Global conservative and multipeakon conservative solutions for the two-component Camassa-Holm system

Yujuan Wang and Yongduan Song[*]

[*]Correspondence:
ydsong@cqu.edu.cn
School of Automation, Chongqing
University, Chongqing, 400044,
P.R. China

Abstract

The continuation of solutions for the two-component Camassa-Holm system after wave breaking is studied in this paper. The global conservative solution is derived first, from which a semigroup and a multipeakon conservative solution are established. In developing the solution, a system transformation based on a skillfully defined characteristic and a set of newly introduced variables is used. It is the transformation, together with the associated properties, that allows for the establishment of the results for continuity of the solution beyond collision time.

Keywords: two-component Camassa-Holm system; Lagrangian system; global conservative solutions; conservative multipeakon solutions

1 Introduction

Because of its capabilities of describing the dynamic behavior of water wave, the following Camassa-Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

modeling the unidirectional propagation of shallow water waves in irrotational flow over a flat bottom, with $u(t, x)$ representing the fluid velocity at time t in the horizontal direction, has attracted considerable attention [1–10]. The CH equation is a quadratic order water wave equation in an asymptotic expansion for unidirectional shallow water waves described by the incompressible Euler equations, which was found earlier by Fuchssteiner and Fokas [1] as a bi-Hamiltonian generalization of the KdV equation. It is completely integrable [2, 3] and possesses an infinite number of conservation laws. A remarkable property of the CH equation is the existence of the non-smooth solitary wave solutions called peakons [2, 11]. The peakon $u(t, x) = ce^{-|x-ct|}$, $c \neq 0$, is smooth except at its crest and the tallest among all waves of fixed energy. Another remarkable fact for the CH equation is that it can model wave breaking [2, 12], which means that the solution remains bounded while its slope becomes unbounded in finite time [12, 13], setting it apart from the classical soliton equations such as KdV. After wave breaking, the solutions of the CH equation can be continued uniquely as either global conservative [4–6] or global dissipative solutions [7].

Considered herein is the two-component Camassa-Holm (CH2) shallow water system [14–16]

$$
\begin{cases}
m_t + um_x + 2u_x m + \sigma \rho \rho_x = 0, & t > 0, x \in R, \\
\rho_t + (u\rho)_x = 0, & t > 0, x \in R, \\
u(0, x) = u_0(x), & x \in R, \\
\rho(0, x) = \rho_0(x), & x \in R,
\end{cases}
\tag{1.2}
$$

with $\sigma = 1$ and $m = \sigma_1 u - u_{xx}$, $\sigma_1 = 1$ (or in the 'short wave' limit, $\sigma_1 = 0$), which is an extension of the CH equation by combining its integrability property with compressibility or free-surface elevation dynamics in its shallow water interpretation [11, 17]. This system appeared originally in [14] as could be identified with the first negative flow of AKNS hierarchy, and then it was derived by Constantin and Ivanov [16] in the context of shallow water theory, with $u(x, t)$ representing the horizontal velocity of the fluid and $\rho(x, t)$ in connection with the free-surface elevation from equilibrium with the boundary assumptions $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$. It is formally integrable [14–16] in the sense that it can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter ζ:

$$
\Psi_{xx} = \left(-\sigma \zeta^2 \rho^2 + \zeta m + \frac{\sigma_1}{4} \right) \Psi, \qquad \Psi_t = \left(\frac{1}{2\zeta} - u \right) \Psi_x + \frac{1}{2} u_x \Psi.
$$

It also has a bi-Hamiltonian structure corresponding to the Hamiltonian

$$
H_1 = \frac{1}{2} \int \left(um + (\rho - 1)^2 \right) dx
$$

and the Hamiltonian

$$
H_2 = \frac{1}{2} \int \left(u(\rho - 1)^2 + 2u(\rho - 1) + u^3 + uu_x^2 \right) dx.
$$

The Cauchy problem for the two-component Camassa-Holm system has been studied extensively [18–24]. It was shown that the CH2 system is locally well posed with initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s > \frac{3}{2}$ [18]. The system also has global strong solutions which blow-up in finite time [19, 21, 22] and a global weak solution [23]. However, the problem about continuation of the solutions beyond wave breaking, although interesting and important, has not been explicitly addressed yet. In our recent work [20], we studied the continuation beyond wave breaking by applying an approach that reformulated system (1.2) as a semilinear system of O.D.E. taking values in a Banach space. Such treatment makes it possible to investigate the continuity of the solution beyond collision time, leading to a global conservative solution where the energy is conserved for almost all times.

It should be stressed that both global conservation and multipeakon conservation are two important aspects worthy of investigation. To our best knowledge, however, little effort has been made in studying the multipeakon conservation associated with the CH2 system in the literature. As a compliment and extension to the previous work [20], we develop a novel approach in this work to construct the multipeakon conservative solution for the CH2 system. Different from the work [20], we reformulate the problem by utilizing

a skillfully defined characteristic and a new set of variables, of which the associated energy serves as an additional variable to be introduced such that a well-posed initial-value problem can be obtained, making it convenient to study the dynamic behavior of wave breaking. Because of the introduction of the new variables, we are able to establish the multipeakon conservative solution from the global conservative solution for the CH2 system.

Some related earlier works [4, 5] studied the global existence of solutions to the CH equation. However, the system considered in this work is a heavily coupled one, in which the mutual effect between the two components makes the analysis quite complicated and involved as compared with the system with a single component as studied in [4, 5]. The key and novel effort made in this work to circumvent the difficulty is the utilization of the skillfully defined characteristic and the new set of variables, as well as careful estimates for each iterative approximate component of the solutions, which allows us to establish the global conservative solutions of system (1.2). It is shown that the multipeakon structure is preserved by the semigroup of a global conservative solution and the multipeakon solution is obtained by carefully computing the convolution equations P_i and P_{xi} ($i = 1, \ldots, n$), where, in contrast to the existing works, the inherent mutual effect between the two components is well reflected.

The remainder of this paper is organized as follows. Section 2 presents the transformation from the original system to a Lagrangian semilinear system. The global solutions of the equivalent semilinear system are obtained in Section 3, which are transformed into the global conservative solutions of the original system in Section 4. Finally, we establish the multipeakon conservative solutions for the original system in Section 5.

2 The original system and the equivalent Lagrangian system

We first present the original system. For simplicity, we consider here the associated evolution for positive times (of course, one would get similar results for negative times just by changing the initial condition u_0 into $-u_0$). Let us introduce an operator $\Lambda = (1 - \partial_x^2)^{-1}$, which can be expressed by its associated Green's function $G = \frac{1}{2} e^{-|x|}$ such as $\Lambda f(x) = G * f(x) = \frac{1}{2} \int_R e^{-|x-x'|} f(x') \, dx'$ for all $f \in L^2(R)$. Thus, we can rewrite Eq. (1.2) as a form of a quasi-linear evolution equation:

$$
\begin{cases}
u_t + uu_x + \partial_x G * (u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} v^2 + v) = 0, & t > 0, x \in R, \\
v_t + (uv)_x + u_x = 0, & t > 0, x \in R, \\
u(0, x) = u_0(x), & x \in R, \\
v(0, x) = v_0(x), & x \in R,
\end{cases}
$$

where $v = \rho - 1$. If we define P as

$$
P(t, x) = G * \left(u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} v^2 + v \right) = \frac{1}{2} \int_R e^{-|x-x'|} \left(u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} v^2 + v \right) (t, x') \, dx',
$$

then Eq. (1.2) can be rewritten as

$$
\begin{cases}
u_t + uu_x + P_x = 0, & t > 0, x \in R, \\
v_t + uv_x + vu_x + u_x = 0, & t > 0, x \in R, \\
u(0, x) = u_0(x), & x \in R, \\
v(0, x) = v_0(x), & x \in R.
\end{cases}
\tag{2.1}
$$

Moreover, for regular solutions, we have that the total energy

$$E(t) = \int_R u^2 + u_x^2 + v^2 \, dx \qquad (2.2)$$

is constant in time. Thus, Eq. (2.1) possesses the H^1-norm conservation law defined as

$$\|z\|_{H^1} = \|u\|_{H^1} + \|v\|_{L^2} = \left(\int_R \left[u^2 + u_x^2 + v^2 \right] dx \right)^{1/2},$$

where $z = (u, v)$. Since $z = (u, v) \in H^1 \times [L^2 \cap L^\infty]$, Young's inequality ensures $P \in H^1$.

We reformulate system (2.1) into a Lagrangian equivalent semilinear system as follows.

Let $z(t, x) = (u, v)(t, x)$ denote the solution of system (2.1). For given initial data $y(0, \xi)$, we define the corresponding characteristic $y(t, \xi)$ as the solution of

$$y_t(t, \xi) = u(t, y(t, \xi)), \qquad (2.3)$$

and define the Lagrangian cumulative energy distribution H as

$$H(t, \xi) = \int_{-\infty}^{y(t,\xi)} \left(u^2 + u_x^2 + v^2 \right)(t, x) \, dx. \qquad (2.4)$$

It is not hard to check that

$$\left(u^2 + u_x^2 + v^2 \right)_t + \left(u \left(u^2 + u_x^2 + v^2 \right) \right)_x = \left(u^3 - 2uP \right)_x. \qquad (2.5)$$

Then it follows from (2.3) and (2.5) that

$$\frac{dH}{dt} = \left[\left(u^3 - 2uP \right) \left(t, y(t, \xi) \right) \right]_{-\infty}^{\xi}. \qquad (2.6)$$

Throughout the following, we use the notation

$$U(t, \xi) = u(t, y(t, \xi)), \qquad V(t, \xi) = v(t, y(t, \xi)), \qquad N(t, \xi) = u_x(t, y(t, \xi)).$$

After the change of variables $x = y(t, \xi)$ and $x' = y(t, \xi')$, we obtain the following expressions for P_x and P, namely

$$P(\xi) = P(y(\xi)) = \frac{1}{2} \int_R e^{-|y(\xi) - y(\xi')|} \left[\left(U^2 + \frac{1}{2} U_x^2 + \frac{1}{2} V^2 + V \right) y_\xi \right](\xi') \, d\xi',$$

$$P_x(\xi) = P_x(y(\xi)) = -\frac{1}{2} \int_R \text{sgn}(\xi - \xi') e^{-|y(\xi) - y(\xi')|}, \qquad (2.7)$$

$$\left[\left(U^2 + \frac{1}{2} U_x^2 + \frac{1}{2} V^2 + V \right) y_\xi \right](\xi') \, d\xi',$$

where we have dropped the variable t for simplicity and taken that y is an increasing function for any fixed time t for granted (the validity will be proved later). Using

$H_\xi = (u^2 + u_x^2 + v^2) \circ yy_\xi$, we can rewrite P_x and P in (2.7) as

$$P_x(\xi) = -\frac{1}{4} \int_R \mathrm{sgn}(\xi - \xi') e^{-|y(\xi)-y(\xi')|} \big[H_\xi + (U^2 + 2V)y_\xi \big](\xi') \, d\xi',$$

$$P(\xi) = \frac{1}{4} \int_R e^{-|y(\xi)-y(\xi')|} \big[H_\xi + (U^2 + 2V)y_\xi \big](\xi') \, d\xi'. \tag{2.8}$$

From the definition of the characteristic, it follows that

$$U_t(t,\xi) = u_t(t,y) + u_x(t,y)y_t(t,\xi) = -P_x \circ y(t,\xi),$$

$$V_t(t,\xi) = v_t(t,y) + v_x(t,y)y_t(t,\xi) = -\big[(v+1)u_x\big] \circ y(t,\xi),$$

$$N_t(t,\xi) = u_{xt}(t,y) + u_{xx}(t,y)y_t(t,\xi) = \left(u^2 - \frac{1}{2}u_x^2 + \frac{1}{2}v^2 + v - P \right) \circ y(t,\xi). \tag{2.9}$$

Let us introduce another variable $\varsigma(t,\xi)$ such that $\varsigma(t,\xi) = y(t,\xi) - \xi$ (it will turn out that $\varsigma \in L^\infty(R)$). With these new variables, we now derive an equivalent system of equations (2.1),

$$\begin{cases} \varsigma_t = U, \\ U_t = -P_x, \\ V_t = -(V+1)N, \\ N_t = U^2 - \frac{1}{2}N^2 + \frac{1}{2}V^2 + V - P, \\ H_t = U^3 - 2UP, \end{cases} \tag{2.10}$$

where P and P_x are given by (2.8). Differentiating (2.10) w.r.t. ξ yields

$$\begin{cases} \varsigma_{\xi t} = U_\xi, \\ U_{\xi t} = \frac{1}{2}H_\xi + (\frac{1}{2}U^2 + V - P)y_\xi, \\ V_{\xi t} = -V_\xi N - (V+1)N_\xi, \\ N_{\xi t} = 2UU_\xi + (V+1)V_\xi - NN_\xi - P_x y_\xi, \\ H_{\xi t} = (3U^2 - 2P)U_\xi - 2UP_x y_\xi, \end{cases} \tag{2.11}$$

which is semilinear w.r.t. the variables y_ξ, U_ξ, V_ξ, N_ξ and H_ξ.

System (2.10) can be regarded as an O.D.E. in the Banach space E given by

$$E = W \times H^1 \times \big[L^2 \cap L^\infty \big] \times \big[L^2 \cap L^\infty \big] \times W,$$

endowed with the norm

$$\|X\|_E = \|\varsigma\|_W + \|U\|_{H^1} + \|V\|_{L^2} + \|V\|_{L^\infty} + \|N\|_{L^2} + \|N\|_{L^\infty} + \|H\|_W$$

for any $X = (\varsigma, U, V, N, H) \in E$. Here W is a Banach space defined as

$$W = \big\{ f \in C(R) \cap L^\infty(R) | f_\xi \in L^2(R) \big\},$$

with the norm $\|f\|_W = \|f\|_{L^\infty(R)} + \|f_\xi\|_{L^2(R)}$. Note that $H^1(R) \subset W$.

3 Global solutions of the equivalent system

In this section, we prove that the equivalent system admits a unique global solution. We first obtain the Lipschitz bounds we need on P and P_x.

Lemma 3.1 (See [5]) *Let $\Re_1 : E \to W$ and $\Re_2 : E \to H^1$, or $\Re_2 : E \to W$ be two locally Lipschitz maps. Then the product $X \to \Re_1(X)\Re_2(X)$ is also a locally Lipschitz map from E to H^1, or from E to W.*

Lemma 3.2 *For any given $X = (\varsigma, U, V, N, H) \in E$, P and P_x defined by (2.8) are locally Lipschitz continuous from E to $H^1(R)$. Moreover, we have*

$$P_{x\xi} = -\frac{1}{2}H_\xi + \left(P - \frac{1}{2}U^2 - V \right)(1 + \varsigma_\xi), \qquad P_\xi = P_x(1 + \varsigma_\xi). \tag{3.1}$$

Proof We write

$$P_x(\xi) = P_x^1(X)(\xi) + P_x^2(X)(\xi)$$

$$= -\frac{e^{-\varsigma(\xi)}}{4} \int_R \chi_{\{\xi'<\xi\}} e^{-|\xi-\xi'|} e^{\varsigma(\xi')} \big[H_\xi + \big(U^2 + 2V \big)(1 + \varsigma_\xi) \big](\xi')\, d\xi'$$

$$+ \frac{e^{\varsigma(\xi)}}{4} \int_R \chi_{\{\xi'>\xi\}} e^{-|\xi-\xi'|} e^{-\varsigma(\xi')} \big[H_\xi + \big(U^2 + 2V \big)(1 + \varsigma_\xi) \big](\xi')\, d\xi', \tag{3.2}$$

where χ_Ω denotes the indicator function of a given set Ω, and P_x^1, P_x^2 are the operators which correspond to the two terms of the last identity in (3.2). We rewrite P_x^1 as

$$P_x^1(X)(\xi) = -\frac{e^{-\varsigma(\xi)}}{2} \Lambda \circ R(X)(\xi), \tag{3.3}$$

where R is the operator from E to $L^2(R)$ given by

$$R(X)(\xi) = \chi_{\{\xi'<\xi\}} e^\varsigma \big[H_\xi + \big(U^2 + 2V \big)(1 + \varsigma_\xi) \big].$$

Since the operator Λ (defined as in Section 2) is linear and continuous from $H^{-1}(R)$ to $H^1(R)$, and $L^2(R)$ is continuously embedded in $H^{-1}(R)$, we have $\Lambda \circ R(X) \in H^1$. It is not hard to know that R is locally Lipschitz from E into $L^2(R)$ and therefore from E into $H^{-1}(R)$. Thus, $\Lambda \circ R$ is locally Lipschitz from E to $H^1(R)$. Since the mapping $X \to e^{-\varsigma}$ is locally Lipschitz from E to W, it then follows from Lemma 3.1 that P_x^1 is locally Lipschitz from E to $H^1(R)$. Similarly, P_x^2 is also locally Lipschitz and therefore P_x is locally Lipschitz from E to $H^1(R)$. We can obtain that P defined by (2.8) is locally Lipschitz continuous from E to $H^1(R)$ in the same way. By using the chain rule, the formulas in (3.1) are obtained by direct computation, see [25, p.129]. □

Theorem 3.1 *Let any $\bar{X} = (\bar\varsigma, \bar{U}, \bar{V}, \bar{N}, \bar{H}) \in E$ be given. System (2.10) admits a unique local solution defined on some time interval $[0, T]$, where T depends only on $\|\bar{X}\|_E$.*

Proof To establish the local existence of solutions, one proceeds as in Lemma 3.2, then obtains that $F(X)$, which is defined by

$$F(X) = \left(U, -P_x, -(V+1)N, U^2 - \frac{1}{2}N^2 + \frac{1}{2}V^2 + V - P, U^3 - 2UP \right)$$

with $X = (\varsigma, U, V, N, H)$, is Lipschitz continuous on any bounded set of E. We rewrite the solutions of system (2.10) as

$$X(t) = \bar{X} + \int_0^t F\big(X(\tau)\big)\, d\tau. \tag{3.4}$$

Then the theorem follows from the standard contraction argument on Banach spaces. $\square$

Theorem 3.1 gives us the existence of local solutions to (2.10) for initial data in E. It remains to prove that the local solutions can be extended to global solutions. Note that the global solutions of (2.10) may not exist for all initial data in E. However, they exist when the initial data $\bar{X} = (\bar{\varsigma}, \bar{U}, \bar{V}, \bar{N}, \bar{H})$ belongs to the set Γ, which is defined as follows.

Definition 3.1 The set Γ is composed of all $(\varsigma, U, V, N, H) \in E$ such that

(i) $(\varsigma, U, V, N, H) \in \big[W^{1,\infty}(R)\big]^5,$ $\tag{3.5a}$

(ii) $y_\xi \geq 0, H_\xi \geq 0, y_\xi + H_\xi > 0$ almost everywhere, and $\lim\limits_{\xi \to -\infty} H(\xi) = 0,$ $\tag{3.5b}$

(iii) $y_\xi H_\xi = y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2$ almost everywhere, $\tag{3.5c}$

with $\varsigma(\xi) = y(\xi) - \xi$, where $W^{1,\infty}(R) = \{f \in C(R) \cap L^\infty(R) | f_\xi \in L^\infty(R)\}$.

The global existence of the solution for initial data in Γ relies essentially on the fact that the set Γ is preserved by the flow as the next lemma shows.

Lemma 3.3 *Given initial data $\bar{X} = (\bar{\varsigma}, \bar{U}, \bar{V}, \bar{N}, \bar{H}) \in \Gamma$, for some $T > 0$, we consider the local solution $X(t) = (\varsigma, U, V, N, H)(t) \in C([0, T], E)$ of system (2.10) given by Theorem* 3.1. *We have*
  (i) *$X(t) \in \Gamma$ for all $t \in [0, T]$,*
  (ii) *$y_\xi(t, \xi) > 0$ for a.e. $t \in [0, T]$ and a.e. $\xi \in R$,*
  (iii) *$\lim_{\xi \to \pm\infty} H(t, \xi) = H(0, \pm\infty) = 0$ for all $t \in [0, T]$.*

Proof (i) For given initial data $\bar{X} = (\bar{\varsigma}, \bar{U}, \bar{V}, \bar{N}, \bar{H}) \in E \cap [W^{1,\infty}(R)]^5$, to ensure that the solution $X = (\varsigma, U, V, N, H)$ of (2.10) also belongs to $E \cap [W^{1,\infty}(R)]^5$, we have to specify the initial conditions for (2.11). Let Ω be the following set:

$$\Omega = \big\{ \xi \in R \big| \big|\bar{\varsigma}_\xi(\xi)\big| \leq \|\bar{\varsigma}_\xi\|_{L^\infty}, \big|\bar{U}_\xi(\xi)\big| \leq \|\bar{U}_\xi\|_{L^\infty}, \big|\bar{V}_\xi(\xi)\big| \leq \|\bar{V}_\xi\|_{L^\infty},$$
$$\big|\bar{N}_\xi(\xi)\big| \leq \|\bar{N}_\xi\|_{L^\infty}, \big|\bar{H}_\xi(\xi)\big| \leq \|\bar{H}_\xi\|_{L^\infty} \big\}.$$

Note that $\text{meas}(\Omega^c) = 0$. For $\xi \in \Omega$, we take $(\varsigma_\xi, U_\xi, V_\xi, N_\xi, H_\xi)(0, \xi) = (\bar{\varsigma}_\xi, \bar{U}_\xi, \bar{V}_\xi, \bar{N}_\xi, \bar{H}_\xi)(\xi)$. For $\xi \in \Omega^c$, we define $(\varsigma_\xi, U_\xi, V_\xi, N_\xi, H_\xi)(0, \xi) = (0, 0, 0, 0, 0)$. We consider U, P and P_x as given functions in $C([0, T], H^1(R))$, which is guaranteed by Lemma 3.2 and V, N in $C([0, T], [L^2(R) \cap L^\infty(R)])$. Thus system (2.11) is affine (it consists of a sum of a linear transformation and a constant) and, therefore, by using a contraction argument, it admits a unique local solution defined on some time interval $[0, T]$. Thus, for the given initial condition $\bar{X} \in E \cap [W^{1,\infty}(R)]^5$, the solution of (2.10) given by Theorem 3.1 also belongs to $E \cap [W^{1,\infty}(R)]^5$, which implies that $X(t)$ satisfies (3.5a) for all $t \in [0, T]$. We claim that

(3.5c) holds for any $\xi \in \Omega$ and therefore almost everywhere. Consider a fixed $\xi \in \Omega$ and drop it in the notation. On the one hand, it follows from (2.11) that

$$
\begin{aligned}
(y_\xi H_\xi)_t &= y_{\xi t} H_\xi + y_\xi H_{\xi t} \\
&= U_\xi H_\xi + y_\xi \big[\big(3U^2 - 2P - 2AU\big) U_\xi - 2UP_x y_\xi \big] \\
&= U_\xi H_\xi + 3U^2 U_\xi y_\xi - 2P U_\xi y_\xi - 2AU U_\xi y_\xi - 2UP_x y_\xi^2,
\end{aligned}
$$

and on the other hand,

$$
\begin{aligned}
\big(y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2\big)_t &= 2y_\xi y_{\xi t} U^2 + 2y_\xi^2 UU_t + 2y_\xi y_{\xi t} V^2 + 2y_\xi^2 VV_t + 2U_\xi U_{\xi t} \\
&= 3U^2 U_\xi y_\xi - 2UP_x y_\xi^2 + U_\xi H_\xi - 2AU U_\xi y_\xi - 2P U_\xi y_\xi.
\end{aligned}
$$

Thus, $(y_\xi H_\xi)_t = (y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2)_t$. Notice that $y_\xi H_\xi(0) = (y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2)(0)$, which implies that $y_\xi H_\xi(t) = (y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2)(t)$ for all $t \in [0, T]$. Thus, (3.5c) holds. It remains to prove that the inequalities in (3.5b) hold. Set $t^* = \sup\{t \in [0, T] | y_\xi(t') \geq 0 \text{ for all } t' \in [0, t]\}$. Assume that $t^* < T$. Since $y_\xi(t)$ is continuous w.r.t. t, we have $y_\xi(t^*) = 0$. It follows from (3.5c) that $U_\xi(t^*) = 0$. Furthermore, (2.11) implies $y_{\xi t}(t^*) = U_\xi(t^*) = 0$ and $y_{\xi tt}(t^*) = U_{\xi t}(t^*) = \frac{1}{2} H_\xi(t^*)$. If $H_\xi(t^*) = 0$, then $(y_\xi, U_\xi, H_\xi)(t^*) = (0, 0, 0)$, which implies $(y_\xi, U_\xi, H_\xi)(t) = 0$ for all $t \in [0, T]$ by the uniqueness of the solution of system (2.11). This contradicts the fact that $y_\xi(0) + H_\xi(0) > 0$ for all $\xi \in \Omega$. If $H_\xi(t^*) < 0$, then $y_{\xi tt}(t^*) < 0$. Since $y_\xi(t^*) = y_{\xi t}(t^*) = 0$, there exists a neighborhood ϖ of t^* such that $y_\xi(t) < 0$ for all $t \in \varpi / \{t^*\}$. This contradicts the definition of t^*. Hence, $H_\xi(t^*) > 0$. We now have $y_{\xi tt}(t^*) > 0$, which conversely implies $y_\xi(t) > 0$ for all $t \in \varpi / \{t^*\}$, which contradicts the fact that $t^* < T$. Thus, we have proved $y_\xi(t) \geq 0$ for all $t \in [0, T]$. We now prove that $H_\xi \geq 0$ for all $t \in [0, T]$. This follows from (3.5c) when $y_\xi(t) > 0$. If $y_\xi(t) = 0$, then $U_\xi(t) = 0$ from (3.5c). As we have seen, $H_\xi < 0$ would imply that $y_\xi(t') < 0$ for some t' in a punctured neighborhood of t, which is impossible. Hence, $H_\xi \geq 0$ for all $t \in [0, T]$. Now we have $y_\xi(t) + H_\xi(t) \geq 0$ for all $t \in [0, T]$. If $y_\xi(t') + H_\xi(t') = 0$ for some t', it then follows that $(y_\xi, U_\xi, H_\xi)(t') = 0$, which implies $(y_\xi, U_\xi, H_\xi)(t) = 0$ for all $t \in [0, T]$, which contradicts the fact that $y_\xi(0) + H_\xi(0) > 0$ for all $\xi \in \Omega$. Hence, $y_\xi(t) + H_\xi(t) > 0$.

(ii) Define the set $N = \{(t, \xi) \in [0, T] \times R | y_\xi(t, \xi) = 0\}$. It follows from Fubini's theorem that

$$
\text{meas}(N) = \int_R \text{meas}(N_\xi) \, d\xi = \int_{[0, T]} \text{meas}(N_t) \, dt, \tag{3.6}
$$

where $N_\xi = \{t \in [0, T] | y_\xi(t, \xi) = 0\}$ and $N_t = \{\xi \in R | y_\xi(t, \xi) = 0\}$. From the above proof, we know that for all $\xi \in \Omega$, N_ξ consists of isolated points that are countable. This means that $\text{meas}(N_\xi) = 0$. It follows from (3.6), and since $\text{meas}(\Omega^c) = 0$, that

$$
\text{meas}(N_t) = 0 \quad \text{for almost every } t \in [0, T].
$$

This implies that $y_\xi(t, \xi) > 0$ for almost all t and therefore $y(t, \xi)$ is strictly increasing and invertible w.r.t. ξ.

(iii) For any given $t \in [0, T]$, since $H_\xi(t) \geq 0$ and $H(t, \xi) \in L^\infty(R)$, we know that $H(t, \pm\infty)$ exist. We have

$$H(t, \xi) = H(0, \xi) + \int_0^t \left(U^3 - 2PU\right)(\tau, \xi)\, d\tau. \tag{3.7}$$

Let $\xi \to \pm\infty$. Since U, P are bounded in $L^\infty([0, T] \times R)$ and $\lim_{\xi \to \pm\infty} U(t, \xi) = 0$ as $U(t, \cdot) \in H^1(R)$, it then follows from (3.7) that $H(t, \pm\infty) = H(0, \pm\infty)$ for all $t \in [0, T]$. Since $\bar{X} \in \Gamma$, it follows that $H(0, \pm\infty) = 0$ for all $t \in [0, T]$. $\qquad \square$

Theorem 3.2 *For any initial data $\bar{X} = (\bar{y}, \bar{U}, \bar{V}, \bar{N}, \bar{H}) \in \Gamma$, there exists a unique global solution $X(t) = (y, U, V, N, H)(t) \in C^1(R^+, E)$ for system (2.10). Moreover, for all $t \geq 0$, if we equip Γ with the topology endowed with the E-norm, then the map $S_t : \Gamma \to \Gamma \times R^+$ defined as*

$$S_t(\bar{X}) = X(t)$$

is a continuous semigroup.

Proof Let $(\varsigma, U, V, N, H) \in C([0, T], E)$ be a local solution of (2.10) with initial data $(\bar{\varsigma}, \bar{U}, \bar{V}, \bar{N}, \bar{H})$. To obtain the global existence of solutions, it suffices to show that

$$\sup_{t \in [0,T)} \left\| \varsigma(t, \cdot), U(t, \cdot), V(t, \cdot), N(t, \cdot), H(t, \cdot) \right\|_E < \infty. \tag{3.8}$$

Since $H(t, \xi)$ is an increasing function w.r.t. ξ for all t and $\lim_{\xi \to \infty} H(t, \xi) = \lim_{\xi \to \infty} H(0, \xi)$, we have $\sup_{t \in [0,T)} \|H(t, \cdot)\|_{L^\infty(R)} = \|\bar{H}\|_{L^\infty(R)} < \infty$. We consider a fixed $t \in [0, T)$ and drop it for simplification. Since $U_\xi(\xi) = 0$ when $y_\xi(\xi) = 0$, and $y_\xi(\xi) > 0$ for a.e. ξ, it follows from (3.5c) that

$$U^2(\xi) = 2 \int_{-\infty}^{\xi} U(\xi') U_\xi(\xi')\, d\xi' = 2 \int_{\{\xi' < \xi \,|\, y_\xi(\xi') > 0\}} U(\xi') U_\xi(\xi')\, d\xi'$$

$$\leq \int_{\{\xi' < \xi \,|\, y_\xi(\xi') > 0\}} \left(y_\xi U^2 + U_\xi^2/y_\xi\right)(\xi')\, d\xi' \leq \int_R H_\xi(\xi')\, d\xi' = H(\xi),$$

which implies

$$\sup_{t \in [0,T)} \left\| U^2(t, \cdot) \right\|_{L^\infty} \leq \sup_{t \in [0,T)} \left\| H(t, \cdot) \right\|_{L^\infty(R)} = \|\bar{H}\|_{L^\infty(R)} < \infty,$$

and therefore

$$\sup_{t \in [0,T)} \left\| U(t, \cdot) \right\|_{L^\infty} < \infty.$$

We can obtain from the governing equation (2.10) that

$$\left| \varsigma(t, \xi) \right| \leq \left| \varsigma(0, \xi) \right| + \sup_{t \in [0,T)} \left\| U(t, \cdot) \right\|_{L^\infty} T.$$

Thus, $\sup_{t\in[0,T)}\|\varsigma(t,\cdot)\|_{L^\infty} < \infty$. The governing equation (2.10) also implies that $\sup_{t\in[0,T)}\|V(t,\cdot)\|_{L^\infty} < \infty$ and $\sup_{t\in[0,T)}\|N(t,\cdot)\|_{L^\infty} < \infty$.

From the identity $H_\xi = (U^2 + U_x^2 + V^2)y_\xi$, we can deduce that

$$\left|\left(U^2 + 2V\right)y_\xi\right| \le \left(U^2 + V^2 + 1\right)y_\xi \le H_\xi + y_\xi,$$

which implies that

$$\begin{aligned}
|P_x| &\le \frac{1}{2}\left|\int_R e^{-|y(\xi)-y(\xi')|}\left[H_\xi + \left(U^2 + 2V\right)y_\xi\right](\xi')\,d\xi'\right| \\
&\le \frac{1}{2}\left|\int_R e^{-|y(\xi)-y(\xi')|}[2H_\xi + y_\xi](\xi')\,d\xi'\right| \\
&\le C\left(\sup_{t\in[0,T)}\left\|H(t,\cdot)\right\|_{L^\infty(R)} + \sup_{t\in[0,T)}\left\|\varsigma(t,\cdot)\right\|_{L^\infty}\right) < \infty.
\end{aligned}$$

Therefore, $\|P_x\|_{L^\infty} < \infty$. Similarly, we obtain $\|P_x\|_{L^2} < \infty$ and the bounds hold for P. Let

$$Z(t) = \left\|U(t,\cdot)\right\|_{L^2} + \left\|U_\xi(t,\cdot)\right\|_{L^2} + \left\|V(t,\cdot)\right\|_{L^2} + \left\|N(t,\cdot)\right\|_{L^2} + \left\|\varsigma_\xi(t,\cdot)\right\|_{L^2} + \left\|H_\xi(t,\cdot)\right\|_{L^2}.$$

After taking the L^2-norms on both sides of (2.10) and (2.11), we obtain

$$Z(t) \le Z(0) + C\int_0^t Z(\tau)\,d\tau.$$

It follows from Gronwall's lemma that $\sup_{t\in[0,T)}Z(t) < \infty$, which implies that S_t is a continuous semigroup by the standard O.D.E. theory. □

4 Global solutions for the original system

We transform the global solution of the equivalent system (2.10) into the global conservative solution of the original system (2.1) in this section. It suffices to establish the correspondence between the Lagrangian equivalent system and the original system.

We first introduce a set G as the set of relabeling functions defined by

$$G = \left\{f \text{ is invertible}\,|\,f - \text{Id} \text{ and } f^{-1} - \text{Id} \text{ both belong to } W^{1,\infty}(R)\right\},$$

where Id denotes the identity function. For any $\alpha > 1$, we define the subsets G_α of G as

$$G_\alpha = \left\{f \in G\,|\,\|f - \text{Id}\|_{W^{1,\infty}(R)} + \left\|f^{-1} - \text{Id}\right\|_{W^{1,\infty}(R)} \le \alpha\right\},$$

with a useful property: If $f \in G_\alpha$ ($\alpha \ge 0$), then $1/(1+\alpha) \le f_\xi \le 1 + \alpha$ almost everywhere. Conversely, if f is absolutely continuous, $f - \text{Id} \in L^\infty(R)$ and there exists $c \ge 1$ such that $1/c \le f_\xi \le c$ almost everywhere, then $f \in G_\alpha$ for some α depending only on c and $\|f - \text{Id}\|_{L^\infty(R)}$. We now define the subsets F and F_α of Γ such that

$$F = \left\{X = (y, U, V, H) \in \Gamma\,|\,y + H \in G\right\},$$

$$F_\alpha = \left\{X = (y, U, V, H) \in \Gamma\,|\,y + H \in G_\alpha\right\}.$$

With the above useful property of G_α, it is not hard to prove that the space F is preserved by the governing equation (2.10).

Notice that the map $\Phi : G \times F \to F$ given by $\Phi(f, X) = X \circ f$ defines a group action of G on F, we then consider the quotient space F/G of F w.r.t. the group action. The equivalence relation on F is defined as: for any $X, X' \in F$, if there exists $f \in G$ such that $X' = X \circ f$, we claim that X and X' are equivalent. We denote the projection $\Pi : F \to F/G$ by $\Pi(X) = [X]$. For any $X = (y, U, V, N, H) \in F$, we introduce the map $K : F \to F_0$ given by $K(X) = X \circ (y + H)^{-1}$. It is not hard to prove that $K(X) = X$ when $X \in F_0$, and $K(X \circ f) = K(X)$ for any $X \in F$ and $f \in G$. Hence, we can define the map $\tilde{K} : F/G \to F_0$ as $\tilde{K}([X]) = K(X)$ for any representative $[X] \in F/G$ of $X \in F$. For any $X \in F_0$, we have $\tilde{K} \circ \Pi(X) = K(X) = X$. Hence, $\tilde{K} \circ \Pi_{|F_0} = \mathrm{Id}_{|F_0}$. Note that any topology defined on F_0 is naturally transported into F/G by this isomorphism, that is, if we equip F_0 with the metric induced by the E-norm, *i.e.*, $d_{F_0}(X, X') = \|X - X'\|_E$ for all $X, X' \in F_0$, which is complete, then for any $[X], [X'] \in F/G$, the topology on F/G is defined by a complete metric given by $d_{F/G}([X], [X']) = \|K(X) - K(X')\|_E$.

For any initial data $\bar{X} \in F$, we denote the continuous semigroup with the solution $X(t)$ of system (2.10) by $S : F \times R^+ \to F$. As we indicated earlier, Eq. (2.1) is invariant w.r.t. relabeling. That is, $t > 0$, $S_t(X \circ f) = S_t(X) \circ f$ for any $X \in F$ and $f \in G$. Thus, the map $\tilde{S}_t : F/G \to F/G$ defined by $\tilde{S}_t([X]) = [S_t X]$ is valid, which generates a continuous semigroup.

To derive the correspondence between the Lagrangian equivalent system and the original system, we have to consider the space D, which characterizes the solutions in the original system:

$$D = \left\{ (z, \mu) \,|\, z \in H^1(R) \times \left[L^2(R) \times L^\infty(R) \right] \text{ and } \mu_{\mathrm{ac}} = \left(u^2 + u_x^2 + v^2 \right) dx \right\},$$

where $z = (u, v)$ and μ is a positive finite Radon measure with μ_{ac} as its absolute continuous part.

We now establish a bijection between F/G and D to transport the continuous semigroup obtained in the Lagrangian equivalent system (functions in F/G) into the original system (functions in D).

We first introduce the mapping M, which corresponds to the transformation from the Lagrangian equivalent system into the original system. In the other direction, we obtain the energy density μ in the original system, by pushing forward by y the energy density $H_\xi \, d\xi$ in the Lagrangian equivalent system, where the push-forward $f_\# \nu$ of a measure ν by a measurable function f is defined by

$$f_\# \nu(B) = \nu\left(f^{-1}(B) \right)$$

for all Borel set B. Let (z, μ) be defined as

$$z(x) = Z(\xi) \quad \text{for any } \xi \text{ such that } x = y(\xi), \tag{4.1a}$$

$$\mu = y_\#(H_\xi \, d\xi), \tag{4.1b}$$

where $z(x) = (u, v)(x)$, $Z(\xi) = (U, V)(\xi)$. We have that $(z, \mu) \in D$, which does not depend on the representative $X = (y, U, V, N, H) \in F$ of $[X]$ we choose. We denote by $M : F/G \to D$

the mapping to any $[X] \in F/G$ and $(z, \mu) \in D$ given by (4.1a) and (4.1b), which transforms the Lagrangian equivalent system into the original system.

We are led to the mapping $L : D \to F/G$, which conversely transforms the original system into the Lagrangian equivalent system defined as follows.

Definition 4.1 For any $(z, \mu) \in D$, let

$$y(\xi) = \sup\{y | \mu(-\infty, y) + y < \xi\}, \tag{4.2a}$$

$$U(\xi) = u \circ y(\xi), \qquad V(\xi) = v \circ y(\xi), \qquad N(\xi) = u_x \circ y(\xi), \tag{4.2b}$$

$$H(\xi) = \xi - y(\xi), \tag{4.2c}$$

where $z = (u, v)$. We define $L(z, \mu) \in F/G$ as the equivalence class of (y, U, V, N, H).

Remark 4.1 Note that $X = (y, U, V, N, H) \in E$, which satisfies (3.5a)-(3.5c) from the definition of y, U, V, N, H in (4.2a)-(4.2c). Moreover, by the definition (4.2c), we have that $y + H = \mathrm{Id}$. Thus, $X = (y, U, V, N, H) \in F_0$.

We claim that the transformation from the original system into the Lagrangian equivalent system is a bijection.

Theorem 4.1 *The maps M and L are invertible, that is,*

$$L \circ M = \mathrm{Id}_{F/G}, \qquad M \circ L = \mathrm{Id}_D.$$

Proof Let $[X] \in F/G$ be given. We consider $X = (y, U, V, N, H) = \tilde{K}([X])$ for a representative of $[X]$ and (z, μ) given by (4.1a) and (4.1b) for this particular X. From the definition of $\tilde{K}$, we have $X \in F_0$. Let $\bar{X} = (\bar{y}, \bar{U}, \bar{V}, \bar{N}, \bar{H})$ be the representative of $L(z, \mu)$ in F_0 given by (4.2a)-(4.2c). To derive $L \circ M = \mathrm{Id}_{F/G}$, it suffices to show that $(\bar{y}, \bar{U}, \bar{V}, \bar{N}, \bar{H}) = (y, U, V, N, H)$. Let

$$g(x) = \sup\{\xi \in R | y(\xi) < x\}. \tag{4.3}$$

Using the fact that y is increasing and continuous, it follows that

$$y(g(x)) = x \tag{4.4}$$

and $y^{-1}((-\infty, x)) = (-\infty, g(x))$. From (4.1b) and since $H(-\infty) = 0$, for any $x \in R$, we get

$$\mu((-\infty, x)) = \int_{y^{-1}((-\infty, x))} H_\xi \, d\xi = \int_{-\infty}^{g(x)} H_\xi \, d\xi = H(g(x)).$$

Since $X \in F_0$ and $y + H = \mathrm{Id}$, we have

$$\mu((-\infty, x)) + x = g(x). \tag{4.5}$$

From the definition of $\bar{y}$, it follows that

$$\bar{y}(\xi) = \sup\{x \in R | g(x) < \xi\}. \tag{4.6}$$

For any given $\xi \in R$, using the fact that y is increasing and (4.4), it follows that $\bar{y}(\xi) \leq y(\xi)$. If $\bar{y}(\xi) < y(\xi)$, there then exists x such that $\bar{y}(\xi) < x < y(\xi)$ and (4.6) implies that $g(x) \geq \xi$. Conversely, since y is increasing, we have $x = y(g(x)) < y(\xi)$, which implies that $g(x) < \xi$. This is a contradiction. Hence, we have that $\bar{y} = y$. Since $y + H = \mathrm{Id}$, it follows directly from the definitions that $\bar{H} = H$, $\bar{U} = U$, $\bar{V} = V$ and $\bar{N} = N$. Hence, $L \circ M = \mathrm{Id}_{F/G}$.

Let $(z, \mu) \in D$ be given and (y, U, V, N, H) be the representative of $L(z, \mu)$ in F_0 given by (4.2a)-(4.2c). Then, let $(\bar{z}, \bar{\mu}) = M \circ L(z, \mu)$. Let g be defined as before by (4.3). The same computation that leads to (4.5) now gives

$$\bar{\mu}\big((-\infty, x)\big) + x = g(x). \tag{4.7}$$

Given $\xi \in R$, we consider an increasing sequence x_i converging to $y(\xi)$, which is guaranteed by (4.2a), and such that $\mu((-\infty, x_i)) + x_i < \xi$. Let i tend to infinity. Since $F(x) = \mu((-\infty, x))$ is lower semi-continuous, we have $\mu((-\infty, y(\xi))) + y(\xi) \leq \xi$. Take $\xi = g(x)$ and then we get

$$\mu\big((-\infty, x)\big) + x \leq g(x). \tag{4.8}$$

By the definition of g, there exists an increasing sequence ξ_i converging to $g(x)$ such that $y(\xi_i) < x$. It follows from the definition of y in (4.2a) that $\mu((-\infty, x)) + x \geq \xi_i$. Passing to the limit, we obtain $\mu((-\infty, x)) + x \geq g(x)$ which, together with (4.8), yields

$$\mu\big((-\infty, x)\big) + x = g(x). \tag{4.9}$$

We obtain that $\bar{\mu} = \mu$ by comparing (4.9) and (4.7). It is clear from the definitions that $\bar{z} = z$. Hence, $(\bar{z}, \bar{\mu}) = (z, \mu)$ and $M \circ L = \mathrm{Id}_D$. $\qquad \square$

Our next task is to transport the topology defined in F/G into D, which is guaranteed by the fact that we have established a bijection between the two equivalent systems and then obtained a continuous semigroup of solutions for the original system.

Let us define the distance d_D on D as

$$d_D\big((z, \mu), (\bar{z}, \bar{\mu})\big) = d_{F/G}\big(L(z, \mu), L(\bar{z}, \bar{\mu})\big),$$

which makes the bijection L between D and F/G into an isometry. Since F/G equipped with $d_{F/G}$ is a complete metric space, it is not hard to know that D equipped with the metric d_D is also a complete metric space. For each $t \in R$, we define the mapping $T_t : D \to D$ as

$$T_t = M \tilde{S}_t L.$$

Theorem 4.2 *Given $(\bar{z}, \bar{\mu}) \in D$, if we denote $t \to (z, \mu)(t) = T_t(\bar{z}, \bar{\mu})$ the corresponding trajectory, then $z = (u, v)$ is a weak solution of the two-component Camassa-Holm equations* (2.1), *which constructs a continuous semigroup. Moreover, μ is a weak solution of the following transport equation*:

$$\mu_t + (u\mu)_x = \big(u^3 - 2Pu\big)_x. \tag{4.10}$$

Furthermore, we have

$$\mu(t)(R) = \mu(0)(R) \quad \text{for all } t \tag{4.11}$$

and

$$\mu(t)(R) = \mu_{\text{ac}}(t)(R) = \|z(t)\|_{H^1}^2$$
$$= \|u(t)\|_{H^1}^2 + \|v(t)\|_{L^2}^2 = \mu(0)(R) \quad \text{for almost all } t. \tag{4.12}$$

Thus, the unique solution described here is a conservative weak solution of system (2.1).

Proof To prove that $z = (u, v)$ is a weak solution of the original system (2.1), it suffices to show that, for all $\phi \in C^\infty(R^+ \times R)$ with compact support,

$$\int_{R^+ \times R} (-u\phi_t + uu_x\phi)(t, x)\, dx\, dt = -\int_{R^+ \times R} (P_x\phi)(t, x)\, dx\, dt,$$
$$\int_{R^+ \times R} (-v\phi_t + uv_x\phi)(t, x)\, dx\, dt = -\int_{R^+ \times R} \left[(v + 1)u_x\phi\right](t, x)\, dx\, dt, \tag{4.13}$$

where P_x is given by (2.1). We denote by the solution $(y, U, V, N, H)(t)$ of (2.10) a representative of $L(z(t), \mu(t))$. On the one hand, since $y(t, \xi)$ is Lipschitz and invertible w.r.t. ξ for almost all t, we can use the change of variables $x = y(t, \xi)$, then we get

$$\int_{R^+ \times R} (-u\phi_t + uu_x\phi)(t, x)\, dx\, dt$$
$$= \int_{R^+ \times R} \left[-(Uy_\xi)(t, \xi)\phi_t\big(t, y(t, \xi)\big) + (UU_\xi)(t, \xi)\phi\big(t, y(t, \xi)\big)\right] d\xi\, dt. \tag{4.14}$$

Since $y_t = U$ and $y_{\xi t} = U_\xi$, it then follows from (2.10) that

$$\int_{R^+ \times R} \left[-Uy_\xi \phi_t(t, y) + UU_\xi\phi(t, y)\right] d\xi\, dt$$
$$= \frac{1}{4}\int_{R^+ \times R^2} \left\{\text{sgn}(\xi - \xi')e^{-|y(\xi) - y(\xi')|}\left[H_\xi + \left(U^2 + 2V\right)y_\xi\right]\right\}(\xi')$$
$$\cdot \phi\big(t, y(\xi)\big)y_\xi(\xi)\, d\xi'\, d\xi\, dt. \tag{4.15}$$

On the other hand, using the change of variables $x = y(t, \xi)$ and $x' = y(t, \xi')$, and since y is an increasing function, we have

$$-\int_{R^+ \times R} (P_x\phi)(t, x)\, dx\, dt$$
$$= \frac{1}{2}\int_{R^+ \times R^2} \left[\text{sgn}(\xi - \xi')e^{-|y(\xi) - y(\xi')|}\left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}v^2 + v\right)\right]\big(t, y(\xi')\big)$$
$$\cdot \phi\big(t, y(\xi)\big)y_\xi(\xi')y_\xi(\xi)\, d\xi'\, d\xi\, dt.$$

It follows from the identity (3.5c) that

$$-\int_{R^+ \times R} (P_x\phi)(t, x)\, dx\, dt$$
$$= \frac{1}{4}\int_{R^+ \times R^2} \left\{\text{sgn}(\xi - \xi')e^{-|y(\xi) - y(\xi')|}\left[H_\xi + \left(U^2 + 2V\right)y_\xi\right]\right\}(\xi')$$
$$\cdot \phi\big(t, y(\xi)\big)y_\xi(\xi)\, d\xi'\, d\xi\, dt. \tag{4.16}$$

By comparing (4.15) and (4.16), we know that

$$\int_{R^+ \times R} \left[-U y_\xi \phi_t(t, y) + U U_\xi \phi(t, y) \right] d\xi \, dt = - \int_{R^+ \times R} (P_x \phi)(t, x) \, dx \, dt.$$

Thus, the first identity in (4.13) follows directly from (4.14) and the second identity in (4.13) follows in the same way. It is not hard to check that $\mu(t)$ is the solution of (4.10). From the definition μ in (4.1b), we can get that

$$\mu(t)(R) = \int_R H_\xi \, d\xi = H(t, \infty),$$

which is constant in time from Lemma 3.3(iii). Thus, we have proved (4.11).

Since $y_\xi(t, \xi) > 0$ a.e. for almost every $\xi \in R$, it then follows from (3.5c) that

$$\mu(t)(B) = \int_{y^{-1}(B)} H_\xi \, d\xi = \int_{y^{-1}(B)} \left(U^2 + U_\xi^2/y_\xi^2 + V^2 \right) y_\xi \, d\xi \tag{4.17}$$

for any Borel set B. Since y is one-to-one and $u_x \circ y y_\xi = U_\xi$ almost everywhere, then (4.17) implies that

$$\mu(t)(B) = \int_B \left(u^2 + u_x^2 + v^2 \right)(t, x) \, dx.$$

Thus, (4.12) holds and the proof is completed. □

5 Multipeakon solutions of the original system

We derive a new system of ordinary differential equations for the multipeakon solutions which is well posed even when collisions occur in this section, and the variables (y, U, V, N, H) are used to characterize multipeakons in a way that avoids the problems related to blowing up.

Solutions of the two-component Camassa-Holm system may experience wave breaking in the sense that the solution develops singularities in finite time, while keeping the H^1 norm finite. Extending the solution beyond wave breaking imposes significant challenge as can be illustrated in the case of multipeakons given by

$$u(t, x) = \sum_{i=1}^n p_i(t) e^{-|x - q_i(t)|}, \tag{5.1}$$

where $(p_i(t), q_i(t))$ satisfy the explicit system of ordinary differential equations

$$\begin{cases} \dot{p}_i = \sum_{j=1}^n p_i p_j \operatorname{sgn}(q_i - q_j) e^{-|q_i - q_j|}, \\ \dot{q}_i = \sum_{j=1}^n p_j e^{-|q_i - q_j|}. \end{cases} \tag{5.2}$$

Peakons interact in a way similar to that of solitons of the CH equation, and wave breaking may appear when at least two of the q_i coincide. Clearly, if the q_i remain distinct, system (5.2) allows for a global smooth solution. In the case where $p_i(0)$ has the same sign for all $i = 1, 2, \ldots, n$, the $q_i(t)$ remain distinct, and (5.2) admits a unique global solution. In this

case, the peakons are traveling in the same direction. However, when two peakons have opposite signs, collisions may occur, and if so, system (5.2) blows up.

We consider initial data $\bar{z} = (\bar{u}, \bar{v})$ given by

$$\begin{cases} \bar{u}(x) = \sum_{i=1}^{n} p_i e^{-|x-\xi_i|}, \\ \bar{v}(x) = \sum_{i=1}^{n} r_i e^{-|x-\xi_i|}. \end{cases} \tag{5.3}$$

Without loss of generality, we assume that the p_i and r_i are all nonzero, and that the ξ_i are all distinct. From Theorem 4.2 we know that there exists a unique and global weak solution with initial data (5.3), and the aim is to characterize this solution explicitly. We consider the following characterization of multipeakons. The multipeakons are given as continuous solutions u defined on intervals $[x_i, x_{i+1}]$ as the solutions of the Dirichlet problem

$$u - u_{xx} = 0, \qquad u(x_i) = u_i, \qquad u(x_{i+1}) = u_{i+1},$$

where the variables x_i denote the position of the peaks, and the variables u_i denote the values of u at the peaks. In the following, we will show that this property persists for conservative solutions.

Let us define $\bar{X} = (\bar{y}, \bar{U}, \bar{V}, \bar{N}, \bar{H})$ as

$$\bar{y}(\xi) = \xi, \tag{5.4a}$$

$$\bar{U}(\xi) = \bar{u}(\xi), \tag{5.4b}$$

$$\bar{V}(\xi) = \bar{v}(\xi), \tag{5.4c}$$

$$\bar{N}(\xi) = \bar{u}_x(\xi), \tag{5.4d}$$

$$\bar{H}(\xi) = \int_{-\infty}^{\xi} \left(u^2 + u_x^2 + v^2 \right) dx, \tag{5.4e}$$

which is a representative of $z = (u, v)$ in the Lagrangian equivalent system, that is, $[\bar{X}] = L(\bar{z}, (\bar{u}^2 + \bar{u}_x^2 + \bar{v}^2) dx)$. Let $A = R \backslash \{\xi_1, \ldots, \xi_n\}$. We claim that the functions $\bar{U}$, $\bar{V}$, $\bar{N}$ and $\bar{H}$ belong to $C^2(A)$ and even belong to $C^\infty(A)$, as the next lemma shows.

Lemma 5.1 *For given initial data $\bar{X} = (\bar{y}, \bar{U}, \bar{V}, \bar{N}, \bar{H}) \in F$ such that $\bar{X} \in [C^2(A)]^5$, the associated solution $X = (y, U, V, W, H)$ of (2.10) belongs to $C^1(R^+, [C^2(A)]^5)$.*

Proof To prove this lemma, one proceeds as in Theorem 3.1 by using the contraction argument. The Banach space E is replaced by

$$\bar{E} = E \cap \left[C^2(A) \right]^5,$$

endowed with the norm

$$\|X\|_{\bar{E}} = \|X\|_E + \|y - \mathrm{Id}\|_{W^{2,\infty}(A)} + \|U\|_{W^{2,\infty}(A)} + \|V\|_{W^{2,\infty}(A)} + \|N\|_{W^{2,\infty}(A)} + \|H\|_{W^{2,\infty}(A)}.$$

It suffices to show that P and P_i are Lipschitz from bounded sets of $\bar{E}$ into $H^1(R) \cap C^2(A)$. Given a bounded set $B = \{X \in \bar{E} | \|X\|_{\bar{E}} \le C_B\}$, where C_B is a positive constant, it follows

from Lemma 3.2 that

$$\left\| P_x(X) - P_x(\bar{X}) \right\|_{L^\infty(R)} \le C \| X - \bar{X} \|_E \le C \| X - \bar{X} \|_{\bar{E}}$$

for a constant C depending only on C_B. From the derivative of P_x given by (3.1) and Lemma 3.1, we have that P_x is locally Lipschitz from $\bar{E}$ into $C^1(A)$. Similarly, we obtain the same result for P. We compute the derivative of P_ξ and $P_{x\xi}$ on A as follows:

$$
\begin{aligned}
&P_{x\xi\xi} = -H_{\xi\xi}/2 + (P_x y_\xi - U U_\xi - V_\xi) y_\xi + \left(P - U^2/2 - V \right) y_{\xi\xi}, \\
&P_{\xi\xi} = -H_\xi y_\xi/2 + \left(P - U^2/2 - V \right) y_\xi^2 + P_x y_{\xi\xi}.
\end{aligned}
\tag{5.5}
$$

Since $P_{\xi\xi}$ and $P_{x\xi\xi}$ are locally Lipschitz maps from $\bar{E}$ into $C(A)$, we have that P and P_x are locally Lipschitz from $\bar{E}$ into $C^2(A)$. The local solution of (2.10) in $\bar{E}$ then can be obtained by the standard contraction argument. As we know, as far as global existence is concerned, $\| X \|_{W^{1,\infty}(R)}$ does not blow up with initial data in $W^{1,\infty}(R)$ (see Lemma 3.3(i)). For any $\xi \in A$, we have that

$$
\begin{aligned}
&y_{\xi\xi t} = U_{\xi\xi}, \\
&U_{\xi\xi t} = \frac{1}{2} H_{\xi\xi} - (P_x y_\xi - U U_\xi - V_\xi) y_\xi - \left(P - \frac{1}{2} U^2 - V \right) y_{\xi\xi}, \\
&V_{\xi\xi t} = -V_{\xi\xi} N - 2 V_\xi N_\xi - V N_{\xi\xi} - N_{\xi\xi}, \\
&N_{\xi\xi t} = 2 U_\xi^2 + 2 U U_\xi - N_\xi^2 - N N_{\xi\xi} + V_\xi^2 + V V_{\xi\xi} + V_{\xi\xi} + \frac{1}{2} H_\xi y_\xi \\
&\qquad - \left(P - \frac{1}{2} U^2 - V \right) y_\xi^2 + P_x y_{\xi\xi}, \\
&H_{\xi\xi t} = \left(3 U^2 - 2P \right) U_{\xi\xi} - 2 U P_x y_{\xi\xi} + 6 U U_\xi^2 - 4 U_\xi P_x y_\xi \\
&\qquad + U H_\xi y_\xi - 2 U P y_\xi^2 + U^3 y_\xi^2 + 2 U V y_\xi^2.
\end{aligned}
\tag{5.6}
$$

System (5.6) is affine w.r.t. $y_{\xi\xi}$, $U_{\xi\xi}$, $V_{\xi\xi}$, $N_{\xi\xi}$, $H_{\xi\xi}$. Hence, on any interval $[0, T)$, we have

$$\left\| X_{\xi\xi}(t, \cdot) \right\|_{L^\infty(A)} \le \left\| X_{\xi\xi}(0, \cdot) \right\|_{L^\infty(A)} + C + C \int_0^t \left\| X_{\xi\xi}(\tau, \cdot) \right\|_{L^\infty(A)} d\tau,$$

where C is a constant depending only on $\sup_{t \in [0,T)} \| X \|_{W^{1,\infty}(R)}$, which is bounded. Thus $\| X \|_{W^{2,\infty}(A)}$ does not blow up from Gronwall's lemma, and therefore the solution is globally defined in $\bar{E}$. □

Theorem 5.1 *Let the initial data be given in* (5.3). *The solution given by Theorem* 4.2 *satisfies* $u - u_{xx} = 0$ *between the peaks.*

Proof Assuming that $y_\xi(t, \xi) \ne 0$, we have

$$u_x \circ y = \frac{U_\xi}{y_\xi}, \qquad u_{xx} \circ y = \left(\frac{U_\xi}{y_\xi} \right)_\xi \frac{1}{y_\xi} = \frac{U_{\xi\xi} y_\xi - y_{\xi\xi} U_\xi}{y_\xi^3}.$$

Hence,

$$(u - u_{xx}) \circ y = \left(Uy_\xi^3 - U_{\xi\xi}y_\xi + y_{\xi\xi}U_\xi\right)/y_\xi^3. \qquad (5.7)$$

Let

$$M = Uy_\xi^3 - U_{\xi\xi}y_\xi + y_{\xi\xi}U_\xi. \qquad (5.8)$$

For a given $\xi \in A$, differentiating (5.8) w.r.t. t, we obtain, by using (2.10), (2.11) and (5.6), that

$$
\begin{aligned}
dM/dt &= 3Uy_\xi^2 y_{\xi t} + U_t y_\xi^3 - U_{\xi\xi t}y_\xi - U_{\xi\xi}y_{\xi t} + y_{\xi\xi t}U_\xi + y_{\xi\xi}U_{\xi t} \\
&= -P_x y_\xi^3 + 3y_\xi^2 UU_\xi - H_{\xi\xi}y_\xi/2 + (Qy_\xi - UU_\xi + AU_\xi - V_\xi)y_\xi^2 \\
&\quad + \left(P - U^2/2 - V\right)y_{\xi\xi}y_\xi - U_{\xi\xi}U_\xi + U_{\xi\xi}U_\xi + y_{\xi\xi}\left[H_\xi/2 + \left(U^2/2 + V - P\right)y_\xi\right] \\
&= 2UU_\xi y_\xi^2 - H_{\xi\xi}y_\xi/2 + H_\xi y_{\xi\xi}/2 - V_\xi y_\xi^2. \qquad (5.9)
\end{aligned}
$$

We differentiate (3.5c) w.r.t. ξ and get

$$y_{\xi\xi}H_\xi + y_\xi H_{\xi\xi} = 2y_\xi y_{\xi\xi}U^2 + 2y_\xi^2 UU_\xi + 2U_\xi U_{\xi\xi} + 2y_\xi y_{\xi\xi}V^2 + 2y_\xi^2 VV_\xi. \qquad (5.10)$$

After inserting the value of $y_\xi H_{\xi\xi}$ given by (5.10) into (5.9) and multiplying the equation by y_ξ, we obtain that

$$y_\xi \cdot dM/dt = UU_\xi y_\xi^3 + U_\xi^2 y_{\xi\xi} - U_\xi U_{\xi\xi}y_\xi.$$

It follows from (3.5c) and since $y_{\xi t} = U_\xi$ that

$$y_\xi \cdot dM/dt = y_{\xi t} \cdot M. \qquad (5.11)$$

We claim that M/y_ξ is C^1 in time. Indeed, we have

$$\frac{M}{y_\xi} = Uy_\xi^2 - U_{\xi\xi} + \frac{y_{\xi\xi}U_\xi}{y_\xi} = Uy_\xi^2 - U_{\xi\xi} + \frac{y_{\xi\xi}U_\xi}{y_\xi + H_\xi} + \frac{y_{\xi\xi}NH_\xi}{y_\xi + H_\xi} = \frac{J(X, X_\xi, X_{\xi\xi})}{y_\xi + H_\xi}$$

for some polynomial J. Since $X \in C^1(R, \bar{E})$, we have X, X_ξ and $X_{\xi\xi}$ are C^1 in time. Since $X(t)$ remains in Γ for all t, from (3.5b), we have $y_\xi + H_\xi > 0$ and therefore $1/(y_\xi + H_\xi)$ is C^1 in time, which implies that M/y_ξ is C^1 in time. For any time t such that $y_\xi(t) \neq 0$, we have

$$\frac{d}{dt}\left(\frac{M}{y_\xi}\right) = \frac{M_t y_\xi - y_{\xi t}M}{y_\xi^2} = 0.$$

Hence,

$$M(t, \xi) = K(\xi)y_\xi(t, \xi)$$

for some constant $K(\xi)$ independent of time. This leads to

$$y_\xi^2(u - u_{xx}) \circ y = K(\xi),$$

which corresponds to the conservation of spatial angular momentum. For the multipeakons at time $t = 0$, we have $y(0, \xi) = \xi$ and $(u - u_{xx})(0, \xi) = 0$ for all $\xi \in A$. Hence,

$$M/y_\xi(t, \xi) = 0 \tag{5.12}$$

for all time t and all $\xi \in A$. Thus, $(u - u_{xx})(t, \xi) = 0$. $\qquad\square$

For solutions with multipeakon initial data, we have the following result: If $y_\xi(t, \xi)$ vanishes at some point $\bar{\xi}$ in the interval (ξ_i, ξ_{i+1}), then $y_\xi(t, \xi)$ vanishes everywhere in (ξ_i, ξ_{i+1}). Furthermore, for the given initial multipeakon solution $\bar{z}(x) = (\bar{u}, \bar{v})(x) = (\sum_{i=1}^n p_i e^{-|x-\xi_i|}, \sum_{i=1}^n r_i e^{-|x-\xi_i|})$, let (y, U, V, N, H) be the solution of system (2.10) with initial data $(\bar{y}, \bar{U}, \bar{V}, \bar{N}, \bar{H})$ given by (5.4a)-(5.4e), then between adjacent peaks, if $x_i = y(t, \xi_i) \neq x_{i+1} = y(t, \xi_{i+1})$, the solution $z(t, x) = (u, v)(t, x)$ is twice differentiable with respect to the space variable, and for $x \in (x_i, x_{i+1})$, we have that $(u - u_{xx}) = 0$.

We now start the derivation of a system of ordinary differential equations for multipeakons.

For each $i = 1, 2, \ldots, n$, we have, from (2.10), that

$$\begin{cases} dy_i/dt = u_i, \\ du_i/dt = -P_{xi}, \\ dv_i/dt = -(v_i + 1)u_{xi}, \\ du_{xi}/dt = u_i^2 - u_{xi}^2/2 + v_i^2/2 + v_i - P_i, \\ dH_i/dt = u_i^3 - 2u_i P_i, \end{cases} \tag{5.13}$$

where $(y_i, u_i, v_i, u_{xi}, H_i) = (y, U, V, N, H)(t, \xi_i)$, $P_i = P(t, \xi_i)$, $P_{xi} = P_x(t, \xi_i)$, respectively. By using the change of variables $x = y(t, \xi)$, P_i and P_{xi} can be rewritten as

$$P_i = 1/2 \cdot \int_R e^{-|y_i - x|} \left(u^2 + u_x^2/2 + v^2/2 + v \right) dx,$$
$$P_{xi} = -1/2 \cdot \int_R \operatorname{sgn}(y_i - x) e^{-|y_i - x|} \left(u^2 + u_x^2/2 + v^2/2 + v \right) dx. \tag{5.14}$$

For $x \in [y_i, y_{i+1}]$, $i = 1, 2, \ldots, n-1$, we write $z = (u, v)$ as

$$z(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} A_i e^x + B_i e^{-x} \\ C_i e^x + D_i e^{-x} \end{pmatrix}. \tag{5.15}$$

The constants A_i, B_i, C_i and D_i depend on u_i, u_{i+1}, v_i, v_{i+1}, y_i and y_{i+1} and read

$$A_i = \frac{e^{-\bar{y}_i}}{2} \left[\frac{\bar{u}_i}{\cosh(\delta y_i)} + \frac{\delta u_i}{\sinh(\delta y_i)} \right], \qquad B_i = \frac{e^{\bar{y}_i}}{2} \left[\frac{\bar{u}_i}{\cosh(\delta y_i)} - \frac{\delta u_i}{\sinh(\delta y_i)} \right],$$
$$C_i = \frac{e^{-\bar{y}_i}}{2} \left[\frac{\bar{v}_i}{\cosh(\delta y_i)} + \frac{\delta v_i}{\sinh(\delta y_i)} \right], \qquad D_i = \frac{e^{\bar{y}_i}}{2} \left[\frac{\bar{v}_i}{\cosh(\delta y_i)} - \frac{\delta v_i}{\sinh(\delta y_i)} \right], \tag{5.16}$$

where

$$\bar{y}_i = \frac{1}{2}(y_i + y_{i+1}), \qquad \delta y_i = \frac{1}{2}(y_i - y_{i+1}),$$

$$\bar{u}_i = \frac{1}{2}(u_i + u_{i+1}), \qquad \delta u_i = \frac{1}{2}(u_i - u_{i+1}), \tag{5.17}$$

$$\bar{v}_i = \frac{1}{2}(v_i + v_{i+1}), \qquad \delta v_i = \frac{1}{2}(v_i - v_{i+1}).$$

The constants A_i, B_i, C_i and D_i uniquely determine $z = (u, v)$ on the interval $[y_i, y_{i+1}]$. Thus, we can compute

$$\delta H_i = H_{i+1} - H_i = \int_{y_i}^{y_{i+1}} \left(u^2 + u_x^2 + v^2 \right) dx$$

$$= 2\bar{u}_i^2 \tanh(\delta y_i) + 2\delta u_i^2 \coth(\delta y_i) + \bar{v}_i^2 \tanh(\delta y_i)$$

$$+ \delta v_i^2 \coth(\delta y_i) + 4C_i D_i \delta y_i. \tag{5.18}$$

At this point, we can get some more understanding of what is happening at the time of collision. Let t^* be the time when the two peaks located at y_i and y_{i+1} collide, *i.e.*, such that $\lim_{t \to t^*} \delta y_i(t) = 0$. The function $z = (u, v)$ remains continuous because the solution $z = (u, v)$ remains in $H^1 \times [L^2 \cap L^\infty]$ for all time, thus we have $\lim_{t \to t^*} \delta u_i(t) = \lim_{t \to t^*} \delta v_i(t) = 0$. Still, A_i, B_i, C_i and D_i may have a finite limit when t tends to t^*. However, the first derivative blows up, which implies $\lim_{t \to t^*} B_i = -\lim_{t \to t^*} A_i = \infty$ and $\lim_{t \to t^*} D_i = -\lim_{t \to t^*} C_i = \infty$. Thus δu_i and δv_i tend to zero but slower than δy_i. In fact, if we let t tend to t^* in (5.18), to first order in δy_i, we obtain

$$\sqrt{2\delta u_i^2 + \delta v_i^2} = \sqrt{\delta H_i} \cdot \sqrt{\delta y_i} + o(\delta y_i),$$

which implies that δu_i and δv_i tend to zero at the same rate as $\sqrt{\delta y_i}$. We now turn to the computation of P_i given by (5.14). Let us write $z = (u, v)$ as

$$z(t, x) = \left(u(t, x), v(t, x) \right)$$

$$= \left(\sum_{j=0}^{n} \left(A_j e^x + B_j e^{-x} \right) \chi_{(y_j, y_{j+1})}(x), \sum_{j=0}^{n} \left(C_j e^x + D_j e^{-x} \right) \chi_{(y_j, y_{j+1})}(x) \right).$$

We have set $y_0 = -\infty$, $y_{n+1} = \infty$, $u_0 = u_{n+1} = 0$, $v_0 = v_{n+1} = 0$, $A_0 = u_1 e^{-y_1}$, $B_0 = 0$, $A_n = 0$, $B_n = u_n e^{y_n}$ and $C_0 = v_1 e^{-y_1}$, $D_0 = 0$, $C_n = 0$, $D_n = v_n e^{y_n}$. We have

$$u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}v^2 + v = \sum_{j=0}^{n} \left(\frac{3}{2}A_j^2 e^{2x} + A_j B_j + \frac{3}{2}B_j^2 e^{-2x} \right.$$

$$\left. + \frac{1}{2}C_j^2 e^{2x} + C_j D_j + \frac{1}{2}D_j^2 e^{-2x} + C_j e^x + D_j e^{-x} \right) \chi_{(y_j, y_{j+1})}. \tag{5.19}$$

Let

$$k_{ij} = \begin{cases} 1 & \text{if } i > j, \\ -1 & \text{if } i \le j. \end{cases}$$

Inserting (5.19) into (5.14), we obtain

$$P_i = \frac{1}{2} \sum_{j=0}^{n} \int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} \left(\frac{3}{2} A_j^2 e^{2x} + A_j B_j + \frac{3}{2} B_j^2 e^{-2x} \right.$$

$$\left. + \frac{1}{2} C_j^2 e^{2x} + C_j D_j + \frac{1}{2} D_j^2 e^{-2x} + C_j e^x + D_j e^{-x} \right) dx. \tag{5.20}$$

From (5.16) and (5.18), we get

$$A_j^2 = \frac{e^{-2\bar{y}_j}}{\sinh^2(2\delta y_j)} \left[\bar{u}_j^2 \sinh^2(\delta y_j) + 2\bar{u}_j \delta u_j \sinh(\delta y_j) \cosh(\delta y_j) + \delta u_j^2 \cosh^2(\delta y_j) \right]$$

$$= \frac{e^{-2\bar{y}_j}}{2 \sinh(2\delta y_j)} \left[\bar{u}_j^2 \tanh(\delta y_j) + 2\bar{u}_j \delta u_j + \delta u_j^2 \coth(\delta y_j) \right], \tag{5.21}$$

$$B_j^2 = \frac{e^{2\bar{y}_j}}{2 \sinh(2\delta y_j)} \left[\bar{u}_j^2 \tanh(\delta y_j) - 2\bar{u}_j \delta u_j + \delta u_j^2 \coth(\delta y_j) \right], \tag{5.22}$$

$$A_j B_j = \frac{1}{4 \sinh(2\delta y_j)} \left[2\bar{u}_j^2 \tanh(\delta y_j) - 2\delta u_j^2 \coth(\delta y_j) \right], \tag{5.23}$$

$$C_j^2 = \frac{e^{-2\bar{y}_j}}{2 \sinh(2\delta y_j)} \left[\bar{v}_j^2 \tanh(\delta y_j) + 2\bar{v}_j \delta v_j + \delta v_j^2 \coth(\delta y_j) \right], \tag{5.24}$$

$$D_j^2 = \frac{e^{2\bar{y}_j}}{2 \sinh(2\delta y_j)} \left[\bar{v}_j^2 \tanh(\delta y_j) - 2\bar{v}_j \delta v_j + \delta v_j^2 \coth(\delta y_j) \right], \tag{5.25}$$

$$C_j D_j = \frac{1}{4 \sinh(2\delta y_j)} \left[2\bar{v}_j^2 \tanh(\delta y_j) - 2\delta v_j^2 \coth(\delta y_j) \right]. \tag{5.26}$$

It then follows from (5.21)-(5.26) that

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} A_j^2 e^{2x} dx = \frac{e^{-k_{ij} y_i} \cdot e^{k_{ij} \bar{y}_j}}{(2 + k_{ij}) \sinh(2\delta y_j)} \sinh\left((2 + k_{ij})\delta y_j\right)$$

$$\cdot \left[\bar{u}_j^2 \tanh(\delta y_j) + 2\bar{u}_j \delta u_j + \delta u_j^2 \coth(\delta y_j) \right], \tag{5.27}$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} B_j^2 e^{-2x} dx = \frac{e^{-k_{ij} y_i} \cdot e^{k_{ij} \bar{y}_j}}{(k_{ij} - 2) \sinh(2\delta y_j)} \sinh\left((k_{ij} - 2)\delta y_j\right)$$

$$\cdot \left[\bar{u}_j^2 \tanh(\delta y_j) - 2\bar{u}_j \delta u_j + \delta u_j^2 \coth(\delta y_j) \right], \tag{5.28}$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} C_j^2 e^{2x} dx = \frac{e^{-k_{ij} y_i} \cdot e^{k_{ij} \bar{y}_j}}{(2 + k_{ij}) \sinh(2\delta y_j)} \sinh\left((2 + k_{ij})\delta y_j\right)$$

$$\cdot \left[\bar{v}_j^2 \tanh(\delta y_j) + 2\bar{v}_j \delta v_j + \delta v_j^2 \coth(\delta y_j) \right], \tag{5.29}$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} D_j^2 e^{-2x} dx = \frac{e^{-k_{ij} y_i} \cdot e^{k_{ij} \bar{y}_j}}{(k_{ij} - 2) \sinh(2\delta y_j)} \sinh\left((k_{ij} - 2)\delta y_j\right)$$

$$\cdot \left[\bar{v}_j^2 \tanh(\delta y_j) - 2\bar{v}_j \delta v_j + \delta v_j^2 \coth(\delta y_j) \right], \tag{5.30}$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i-x)} A_j B_j \, dx = \frac{e^{-k_{ij}y_i} \cdot e^{k_{ij}\bar{y}_j}}{2\sinh(2\delta y_j)} \sinh(\delta y_j)$$

$$\cdot \left[2\bar{u}_j^2 \tanh(\delta y_j) - 2\delta u_j^2 \coth(\delta y_j) \right], \tag{5.31}$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i-x)} C_j D_j \, dx = \frac{e^{-k_{ij}y_i} \cdot e^{k_{ij}\bar{y}_j}}{2\sinh(2\delta y_j)} \sinh(\delta y_j)$$

$$\cdot \left[2\bar{v}_j^2 \tanh(\delta y_j) - 2\delta v_j^2 \coth(\delta y_j) \right]. \tag{5.32}$$

Therefore, the above formulas (5.27)-(5.32) imply that

$$P_i = \sum_{j=0}^{n} \frac{e^{-k_{ij}y_i} \cdot e^{k_{ij}\bar{y}_j}}{4\cosh(\delta y_j)} \left\{ \left[\bar{u}_j^2 \tanh(\delta y_j) + \delta u_j^2 \coth(\delta y_j) \right] \cdot 2\cosh^2(\delta y_j) \right.$$

$$+ 4k_{ij}\bar{u}_j\delta u_j \sinh^2(\delta y_j) + 2\bar{u}_j^2 \tanh(\delta y_j) + \frac{2}{3}\sinh^2(\delta y_j) \cdot \left[\bar{v}_j^2 \tanh(\delta y_j) + \delta v_j^2 \coth(\delta y_j) \right]$$

$$+ \frac{4}{3}k_{ij}\bar{v}_j\delta v_j \sinh^2(\delta y_j) + 2\bar{v}_j^2 \tanh(\delta y_j)$$

$$+ 2\left[\bar{v}_j - k_{ij}\delta v_j \coth(\delta y_j) \right] \cdot \delta y_j + \left[\bar{v}_j \sinh(\delta y_j) + k_{ij}\delta v_j \cosh(\delta y_j) \right] \cdot 2\cosh(\delta y_j) \right\},$$

which can also be written in the following form

$$P_i = \sum_{j=0}^{n} P_{ij} \tag{5.33}$$

with

$$P_{ij} = \begin{cases} \frac{1}{4}u_1^2 e^{y_1-y_i} + \frac{1}{12}v_1^2 e^{y_1-y_i} + \frac{1}{4}v_1 e^{y_1-y_i}, & \text{for } j = 0, \\ \frac{e^{-k_{ij}y_i} \cdot e^{k_{ij}\bar{y}_j}}{4\cosh(\delta y_j)} \{ [\bar{u}_j^2 \tanh(\delta y_j) + \delta u_j^2 \coth(\delta y_j)] \cdot 2\cosh^2(\delta y_j) + 4k_{ij}\bar{u}_j\delta u_j \sinh^2(\delta y_j) \\ \quad + 2\bar{u}_j^2 \tanh(\delta y_j) + \frac{2}{3}\sinh^2(\delta y_j) \cdot [\bar{v}_j^2 \tanh(\delta y_j) + \delta v_j^2 \coth(\delta y_j)] \\ \quad + \frac{4}{3}k_{ij}\bar{v}_j\delta v_j \cdot \sinh^2(\delta y_j) + 2\bar{v}_j^2 \tanh(\delta y_j) + 2[\bar{v}_j - k_{ij}\delta v_j \coth(\delta y_j)] \cdot \delta y_j \\ \quad + [\bar{v}_j \sinh(\delta y_j) + k_{ij}\delta v_j \cosh(\delta y_j)] \cdot 2\cosh(\delta y_j) \}, & \text{for } j = 1,\ldots,n-1, \\ \frac{1}{4}u_n^2 e^{y_i-y_n} + \frac{1}{12}v_n^2 e^{y_i-y_n} + \frac{1}{4}v_n e^{y_i-y_n}, & \text{for } j = n. \end{cases}$$

We compute P_{ix} in the same way and obtain

$$P_{ix} = -\sum_{j=0}^{n} k_{ij} P_{ij}. \tag{5.34}$$

Now we can summarize the result as follows.

Theorem 5.2 *Let $\bar{y}_i = \xi_i$, $(\bar{u}_i, \bar{v}_i, \bar{u}_{xi}) = (\bar{u}, \bar{v}, \bar{u}_x)(\xi_i)$ and $\bar{H}_i = \int_{-\infty}^{\xi_i} (\bar{u}^2 + \bar{u}_x^2 + \bar{v}^2) \, dx$ for $i = 1,\ldots,n$ with a multipeakon initial data $\bar{z} = (\bar{u}, \bar{v})$ given by (5.3). Then with initial data $(\bar{y}_i, \bar{u}_i, \bar{v}_i, \bar{u}_{xi}, \bar{H}_i)$ there exists a global solution $(y_i, u_i, v_i, u_{xi}, H_i)$ of (5.13), (5.33), (5.34). For each time t, $u(t,x)$ is defined as the solution of the Dirichlet problem $u - u_{xx} = 0$ with boundary conditions $u(t, y_i(t)) = u_i(t)$, $u(t, y_{i+1}(t)) = u_{i+1}(t)$ on each interval $[y_i(t), y_{i+1}(t)]$. Thus,*

$z = (u, v)$ is a conservative solution of the two-component Camassa-Holm system, which is the multipeakon conservative solution.

Authors' contributions
YS conceived of the multipeakon conservative solution of the two-component Camassa-Holm system and participated in its coordination and helped to draft the manuscript. YW participated in the design and coordination of the study, carried out the global conservative solution and the multipeakon conservative solution for the two-component Camassa-Holm system and drafted the manuscript. All authors read and approved the final manuscript.

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