, \xi(t)\bigr) = 0.$$

Differentiating the first equation in (2.2) with respect to x and using the identity $\partial_x^2 G * f = G * f - f$, we have

$$\begin{aligned} u_{tx} + (u-\gamma)u_{xx} = &-\frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + u^2 + (\gamma - A)u \\ &- G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + (\gamma - A)u\right). \end{aligned} \tag{3.1}$$

Since the map $q(t, \cdot)$ given by (2.3) is an increasing diffeomorphism of $\mathbb{R}$, there exists a $x(t) \in \mathbb{S}$ such that $q(t, x(t)) = \xi(t)$. In particular, $x(0) = \xi(0)$. Note that $u_0'(x_0) = \inf_{x\in\mathbb{S}} u_0'(x)$, we can choose $x_0 = \xi(0)$. It follows that $x(0) = \xi(0) = x_0$. By Lemma 2.3 and the condition $\rho_0(x_0) = 0$, we have

$$\rho\bigl(t, \xi(t)\bigr)q_x(t,x) = \rho\bigl(t, q\bigl(t, x(t)\bigr)\bigr)q_x(t,x) = \rho_0\bigl(x(0)\bigr) = \rho_0(x_0) = 0.$$

Thus $\rho(t, \xi(t)) = 0$.

Valuating (3.1) at $(t, \xi(t))$ and using Lemma 2.7, we obtain

$$\frac{dm(t)}{dt} \le -\frac{1}{2}m^2(t) + \frac{1}{2}u^2 + (\gamma - A)u - (\gamma - A)G * u, \tag{3.2}$$

here we used the relations $G * (u^2 + \frac{1}{2}u_x^2) \ge \frac{1}{2}u^2$ and $G * \rho^2 \ge 0$. Note that $\|G\|_{L^1} = 1$. By Lemma 2.4 and Lemma 2.8, we get

$$\|u\|_{L^\infty}^2 \le \frac{e+1}{2(e-1)}\|u\|_{H^1}^2 \le \frac{e+1}{2(e-1)}E_0,$$

$$\bigl|(\gamma - A)u\bigr| \le |\gamma - A|\|u\|_{L^\infty} \le |\gamma - A|\sqrt{\frac{e+1}{2(e-1)}}E_0^{\frac{1}{2}}$$

and

$$\bigl|(\gamma - A)G * u\bigr| \le |\gamma - A|\|G\|_{L^1}\|u\|_{L^\infty} \le |\gamma - A|\sqrt{\frac{e+1}{2(e-1)}}E_0^{\frac{1}{2}}.$$

It follows that

$$\frac{dm(t)}{dt} \le -\frac{1}{2}m^2(t) + K, \tag{3.3}$$

where $K = \frac{e+1}{4(e-1)}E_0 + 2|\gamma - A|\sqrt{\frac{e+1}{2(e-1)}}E_0^{\frac{1}{2}}$. Since $m(0) < -\sqrt{2K}$, Lemma 2.10 implies

$$\lim_{t\to T} m(t) = -\infty \quad \text{with } T = \frac{2u_0'(x_0)}{2K - (u_0'(x_0))^2}.$$

Applying Lemma 2.6, the solution blows up in finite time. □

Theorem 3.2 *Let* $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, *and* T *be the maximal time of the solution* (u, ρ) *to* (2.1) *with the initial data* (u_0, ρ_0). *Assume that* $\int_{\mathbb{S}} u_0(x)\,dx = \frac{a_0}{2}$. *If there is some* $x_0 \in \mathbb{S}$ *such that* $\rho_0(x_0) = 0$ *and for any* $\epsilon > 0$,

$$u_0'(x_0) = \inf_{x\in\mathbb{S}} u_0'(x) < -\sqrt{\frac{\epsilon+2}{24}E_0 + \frac{\epsilon+2}{4\epsilon}a_0^2 + |\gamma - A|\sqrt{\frac{2(\epsilon+2)}{3}E_0 + \frac{4(\epsilon+2)}{\epsilon}a_0^2}},$$

then the corresponding solution to (2.1) *blows up in finite time.*

Proof By Lemma 2.5, we have $\int_{\mathbb{S}} u(t,x)\,dx = \frac{a_0}{2}$. Using Lemma 2.4 and Lemma 2.9, we obtain

$$\|u\|_{L^\infty}^2 \leq \frac{\epsilon+2}{24}E_0 + \frac{\epsilon+2}{4\epsilon}a_0^2,$$
$$\bigl|(\gamma - A)u\bigr| \leq |\gamma - A|\|u\|_{L^\infty} \leq |\gamma - A|\sqrt{\frac{\epsilon+2}{24}E_0 + \frac{\epsilon+2}{4\epsilon}a_0^2}$$

and

$$\bigl|(\gamma - A)G * u\bigr| \leq |\gamma - A|\|G\|_{L^1}\|u\|_{L^\infty} \leq |\gamma - A|\sqrt{\frac{\epsilon+2}{24}E_0 + \frac{\epsilon+2}{4\epsilon}a_0^2}.$$

Following a similar proof in Theorem 3.1, we have

$$\frac{dm(t)}{dt} \leq -\frac{1}{2}m^2(t) + K, \tag{3.4}$$

where $K = \frac{\epsilon+2}{48}E_0 + \frac{\epsilon+2}{8\epsilon}a_0^2 + |\gamma - A|\sqrt{\frac{\epsilon+2}{6}E_0 + \frac{\epsilon+2}{\epsilon}a_0^2}$. Following the same argument as in Theorem 3.1, we deduce that the solution blows up in finite time. □

Letting $a_0 = 0$ and $\epsilon \to 0$ in Theorem 3.2, we have the following result.

Corollary 3.1 *Let* $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, *and* T *be the maximal time of the solution* (u, ρ) *to* (2.1) *with the initial data* (u_0, ρ_0). *Assume that* $\int_{\mathbb{S}} u_0(x)\,dx = 0$. *If there is some* $x_0 \in \mathbb{S}$ *such that* $\rho_0(x_0) = 0$ *and*

$$u_0'(x_0) = \inf_{x\in\mathbb{S}} u_0'(x) < -\sqrt{\frac{E_0}{12} + 2|\gamma - A|\sqrt{\frac{E_0}{3}}},$$

then the corresponding solution to (2.1) *blows up in finite time.*

Remark 3.1 Note that system (2.1) is variational under the transformation $(u,x) \to (-u,-x)$ and $(\rho,x) \to (\rho,-x)$ even $\gamma = 0$. Thus, we cannot get a blow-up result according to the parity of the initial data (u_0, ρ_0) as we usually do.

Next, we will give more insight into the blow-up mechanism for the wave-breaking solution to system (2.1), that is, the blow-up rate for strong solutions to (2.1).

Theorem 3.3 *Let* (u,ρ) *be the solution to system* (2.1) *with the initial data* $(u_0,\rho_0)\in H^s\times H^{s-1}$, $s\geq 2$, *satisfying the assumption of Theorem* 3.1, *and* T *be the maximal time of the solution* (u,ρ). *Then we have*

$$\lim_{t\to T}(T-t)\inf_{x\in\mathbb{S}} u_x(t,x) = -2.$$

Proof As mentioned earlier, here we only need to show that the above theorem holds for $s=3$.

Define now

$$m(t) := \inf_{x\in\mathbb{S}}\bigl[u_x(t,x)\bigr], \quad t\in[0,T).$$

By the proof of Theorem 3.1, there exists a positive constant $K=K(E_0,\gamma,A)$ such that

$$-K\leq \frac{d}{dt}m+\frac{1}{2}m^2\leq K \quad \text{a.e. on } (0,T). \tag{3.5}$$

Let $\varepsilon\in(0,\frac{1}{2})$. Since $\liminf_{t\to T} m(t)=-\infty$ by Theorem 3.1, there is some $t_0\in(0,T)$ with $m(t_0)<0$ and $m^2(t_0)>\frac{K}{\varepsilon}$. Since m is locally Lipschitz, it is then inferred from (3.5) that

$$m^2(t)>\frac{K}{\varepsilon}, \quad t\in[t_0,T). \tag{3.6}$$

A combination of (3.5) and (3.6) enables us to infer

$$\frac{1}{2}+\varepsilon\geq -\frac{\frac{dm}{dt}}{m^2}\geq \frac{1}{2}-\varepsilon \quad \text{a.e. on } (0,T). \tag{3.7}$$

Since m is locally Lipschitz on $[0,T)$ and (3.6) holds, it is easy to check that $\frac{1}{m}$ is locally Lipschitz on (t_0,T). Differentiating the relation $m(t)\cdot\frac{1}{m(t)}=1$, $t\in(t_0,T)$, we get

$$\frac{d}{dt}\left(\frac{1}{m}\right)=-\frac{\frac{dm}{dt}}{m^2} \quad \text{a.e. on } (t_0,T),$$

with $\frac{1}{m}$ absolutely continuous on (t_0,T). For $t\in(t_0,T)$. Integrating (3.7) on (t,T) to obtain

$$\left(\frac{1}{2}+\varepsilon\right)(T-t)\geq -\frac{1}{m(t)}\geq \left(\frac{1}{2}-\varepsilon\right)(T-t), \quad t\in(t_0,T),$$

that is,

$$\frac{1}{\frac{1}{2}+\varepsilon}\leq -m(t)(T-t)\leq \frac{1}{\frac{1}{2}-\varepsilon}, \quad t\in(t_0,T).$$

By the arbitrariness of $\varepsilon\in(0,\frac{1}{2})$ the statement of Theorem 3.3 follows. □

4 Global existence

In this section, we will present a global existence result.

Theorem 4.1 *Let* $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, *and* T *be the maximal time of the solution* (u, ρ) *to* (2.1) *with the initial data* (u_0, ρ_0). *If* $\rho_0(x) \neq 0$ *for all* $x \in \mathbb{S}$, *then the corresponding solution* (u, ρ) *exists globally in time.*

Proof Define

$$m(t) := \inf_{x\in\mathbb{S}}\bigl[u_x(t,x)\bigr], \quad t \in [0, T).$$

By Lemma 2.7, we let $\xi(t) \in \mathbb{S}$ be a point where this infimum is attained. It follows that

$$m(t) = u_x\bigl(t, \xi(t)\bigr) \quad \text{and} \quad u_{xx}\bigl(t, \xi(t)\bigr) = 0.$$

Since the map $q(t, \cdot)$ given by (2.3) is an increasing diffeomorphism of $\mathbb{R}$, there exists an $x(t) \in \mathbb{S}$ such that $q(t, x(t)) = \xi(t)$.

Set $m(t) = u_x(t, \xi(t)) = u_x(t, q(t, x(t)))$ and $\alpha(t) = \rho(t, \xi(t)) = \rho(t, q(t, x(t)))$. Valuating (3.1) at $(t, \xi(t))$ and using Lemma 2.7, we obtain

$$m'(t) = -\frac{1}{2}m^2(t) + \frac{1}{2}\alpha^2(t) + f \quad \text{and} \quad \alpha'(t) = -m(t)\alpha(t), \tag{4.1}$$

where $f = u^2 + (\gamma - A)u - G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + (\gamma - A)u)$. By Lemma 2.4, Lemma 2.8 and $\frac{1}{2\sinh\frac{1}{2}} \leq G(x) \leq \frac{\cosh\frac{1}{2}}{2\sinh\frac{1}{2}}$, we have

$$\begin{aligned}
|f| &\leq \|u\|_{L^\infty}^2 + 2|\gamma - A|\|u\|_{L^\infty} + \|G\|_{L^\infty}\left\|u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right\|_{L^1} \\
&\leq \frac{e+1}{2(e-1)}E_0 + 2|\gamma - A|\sqrt{\frac{e+1}{2(e-1)}}E_0^{\frac{1}{2}} + \frac{\cosh\frac{1}{2}}{2\sinh\frac{1}{2}}E_0 := c_1.
\end{aligned}$$

By Lemmas 2.2-2.3, we know that $\alpha(t)$ has the same sign with $\alpha(0) = \rho_0(x_0)$ for every $x \in \mathbb{R}$. Moreover, there is a constant $\beta > 0$ such that $|\alpha(0)| = \inf_{x\in\mathbb{S}} |\rho_0(x)| \geq \beta > 0$ because of $\rho_0(x) \neq 0$ for all $x \in \mathbb{S}$. Next, we consider the following Lyapunov positive function:

$$w(t) = \alpha(0)\alpha(t) + \frac{\alpha(0)}{\alpha(t)}\bigl(1 + m^2(t)\bigr), \quad t \in [0, T). \tag{4.2}$$

Letting $t = 0$ in (4.2), we have

$$w(0) \leq \|\rho_0\|_{L^\infty}^2 + 1 + \bigl\|u_0'(x)\bigr\|_{L^\infty}^2 := c_2.$$

Differentiating (4.2) with respect to t and using (4.1), we obtain

$$\begin{aligned}
w'(t) &= \frac{\alpha(0)}{\alpha(t)} \cdot 2m(t)\left(f + \frac{1}{2}\right) \\
&\leq \frac{\alpha(0)}{\alpha(t)}\bigl(1 + m^2(t)\bigr)\left(|f| + \frac{1}{2}\right) \\
&\leq w(t)\left(c_1 + \frac{1}{2}\right).
\end{aligned}$$

By Gronwall's inequality, we have

$$w(t) \le w(0)e^{(c_1+\frac{1}{2})t} \le c_2 e^{(c_1+\frac{1}{2})t}$$

for all $t \in [0, T)$. On the other hand,

$$w(t) \ge 2\sqrt{\alpha^2(0)\bigl(1+m^2(t)\bigr)} \ge 2\beta\bigl|m(t)\bigr|, \quad \forall t \in [0, T).$$

Thus,

$$\bigl|m(t)\bigr| \le \frac{1}{2\beta} w(t) \le \frac{c_2}{2\beta} e^{(c_1+\frac{1}{2})t}$$

for all $t \in [0, T)$. It follows that

$$\liminf_{t\to T} m(t) \ge -\frac{c_2}{2\beta} e^{(c_1+\frac{1}{2})T}.$$

This completes the proof by using Lemma 2.6. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This paper is the result of joint work of all authors who contributed equally to the final version of this paper. All authors read and approved the final manuscript.

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