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Energy decay and blow-up of solution for a Kirchhoff equation with dynamic boundary condition

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Abstract
The energy decay and blow-up of a solution for a Kirchhoff equation with dynamic boundary condition are considered. With the help of Nakao's inequality and a stable set, the energy decay of the solution is given. By the convexity inequality lemma and an unstable set, the sufficient condition of blow-up of the solution with negative and small positive initial energy are obtained, respectively.

1 Introduction

The aim of this article is to study the energy decay and blow-up of a solution of the following Kirchhoff-type equation with nonlinear dynamic boundary condition:

$$u_{tt} + u_t + u_{xxxx} - M(\|u_x\|^2)u_{xx} = 0, \quad 0 < x < l, t > 0, \tag{1}$$

$$u(0,t) = u_{xx}(0,t), \quad t > 0, \tag{2}$$

$$u_{xx}(l,t) + u_x(l,t) = 0, \quad t > 0, \tag{3}$$

$$u_{tt}(l,t) + u_t(l,t) - u_{xxx}(l,t) + M(\|u_x\|^2)u_x(l,t) = f(u(l,t)), \tag{4}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad 0 < x < l, \tag{5}$$

here $f(s) = |s|^{p-2}s$, $M(s) = 1 + s^m$, $p > 2$, $m \geq 1$ are positive constants and $\|u_x\|^2 = \int_0^l u_x^2\,dx$.

This problem is based on the equation

$$u_{tt} + u_{xxxx} - \left(\alpha + \beta \int_0^l u_x^2(x,t)\,dx\right)u_{xx} = 0, \tag{6}$$

which was proposed by Woinowsky-Krieger [1, 2] as a model for a vibrating beam with hinged ends, where $u(x,t)$ is the lateral displacement at the space coordinate x and time t. Equation (6) was studied by many authors such as Dickey [3], Ball Rivera [4], Tucsnak [5], Kouemou Patchen [6], Aassila [7], Oliveira and Lima [8]; Wu and Tsai [9] considered the following beam equation:

$$u_{tt} + \alpha\Delta^2 u - M(\|\nabla u\|^2)\Delta u + g(u_t) = f(u), \quad \text{in } \Omega \subset R^n. \tag{7}$$

They obtained the existence and uniqueness, as well as decay estimates, of global solutions and blow-up of solutions for the initial boundary value problem of equation (7) through various approaches and assumptive conditions. Feireisl [10] and Fitzgibbon *et al.* [11] showed the existence of a global attractor and an inertial manifold of equation (6) with damping u_t. Ma [12] studied the existence and decay rates for the solution of equation (6) with nonlinear boundary conditions

$$u_{xxx}(l,t) - M\left(\int_0^l u_x^2(x,t)\,dx\right)u_x(l,t) = f\big(u(l,t)\big) + g\big(u_t(l,t)\big). \tag{8}$$

Pazoto and Menzala [13] were concerned with equation (6) with rotational inertia term $-u_{xxtt}$ and nonlinear boundary condition (8). Santos *et al.* [14] established the existence and exponential decay of the Kirchhoff systems with nonlocal boundary condition. Guedda and Labani [15] gave the sufficient condition of the blow-up of the solution to equation (7) with $g(u_t) = u_t$ and dynamic boundary condition. As the related problem, we mention the following:

$$u_{tt} - M\big(\| \nabla u\|^2\big)\Delta u + Q(t,x,u,u_t) + f(x,u) = 0,$$

we refer the reader to [16–19].

When $f = 0$ and $M(s) = \beta + ks$, problem (1)-(5) comes from the reference [20–22]. In this case, the model describes the weakly damped vibrations of an extensible beam whose ends are a fixed distance apart if one end is hinged while a load is attached to the other end [21]. One can find many references on problem (1)-(5) with $M = 0$ and $f = 0$, for example, Littman and Markus [23], Andrews *et al.* [24], Conrad and Morgul [25], Rao [26].

Dalsen [21, 22] showed the exponential stability of problem (1)-(5) with $m = 1$ and $f = 0$. Park *et al.* [27] discussed the existence of the solution of the Kirchhoff equation with dynamic boundary conditions and boundary differential inclusion. Doronin and Larkin [28] and Gerbi and Said-Houari [29] were concerned with the wave equation with dynamic boundary conditions. Recently, Autuori and Pucci [30] studied the global nonexistence of solutions of the p-Kirchhoff system with dynamic boundary condition.

In this paper, we use the idea of references [31] to get the energy decay and blow-up of the solution for problem (1)-(5). We construct a stable set and an unstable set, which is similar to [32]. By the help of Nakao's inequality, combining it with the stable set, we get the decay estimate. We find that the set of initial data such that the solution of problem (1)-(5) is decay, is smaller than the potential well in [32]. The blow-up properties of the solution of problem (1)-(5) with small positive initial energy and negative initial energy are obtained by using the convexity lemma [33]. These results are different from the results in [29, 30].

2 Assumptions and preliminaries

In this section, we give some preliminaries which are used throughout this work.

We use the standard space $L^p[0,l]$ and the Sobolev space $H_0^1(0,l)$, $H^2(0,l)$ with their usual scalar products and norms. Especially, $\|\cdot\|_p$ denotes the norm of $L^p[0,l]$ and $\|\cdot\|$ the norms $L^2[0,l]$.

We denote $V = \{u | u \in H^2(0,l), u(0) = u_{xx}(0) = 0\}$.

Lemma 2.1

(1) *If* $u \in H_0^1$, *then*

$$|u(l)|^2 \le \|u_x\|^2, \qquad \|u\|^2 \le C_0\|u_x\|^2; \tag{9}$$

(2) *If* $u \in V$ *and* $u_{xx}(l) + u_x(l) = 0$, *then*

$$\|u\|^2 \le C_1\left(\|u_{xx}\|^2 + u_x^2(l,t)\right) \le C_1\left(\|u_{xx}\|^2 + u_{xx}^2(l,t)\right). \tag{10}$$

Proof Since $u(0) = 0$, we have

$$|u(x)|^2 = \left|u(0) + \int_0^x u_x(s)\,ds\right|^2 \le \left|\int_0^l u_x^2(s)\,ds\right|.$$

Take $x = l$, we get the first inequality of (9). Integrating the above inequality over $[0,l]$, we get the second part of (9). From the following equation

$$u_x(x) = u_x(l) - \int_x^l u_{xx}(s)\,ds = -u_{xx}(l) - \int_x^l u_{xx}(s)\,ds$$

and the Cauchy inequality, we can get the result of (10) with the help of (9). □

Lemma 2.2 [34] *Let* $\phi(t)$ *be a non-increasing and nonnegative function on* $[0,\infty)$ *such that*

$$\sup_{s\in[t,t+1]} \phi(s) \le C\big(\phi(t) - \phi(t+1)\big), \quad t > 0, \tag{11}$$

then

$$\phi(t) \le Ce^{-\omega t},$$

where C, ω *are positive constants depending on* $\phi(0)$ *and other known qualities.*

Lemma 2.3 [33] *Suppose that a positive, twice-differentiable function* $H(t)$ *satisfies on* $t \ge 0$ *the inequality*

$$H''(t)H(t) - (\beta+1)\big(H'(t)\big)^2 \ge 0, \tag{12}$$

where $\beta > 0$, *then there is a* $t_1 < t_2 = \frac{H(0)}{\beta H'(0)}$ *such that* $H(t) \to \infty$ *as* $t \to t_1$.

A solution u of problem (1)-(5) means that there exists $T > 0$ such that

$$u \in C\big([0,T],V\big) \cap C^1\big([0,T],H_0^1(0,l) \cap H^2(0,l)\big), \qquad u_{tt} \in C\big(0,T;L^2[0,l]\big),$$

and it satisfies the following identity

$$\int_0^t \Big\{(u_{tt},\varphi) - M\big(\|u_x\|^2\big)(u_x,\varphi_x) + (u_{xx},\varphi_{xx}) + (u_t,\varphi) + u_{tt}(l,\tau)\varphi(l,\tau) + u_t(l,\tau)\varphi(l,\tau) - f(u(l,\tau)\varphi(l,\tau)\Big\}\,d\tau = 0 \tag{13}$$

for all $\varphi \in C((0,T),V) \cap C^1(0,T;L^2(0,l))$.

In this paper, we always assume that a local solution exists for problem (1)-(5). In order to study the energy decay or the blow-up phenomenon of the solution of problem (1)-(5), we define the energy of the solution u of problem (1)-(5) by

$$\begin{aligned} E(t) &= E\bigl(u(t)\bigr) \\ &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u_{xx}\|^2 + \frac{1}{2}\|u_x\|^2 + \frac{k}{2(m+1)}\|u_x\|^{2(m+1)} + \frac{1}{2}\bigl|u_t(l,t)\bigr|^2 \\ &\quad + \frac{1}{2}\bigl|u_{xx}(l,t)\bigr|^2 - \frac{1}{p}\bigl|u(l,t)\bigr|^p. \end{aligned} \tag{14}$$

The initial energy is defined by

$$\begin{aligned} E(0) &= \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|u_{0xx}\|^2 + \frac{1}{2}\|u_{0x}\|^2 + \frac{k}{2(m+1)}\|u_{0x}\|^{2(m+1)} + \frac{1}{2}\bigl|u_t(l,t)\bigr|^2 \\ &\quad + \frac{1}{2}\bigl|u_{0xx}(l)\bigr|^2 - \frac{1}{p}\bigl|u(l,t)\bigr|^p. \end{aligned} \tag{15}$$

Then, after some simple computation, we have

$$\frac{d}{dt}E(t) = -\|u_t\|^2 - \bigl|u_t(l,t)\bigr|^2 < 0. \tag{16}$$

That is to say, $E(t)$ is a non-increasing function on $[0,\infty)$. Moreover, we have

$$E(t) + \int_0^t \bigl[\|u_t\|^2 + \bigl|u_t(l,s)\bigr|^2\bigr]\,ds = E(0). \tag{17}$$

We denote

$$B_1^{-1} = \inf_{u\in V}\frac{\|u_{xx}\|}{|u(l)|}, \tag{18}$$

$$\lambda_0 = B_1^{-\frac{p}{p-2}}, \qquad E_0 = \left(\frac{1}{2} - \frac{1}{p}\right)\lambda_0^2. \tag{19}$$

We can now define a stable set and an unstable set [31]

$$\Sigma_0 = \bigl\{(\lambda,E)\in R^2 \mid 0<\lambda<\lambda_0, 0<E\le E_0\bigr\}, \qquad \Sigma^e = \bigl\{(\lambda,E)\in R^2 \mid \lambda>\lambda_0, 0<E<E_0\bigr\}.$$

3 Energy decay of the solution

In order to get the energy decay of the solution, we introduce the following set:

$$\Sigma_1 = \bigl\{(\lambda,E)\in R^2 \mid 0<\lambda<\lambda_1, 0<E<E_1\bigr\}, \tag{20}$$

where $\lambda_1 = (pB_1^p)^{-\frac{1}{p-2}}$, $E_1 = (\frac{1}{2} - \frac{1}{p})\lambda_1^2$. Obviously, $\Sigma_1 \subset \Sigma_0$.

Adapting the idea of Vitillaro [35], we have the following lemma.

Lemma 3.1 *Suppose that u is the solution of* (1)-(5), $u_0 \in V$, $u_1 \in L^2$ *and* $(\|u_{0xx}\|, E(0)) \in \Sigma_1$, *then* $(\|u_{xx}(t)\|, E(t)) \in \Sigma_1$, *for* $t \ge 0$.

Lemma 3.2 *Under the condition of Lemma* 3.1 *and* $p > 2$, *then, for* $t \geq 0$,

$$\|u_{xx}\|^2 \geq 2\left|u(l,t)\right|^p, \tag{21}$$

$$E(t) \geq \frac{p-1}{2p}\|u_{xx}\|^2 \geq \frac{p-1}{p}\left|u(l,t)\right|^p, \tag{22}$$

$$E(t) \geq C_3\left(\|u_{xx}\|^2 + \left|u_{xx}(l,t)\right|^2\right). \tag{23}$$

Proof By (14) and (18), we have

$$\begin{aligned} E(t) &\geq \frac{1}{2}\|u_{xx}\|^2 - \frac{1}{p}\left|u(l,t)\right|^p \geq \frac{1}{2}\|u_{xx}\|^2 - \left|u(l,t)\right|^p \\ &\geq \frac{1}{2}\|u_{xx}\|^2 - B_1^p\|u_{xx}\|^p = G\left(\|u_{xx}\|\right), \end{aligned} \tag{24}$$

where $G(\lambda) = \frac{1}{2}\lambda^2 - B_1^p\lambda^p$. Note that $G(\lambda)$ has the maximum at $\lambda_1 = (pB_1^p)^{-\frac{1}{p-2}}$ and the maximum value $G(\lambda_1) = E_1$. We see that $G(\lambda)$ is increasing in $(0,\lambda_1)$, decreasing in (λ_1,∞) and $G(\lambda) \to 0$ as $\lambda \to \infty$. Since $\|u_{xx}\| < \lambda_1$, $E(0) < E_1$, then $\|u_{xx}\| < \lambda_1$ for any $t \geq 0$, so $G(\|u_{xx}\|) \geq 0$.

By (24), we have

$$\|u_{xx}\|^2 - \left|u(l,t)\right|^p = \frac{1}{2}\|u_{xx}\|^2 + \left(\frac{1}{2}\|u_{xx}\|^2 - \left|u(l,t)\right|^p\right) \geq \frac{1}{2}\|u_{xx}\|^2 + G\left(\|u_{xx}\|\right),$$

then (21) holds since $G(\|u_{xx}\|) > 0$. Furthermore, we have

$$E(t) \geq \frac{1}{2}\|u_{xx}\|^2 - \frac{1}{p}\left|u(l,t)\right|^p \geq \frac{p-1}{2p}\|u_{xx}\|^2. \tag{25}$$

So (22) holds. Similar to (25), the above equality becomes

$$E(t) \geq \frac{1}{2}\|u_{xx}\|^2 + \frac{1}{2}\left|u_{xx}(l,t)\right|^2 - \frac{1}{p}\left|u(l,t)\right|^p \geq \frac{p-1}{2p}\|u_{xx}\|^2 + \frac{1}{2}\left|u_{xx}(l,t)\right|^2,$$

so (23) holds. □

Theorem 3.3 *Let* $p > 2$, $(\|u_{0xx}\|, E(0)) \in \Sigma_1$, *and u be the solution of problem* (1)-(5), *then there exist two positive constants l and* θ *independent of t such that*

$$E(t) \leq le^{-\theta t}, \quad t \geq 0.$$

Proof From (16), we have

$$\int_t^{t+1}\left[\left\|u_t(s)\right\|^2 + \left|u_t(l,s)\right|^2\right]ds = E(t) - E(t+1). \tag{26}$$

Now, for the above estimate and the mean value theorem, we choose $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that, for $i = 1,2$,

$$\left\|u_t(t_i)\right\| \leq \left(\int_t^{t+1}\left\|u_t(s)\right\|^2 ds\right)^{\frac{1}{2}} \leq C\left(E(t) - E(t+1)\right)^{\frac{1}{2}}, \tag{27}$$

$$\left|u(l,t_i)\right| \le \left(\int_t^{t+1} \left|u(l,s)\right|^2\,ds\right)^{\frac{1}{2}} \le C\big(E(t)-E(t+1)\big)^{\frac{1}{2}}. \tag{28}$$

Multiplying equation (1) by u and integrating over $[0,l]\times[t_1,t_2]$, by the boundary conditions (2) and (3), we have

$$\begin{aligned}
&\int_{t_1}^{t_2}\left[\|u_{xx}\|^2 + \|u_x\|^2 + k\|u_x\|^{2(m+1)} + \left|u_{xx}(l,s)\right|^2 - \left|u(l,s)\right|^p\right] ds \\
&\quad = \int_{t_1}^{t_2}\left[(u_{tt},u) + u_t(l,s)u(l,s) + (u_t,u) + u_{tt}(l,s)u(l,s)\right] ds \\
&\quad \le \sum_{i=1}^{2}\left[\left\|u_t(t_i)\right\|\left\|u(t_i)\right\| + \left|u_t(l,t_i)\right|\left|u(l,t_i)\right|\right] + \int_{t_1}^{t_2}\left[\|u_t\|^2 + u_t^2(l,s)\right] ds \\
&\qquad + \int_{t_1}^{t_2}\left[\|u_t\|\|u\| + \left|u_t(l,s)\right|\left|u(l,s)\right|\right] ds.
\end{aligned} \tag{29}$$

Now, we estimate the terms of the right-hand side of (29). By (10), (23), (27) and the Young inequality, we have

$$\begin{aligned}
I_1 &= \sum_{i=1}^{2}\left\|u_t(t_i)\right\|\left\|u(t_i)\right\| \le C\big(E(t)-E(t+1)\big)^{\frac{1}{2}}\sum_{i=1}^{2}\left\|u(t_i)\right\| \\
&\le \varepsilon E(t) + C(\varepsilon)\big(E(t)-E(t+1)\big).
\end{aligned} \tag{30}$$

It follows from (9), (10), (23), (28) and the Young inequality that

$$\begin{aligned}
I_2 &= \sum_{i=1}^{2}\left|u_t(l,t_i)\right|\left|u(l,t_i)\right| \le C\big(E(t)-E(t+1)\big)^{\frac{1}{2}}\sum_{i=1}^{2}\left|u(l,t_i)\right| \\
&\le \varepsilon E(t) + C(\varepsilon)\big(E(t)-E(t+1)\big).
\end{aligned} \tag{31}$$

From (26), we get

$$I_3 = \int_{t_1}^{t_2}\left[\|u_t\|^2 + u_t^2(l,s)\right] ds \le C\big(E(t)-E(t+1)\big). \tag{32}$$

From the Young inequality, (10), (23), (26) and from the fact that $E(t)$ is non-increasing, we arrive at

$$I_4 = \int_{t_1}^{t_2}\|u_t\|\|u\|\,ds \le \varepsilon E(t) + C(\varepsilon)\big(E(t)-E(t+1)\big). \tag{33}$$

By the Young inequality, (9), (10), (23), (26) and the fact that $E(t)$ is non-increasing, we obtain

$$I_5 = \int_{t_1}^{t_2}\left|u(l,s)\right|\left|u_t(l,s)\right| ds \le \varepsilon E(t) + C(\varepsilon)\big(E(t)-E(t+1)\big). \tag{34}$$

Substituting (30)-(34) into (29), we get the estimate

$$\begin{aligned}&\int_{t_1}^{t_2}\bigl[\|u_{xx}\|^2+\|u_x\|^2+k\|u_x\|^{2(m+1)}+\bigl|u_{xx}(l,s)\bigr|^2-\bigl|u(l,s)\bigr|^p\bigr]\,ds\\&\quad\le\varepsilon E(t)+C(\varepsilon)\bigl(E(t)-E(t+1)\bigr).\end{aligned}\tag{35}$$

On the other hand, it follows from (22) that

$$\begin{aligned}E(t)&=\frac{1}{2}\bigl(\|u_{xx}\|^2-\bigl|u(l,t)\bigr|^p\bigr)+\frac{1}{2}\|u_x\|^2+\frac{k}{2(m+1)}\|u_x\|^{2(m+1)}+\frac{1}{2}\|u_t\|^2\\&\quad+\frac{1}{2}\bigl|u_t(l,t)\bigr|^2+\frac{1}{2}\bigl|u_{xx}(l,t)\bigr|^2+\frac{p-2}{2p}\bigl|u(l,t)\bigr|^p\\&\le\frac{1}{2}\bigl(\|u_{xx}\|^2-\bigl|u(l,t)\bigr|^p\bigr)+\frac{1}{2}\|u_x\|^2+\frac{k}{2(m+1)}\|u_x\|^{2(m+1)}+\frac{1}{2}\|u_t\|^2\\&\quad+\frac{1}{2}\bigl|u_t(l,t)\bigr|^2+\frac{1}{2}\bigl|u_{xx}(l,t)\bigr|^2+\frac{p-2}{2(p-1)}E(t).\end{aligned}\tag{36}$$

Then we have

$$\begin{aligned}\frac{p}{2(p-1)}E(t)&\le\frac{1}{2}\bigl[\|u_{xx}\|^2+\|u_x\|^2+k\|u_x\|^{2(m+1)}+\bigl|u_{xx}(l,t)\bigr|^2-\bigl|u(l,t)\bigr|^p\bigr]\\&\quad+\frac{1}{2}\|u_t\|^2+\frac{1}{2}\bigl|u_t(l,t)\bigr|^2.\end{aligned}\tag{37}$$

Therefore, by (37), (35) and (26), we arrive at

$$\begin{aligned}\int_{t_1}^{t_2}E(s)\,ds&\le C\int_{t_1}^{t_2}\bigl[\|u_{xx}\|^2+\|u_x\|^2+k\|u_x\|^{2(m+1)}\\&\quad+\bigl|u_{xx}(l,s)\bigr|^2-\bigl|u(l,s)\bigr|^p\bigr]\,ds+C\int_{t_1}^{t_2}\bigl(\|u_t\|^2+\bigl|u_t(l,s)\bigr|^2\bigr)\,ds\\&\le\varepsilon_1E(t)+C(\varepsilon_1)\bigl(E(t)-E(t+1)\bigr)\\&\le\varepsilon_1\sup_{s\in[t,t+1]}E(s)+C(\varepsilon_1)\bigl(E(t)-E(t+1)\bigr).\end{aligned}\tag{38}$$

Since $E(t)$ is non-increasing, we choose $t_3\in[t_1,t_2]$ such that

$$E(t_3)\le C\int_{t_1}^{t_2}E(s)\,ds.\tag{39}$$

Then, using (26), $t_3<t+1$, and the fact that $E(t)$ is non-increasing, we have

$$\begin{aligned}E(t)&=E(t+1)+\int_t^{t+1}\bigl[\bigl\|u_t(s)\bigr\|^2+\bigl|u_t(l,s)\bigr|^2\bigr]\,ds\le E(t_3)+\int_t^{t+1}\bigl[\bigl\|u_t(s)\bigr\|^2+\bigl|u_t(l,s)\bigr|^2\bigr]\,ds\\&\le C\int_{t_1}^{t_2}E(s)\,ds+\int_t^{t+1}\bigl[\bigl\|u_t(s)\bigr\|^2+\bigl|u_t(l,s)\bigr|^2\bigr]\,ds.\end{aligned}\tag{40}$$

Since $E(t)$ is non-increasing, combining this with (40), (38) and (26), we have

$$\sup_{s\in[t,t+1]}E(s)\le\varepsilon_1\sup_{s\in[t,t+1]}E(s)+C(\varepsilon_1)\bigl(E(t)-E(t+1)\bigr).\tag{41}$$

Choosing ε_1 sufficiently small, (41) leads to

$$\sup_{s\in[t,t+1]} E(s) \leq C\big(E(t) - E(t+1)\big), \tag{42}$$

then, applying Lemma 2.2, we obtain the energy decay. □

4 Blow-up property

In this section, we show that the solution of problem (1)-(5) blows up in finite time if $E(0) < E_0$.

Lemma 4.1 *Suppose* $p > 2$, $(\|u_{0xx}\|, E(0)) \in \Sigma^e$, *then*

$$\|u_{xx}\| \geq \lambda_0, \qquad 0 < E(t) < E_0, \quad t \geq 0. \tag{43}$$

Proof Since $E(t)$ is non-increasing, and $0 < E(0) < E_0$, then $0 < E(t) < E_0$ for $t \geq 0$. Similar to the proof of Lemma 3.2, we have

$$E(t) \geq \frac{1}{2}\|u_{xx}\|^2 - \frac{1}{p}\big|u(l,t)\big|^p \geq \frac{1}{2}\|u_{xx}\|^2 - \frac{1}{p}B_1^p\|u_{xx}\|^p = g\big(\|u_{xx}\|\big), \tag{44}$$

where $g(\lambda) = \frac{1}{2}\lambda^2 - \frac{1}{p}B_1^p\lambda^p$. Note that $g(\lambda)$ has the maximum at $\lambda_0 = B_1^{-\frac{2p}{p-2}}$ and the maximum value is $g(\lambda_0) = E_0$. It is easy to verify that $g(\lambda)$ is increasing for $0 < \lambda < \lambda_0$, decreasing in $\lambda > \lambda_0$ and $g(\lambda) \to -\infty$ as $\lambda \to +\infty$. Therefore, since $E(0) < E_0$, there exists $\lambda_2 > \lambda_0$ such that $g(\lambda_2) > E(0)$. By (44), we have $g(\|u_{0xx}\|) \leq E(0) = g(\lambda_2)$, which implies $\|u_{0xx}\| \geq \lambda_2$. We claim that

$$\big\|u_{xx}(t)\big\| \geq \lambda_2, \quad t > 0. \tag{45}$$

Otherwise, we suppose that $\|u_{xx}(t_0)\| < \lambda_2$ for some $t_0 > 0$ and by the continuity of $\|u_{xx}\|$, we can choose t_0 such that $\|u_{xx}(t_0)\| < \lambda_0$.

Again the use of (44) leads to

$$E(t_0) \geq g\big(\big\|u_{xx}(t_0)\big\|\big) > g(\lambda_2) = E(0). \tag{46}$$

This is impossible since $E(t) \leq E(0)$ for all $t \geq 0$. Hence, (45) holds. Furthermore, (43) is established since $\lambda_2 > \lambda_0$. □

Theorem 4.2 *Suppose that u is the local solution of problem* (1)-(5), $p > 2(m+1)$, $E(0) < 0$, *then the solution u blows up at some finite time.*

Proof Let

$$\begin{aligned} F(t) = \|u\|^2 + u^2(l,t) + \int_0^t \big[\big\|u(s)\big\|^2 + u^2(l,s)\big]\,ds \\ + (T_0 - t)\big(u_0^2(l) + u_{0xx}^2(l)\big) + \beta(t+t_0)^2, \end{aligned} \tag{47}$$

where t_0, T_0, β are positive constants which will be fixed later (see Levine [33]). Then one finds

$$F'(t) = 2(u,u_t) + 2u(l,t)u_t(l,t) + 2\int_0^t \big[\big(u(s),u_s(s)\big) + u(l,s)u_s(l,s)\big]\,ds + 2\beta(t+t_0). \quad (48)$$

$$\begin{aligned} F''(t) &= 2\|u_t\|^2 + 2u_t^2(l,t) + 2(u,u_t) + 2u(l,t)u_t(l,t) + 2\beta + 2(u,u_{tt}) + 2u(l,t)u_{tt}(l,t) \\ &= 2\|u_t\|^2 + 2u_t^2(l,t) - 2\|u_{xx}\|^2 - 2u_{xx}^2(l,t) - 2\|u_x\|^2 \\ &\quad - 2k\|u_x\|^{2(m+1)} + 2\beta + 2\big|u(l,t)\big|^p. \end{aligned} \quad (49)$$

By (17) and (14), we have

$$\begin{aligned} F''(t) &= F''(t) + 2p\big(E(t) - E(0)\big) + 2p\int_0^t \big[\|u_t\|^2 + \big|u_t(l,s)\big|^2\big]\,ds \\ &= (p+2)\Big[\|u_t\|^2 + u_t^2(l,t) + \int_0^t \big[\big\|u_t(s)\big\|^2 + \big|u_t(l,s)\big|^2\big]\,ds + \beta\Big] \\ &\quad + (p-2)\big[\|u_{xx}\|^2 + u_{xx}^2(l,t) + \|u_x\|^2 + k\|u_x\|^{2(m+1)}\big] - p\big(\beta + 2E(0)\big) \\ &\quad + (p-2)\int_0^t \big[\|u_t\|^2 + \big|u_t(l,s)\big|^2\big]\,ds. \end{aligned} \quad (50)$$

Taking $0 < \beta < -2E(0)$ and noticing $p > 2$, we get

$$F''(t) \ge (p+2)\Big[\|u_t\|^2 + u_t^2(l,t) + \int_0^t \big[\big\|u_t(s)\big\|^2 + u_t^2(l,s)\big]\,ds + \beta\Big]. \quad (51)$$

By the Hölder inequality, we have

$$\begin{aligned} \big[F'(t)\big]^2 \le 4\Big[&\|u\|^2\|u_t\|^2 + u(l,t)u_t(l,t) + \Big(\int_0^t \|u\|^2\,ds\Big)^{\frac{1}{2}}\Big(\int_0^t \|u_t\|^2\,ds\Big)^{\frac{1}{2}} \\ &+ \Big(\int_0^t u^2(l,s)\,ds\Big)^{\frac{1}{2}}\Big(\int_0^t u_t^2(l,s)\,ds\Big)^{\frac{1}{2}} + \beta(t+t_0)^2\Big]. \end{aligned} \quad (52)$$

Denote

$$\begin{aligned} &A_1 = \|u\|^2, \quad A_2 = \int_0^t u^2(l,s)\,ds, \quad A_3 = \int_0^t \|u\|^2\,ds, \quad A_4 = u^2(l,t), \\ &B_1 = \|u_t\|^2, \quad B_2 = \int_0^t u_t^2(l,s)\,ds, \quad B_3 = \int_0^t \|u_t\|^2\,ds, \quad B_4 = u_t^2(l,t). \end{aligned}$$

Then, by (48), (51), (52) and the Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned} &F(t)F''(t) - \frac{p+2}{4}\big[F'(t)\big]^2 \\ &\quad \ge (p+2)\big[A_1 + A_2 + A_3 + A_4 + \beta(t+t_0)^2\big][B_1 + B_2 + B_3 + B_4 + \beta] \\ &\quad\quad - (p+2)\big[A_1^{\frac{1}{2}}B_1^{\frac{1}{2}} + A_2^{\frac{1}{2}}B_2^{\frac{1}{2}} + A_3^{\frac{1}{2}}B_3^{\frac{1}{2}} + A_4^{\frac{1}{2}}B_4^{\frac{1}{2}} + \beta(t+t_0)^2\big]^2 \ge 0. \end{aligned} \quad (53)$$

Take t_0 sufficiently large such that

$$F'(0) = 2(u_0, u_1) + 2u_0(l)u_1(l) + 2\beta t_0. \tag{54}$$

Noticing $F(0) > 0$, by Lemma 2.3, we get the result. □

Theorem 4.3 *Suppose that $u(x,t)$ is the local solution of problem* (1)-(5), $p > 2(m+1)$, *and that either of the following conditions is satisfied*:

(i) $E(0) = 0$ *and* $(u_0, u_1) + u_0(l)u_1(l) > 0$;

(ii) $0 < E(0) < E_0$ *and* $\|u_{0xx}\| > \lambda_0$ (*or* $(\|u_{0xx}\|, E(0)) \in \Sigma^e$);

then the solution u blows up at some finite time.

Proof (i) For $E(0) = 0$, similar to the proof of Theorem 4.2, we take $\beta = 0$ in (51), then (53) holds. Since $F(0) > 0$, $F'(0) = 2(u_0, u_1) + 2u_0(l)u_1(l) > 0$, then the result holds by Lemma 2.3.

(ii) For the case of $0 < E(0) < E_0$, from (48), (49), (50) and (14), we get

$$\begin{aligned} F''(t) &= 2\big[\|u_t\|^2 + u_t^2(l,t) - \|u_{xx}\|^2 - u_{xx}^2(l,t) - \|u_x\|^2 - k\|u_x\|^{2(m+1)} + \beta\big] \\ &\quad + p\big[\|u_t\|^2 + u_t^2(l,t) + \|u_{xx}\|^2 + u_{xx}^2(l,t) + \|u_x\|^2 + k\|u_x\|^{2(m+1)} - 2E(t)\big] \\ &= (p+2)\big[\|u_t\|^2 + u_t^2(l,t)\big] + (p-2)\bigg[\|u_{xx}\|^2 + u_{xx}^2(l,t) + \|u_x\|^2 \\ &\quad + \left(\frac{p}{m+1} - 2\right)k\|u_x\|^{2(m+1)}\bigg] + 2\beta - 2pE(t). \end{aligned} \tag{55}$$

By Lemma 4.1,

$$(p-2)\|u_{xx}\|^2 \geq (p-2)\lambda_0^2 = 2pE_0. \tag{56}$$

Combining (55) with (56), $E(0) < E_0$ and (17), we get

$$\begin{aligned} F''(t) &\geq (p+2)\big[\|u_t\|^2 + u_t^2(l,t)\big] + (p-2)\|u_{xx}\|^2 + 2\beta - 2pE(t) \\ &\geq (p+2)\big[\|u_t\|^2 + u_t^2(l,t)\big] + 2p\big(E_0 - E(0)\big) + 2\beta \\ &\quad + 2p\int_0^t \big[\|u_t\|^2 + u_t^2(l,s)\big]\,ds. \end{aligned} \tag{57}$$

Take $\beta = 2(E_0 - E(0)) > 0$, then $2p(E_0 - E(0)) + 2\beta = (p+2)\beta > 0$, since $p > 2$ and $2p > p + 2$, then (57) can be rewritten

$$F''(t) \geq (p+2)[\|u_t\|^2 + u_t^2(l,t) + \beta + \int_0^t \big[\|u_t\|^2 + u_t^2(l,s)\,ds\big]. \tag{58}$$

The remainder of the proof is the same as the proof of Theorem 4.2. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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