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Attractor bifurcation for the extended Fisher-Kolmogorov equation with periodic boundary condition

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Abstract
In this paper, we study the bifurcation and stability of solutions of the extended Fisher-Kolmogorov equation with periodic boundary condition. We prove that the system bifurcates from the trivial solution to an attractor as parameter crosses certain critical value. The topological structure of the attractor is also investigated.
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1 Introduction

In this paper we work with the extended Fisher-Kolmogorov type equation with periodic boundary condition, which reads

$$\begin{cases} \frac{\partial u}{\partial t} = -\mu \frac{\partial^4}{\partial x^4} u + \alpha \frac{\partial^2}{\partial x^2} u + \lambda u + g(u), & (x,t) \in \mathbb{R} \times (0,\infty), \\ \int_0^{2\pi} u(x,t)\,dx = 0, & t \geq 0, \\ u(x,t) = u(x + 2k\pi, t), & \forall k \in \mathbb{Z}, \\ u(x,0) = u_0, & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u = u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is an unknown function, $\mu > 0$, $\alpha > 0$ are constants, $\lambda \in \mathbb{R}^+$ is the system parameter. $g(s)$ is a polynomial on $s \in \mathbb{R}$, which is given by

$$g(s) = \sum_{k=2}^{p} a_k s^k,$$

where $2 \leq p \in \mathbb{N}$ and a_k are given constants.

The extended Fisher-Kolmogorov (EFK) equation has been proposed as a model for phase transitions and other bistable phenomena [1–3]. It has been extensively studied during past decades. Kalies and van der Vorst [4] considered the steady-state problem; by analyzing the variational structure, they proved the existence of heteroclinic connections, which are the critical points of a certain functional. Also, by the variational method, Tersian and Chaparova [5] derived the existence of periodic and homoclinic solutions. Peletier and Troy [6] were interested in the stationary spatially periodic patterns and showed that the structure of the solutions is enriched by increasing the coefficient of the fourth-order derivative term. The structure of the solution set was also investigated by van den Berg [7], who enumerated all the possible bounded stationary solutions provided this coefficient is small. Rottschäer and Wayne [8] showed that for every positive wavespeed there exists a traveling wave. And they also found the critical wavespeed to discriminate the monotonic solution from the oscillatory one. By an iteration procedure, Luo and Zhang [9] proved that equation (1.1) possesses a global attractor in the Sobolev space H^k for all $k > 0$ provided that $a_p < 0$ and p is an odd number. We refer the interested readers to the references in [4–9] for other results on the EFK equation; see also, among others, [10–13].

Returning to problem (1.1), our main interest in the present paper is the bifurcation and stability of solutions. By using a notion of bifurcation called attractor bifurcation developed by Ma and Wang in [14, 15], a nonlinear attractor bifurcation theory for this problem is established. Work on the topic of attractor bifurcation also can be seen in [16, 17].

The main objectives of this theory include:

(1) existence of attractor bifurcation when the system parameter crosses some critical number,
(2) dynamic stability of bifurcated solutions, and
(3) the topological structure of the bifurcated attractor.

Our main results can be summarized as follows.

1. If $\lambda \leq \mu + \alpha$, the steady state $u = 0$ is locally asymptotically stable.
2. As λ crosses $\mu + \alpha$, *i.e.*, there exists an $\epsilon > 0$ such that for any $\mu + \alpha < \lambda < \lambda + \epsilon$, system (1.1) bifurcates from the trivial solution to an attractor Σ_λ.
3. Σ_λ is homeomorphic to S^1 and consists of exactly one cycle of steady solutions of (1.1).

Moreover, we apply this theory to a model of the population density for single-species and derive biological results.

This article is organized as follows. The preliminaries are given in Section 2. The mathematical setting is presented in Section 3. The mathematical results are given in Section 4. In Section 5 we apply mathematical results to a model of the population density for single-species and derive biological results. In Section 6 we discuss some existing results and compare them with ours. Finally, Section 7 is devoted to the conclusions.

2 Preliminaries

We begin with the definition of attractor bifurcation which was first proposed by Ma and Wang in [14, 15].

Let H and H_1 be two Hilbert spaces, and let $H_1 \hookrightarrow H$ be a dense and compact inclusion. We consider the following nonlinear evolution equations

$$\begin{cases} \frac{du}{dt} = L_\lambda u + G(u), \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $u : [0, \infty) \to H$ is the unknown function, $\lambda \in \mathbb{R}$ is the system parameter, and $L_\lambda : H_1 \to H$ are parameterized linear completely continuous fields depending continuously on λ, which satisfy

$$\begin{cases} L_\lambda = A + B_\lambda & \text{a sectorial operator,} \\ A : H_1 \to H & \text{a linear homeomorphism,} \\ B_\lambda : H_1 \to H & \text{parameterized linear compact operators.} \end{cases} \tag{2.2}$$

Since L_λ is a sectorial operator which generates an analytic semigroup $S_\lambda(t) = \{e^{tL_\lambda}\}_{t\geq 0}$ for any $\lambda \in \mathbb{R}$, we can define fractional power operators $(-L_\lambda)^\mu$ for $0 \leq \mu \leq 1$ with domain $H_\mu = D((-L_\lambda)^\mu)$ such that $H_{\mu_1} \subset H_{\mu_2}$ if $\mu_1 > \mu_2$, and $H_0 = H$ (see [18, 19]).

In addition, we assume that the nonlinear terms $G : H_\alpha \to H$ for some $0 \leq \alpha < 1$ are a family of parameterized C^r bounded operators ($r \geq 1$) such that

$$G(u) = o(\|u\|_{H_\alpha}). \tag{2.3}$$

Definition 2.1 [15] A set $\Sigma \subset H$ is called an invariant set of (2.1) if $S(t)\Sigma = \Sigma$ for any $t \geq 0$. An invariant set $\Sigma \subset H$ of (2.1) is said to be an attractor if Σ is compact, and there exists a neighborhood of $W \subset H$ of Σ such that for any $\varphi \in W$ we have

$$\lim_{t\to\infty} \operatorname{dist}_H(u(t,\varphi), \Sigma) = 0,$$

where $\operatorname{dist}_H(u(t,\varphi), \Sigma) = \inf_{v\in\Sigma} \|u(t,\varphi) - v\|_H$, $\forall t \geq 0$.

Definition 2.2 [15] (1) We say that the solution to equation (2.1) bifurcates from $(u,\lambda) = (0,\lambda_0)$ to an invariant set Σ_λ if there exists a sequence of invariant sets $\{\Sigma_{\lambda_n}\}$ of (2.1) such that $0 \notin \Sigma_{\lambda_n}$, and

$$\lim_{n\to\infty} \lambda_n = \lambda_0,$$

$$\lim_{n\to\infty} \max_{v\in\Sigma_{\lambda_n}} \|v\|_H = 0.$$

(2) If the invariant sets Σ_λ are attractors of (2.1), then the bifurcation is called attractor bifurcation.

To prove the main result, we introduce an important theorem.

Let the eigenvalues (counting multiplicity) of L_λ be given by

$$\beta_k(\lambda) \in \mathbb{C} \quad (k \geq 1),$$

and the principle of exchange of stabilities holds true:

$$\operatorname{Re}\beta_i(\lambda) \begin{cases} < 0, & \text{if } \lambda < \lambda_0, \\ = 0, & \text{if } \lambda = \lambda_0 \ (1 \leq i \leq m), \\ > 0, & \text{if } \lambda > \lambda_0, \end{cases} \tag{2.4}$$

$$\operatorname{Re}\beta_j(\lambda_0) < 0, \quad \forall j \geq m+1. \tag{2.5}$$

Let the eigenspace of L_λ at $\lambda = \lambda_0$ be

$$E_0 = \bigcup_{1\le j\le m}\bigcup_{k=1}^{\infty}\{u, v \in H_1 | (L_{\lambda_0} - \beta_j(\lambda_0))^k w = 0, w = u + iv\}.$$

It is known that $\dim E_0 = m$.

The following attractor bifurcation theorem can be found in [15].

Theorem 2.1 *Let* $H_1 = H = \mathbb{R}^n$, *conditions* (2.4), (2.5) *hold true, and* $u = 0$ *is a locally asymptotically stable equilibrium point of* (2.1) *at* $\lambda = \lambda_0$. *Then the following assertions hold true*:

(1) *Equation* (2.1) *bifurcates from* $(u,\lambda) = (0,\lambda_0)$ *to attractors* Σ_λ *for* $\lambda > \lambda_0$, *with dimension* $m - 1 \le \dim \Sigma_\lambda \le m$, *which is connected as* $m > 1$.

(2) *The attractor* Σ_λ *is a limit of a sequence of* m*-dimensional annuli* A_k *with* $A_{k+1} \subset A_k$; *especially, if* Σ_λ *is a finite simplicial complex, then* Σ_λ *has the homology type of the* $(m-1)$*-dimensional sphere* S^{m-1}.

(3) *For any* $u_\lambda \in \Sigma_\lambda$, u_λ *can be expressed as*

$$u_\lambda = v_\lambda + o(\|v_\lambda\|_{H_1}), \quad v_\lambda \in E_0.$$

(4) *If* $u = 0$ *is globally asymptotically stable for* (2.1) *at* $\lambda = \lambda_0$, *then for any bounded open set* $U \subset H$ *with* $0 \in U$, *there is an* $\epsilon > 0$ *such that* $\lambda_0 < \lambda < \lambda_0 + \epsilon$, *the attractor* Σ_λ *attracts* $U \backslash \Gamma$ *in* H, *where* Γ *is the stable manifold of* $u = 0$ *with codimension* m. *In particular, if* (2.1) *has a global attractor for all* λ *near* λ_0, *then* $U = H$.

Remark 2.1 As H_1 and H are infinite dimensional Hilbert spaces, if (2.1) satisfies conditions (2.2)-(2.5) and $u = 0$ is a locally (global) asymptotically stable equilibrium point of (2.1) at $\lambda = \lambda_0$, then the assertions (1)-(4) of Theorem 2.1 hold; see [14, 15].

To get the structure of the bifurcated solutions, we introduce another theorem.

Let v be a two-dimensional C^r $(r \ge 1)$ vector field given by

$$v_\lambda(x) = \lambda x - F(x) \tag{2.6}$$

for $x \in \mathbb{R}^2$. Here

$$F(x) = F_k(x) + o(|x|^k), \tag{2.7}$$

where F_k is a k-multilinear field, which satisfies the inequality

$$C_1|x|^{k+1} \le \langle F_k(x), x\rangle \le C_2|x|^{k+1} \tag{2.8}$$

for some constants $0 < C_1 < C_2$ and $k = 2m + 1$, $m \ge 1$.

Theorem 2.2 (Theorem 5.10 in [15]) *Under conditions* (2.7), (2.8), *the vector field* (2.6) *bifurcates from* $(x,\lambda) = (0,0)$ *to an attractor* Σ_λ *for* $\lambda > 0$, *which is homeomorphic to* S^1. *Moreover, one and only one of the following conclusions is true*:

(1) Σ_λ *is a period orbit.*
(2) Σ_λ *consists of infinitely many singular points.*
(3) Σ_λ *contains at most* $2(k+1) = 4(m+1)$ *singular points and has* $4N + n$ $(N + n \geq 1)$ *singular points*, $2N$ *of which are saddle points*, $2N$ *of which are stable node points* (*possibly degenerate*), *and* n *of which have index zero*.

3 Mathematical setting

Let

$$H = L^2(0, 2\pi)$$

and

$$H_1 = \left\{ u \in H^4(0, 2\pi) \middle| u(x + 2\pi) = u(x), \int_0^{2\pi} u\,dx = 0 \right\}.$$

We define $L_\lambda = A + B_\lambda : H_1 \to H$ and $G : H_1 \to H$ by

$$\begin{cases} Au = -\mu \frac{d^4}{dx^4} u + \alpha \frac{d^2}{dx^2} u, \\ B_\lambda u = \lambda u, \\ G(u) = g(u). \end{cases} \tag{3.1}$$

Consequently, we have an operator equation which is equivalent to problem (1.1) as follows:

$$\begin{cases} \frac{du}{dt} = L_\lambda u + G(u), \\ u(0) = u_0. \end{cases} \tag{3.2}$$

4 Mathematical results

As mentioned in the introduction, we study in this manuscript attractor bifurcation of the EFK equation under the periodic boundary condition. Then we have the following bifurcation theorem.

Theorem 4.1 *For problem* (1.1), *if* $2a_2^2 + 45\mu a_3 + 9\alpha a_3 < 0$ *is satisfied, then the following assertions hold true*:
(1) *If* $\lambda \leq \mu + \alpha$, *the steady state* $u = 0$ *is locally asymptotically stable.*
(2) *If* $\lambda > \mu + \alpha$, *system* (1.1) *bifurcates from the trivial solution* $u = 0$ *to an attractor* Σ_λ.
(3) Σ_λ *is homeomorphic to* S^1 *and consists of exactly one cycle of steady solutions of* (1.1).
(4) Σ_λ *can be expressed as*

$$\Sigma_\lambda = \left\{ \tilde{x} \cos(x + \theta) + o\big(|\tilde{x}|\big) | \theta \in \mathbb{R} \right\},$$

where $\tilde{x} = \sqrt{\frac{4(16\mu+4\alpha-\lambda)(\mu+\alpha-\lambda)}{3a_3(16\mu+4\alpha-\lambda)+2a_2^2}}$ $(a_2 \neq 0)$, *or* $\tilde{x} = \sqrt{\frac{4(\mu+\alpha-\lambda)}{3a_3}}$ $(a_2 = 0)$, *and* $\mu + \alpha < \lambda < \mu + \alpha + \epsilon$, ϵ *is sufficiently small.*

Proof of Theorem 4.1 We shall prove Theorem 4.1 in four steps.

Step 1. In this step, we study the eigenvalue problem of the linearized equation of (3.2) and find the eigenvectors and the critical value of λ.

Consider the eigenvalue problem of the linear equation,

$$L_\lambda u = \beta u. \tag{4.1}$$

It is not difficult to find that the eigenvalues and the normalized eigenvectors of (4.1) are

$$\begin{cases} \beta_{2k-1} = \beta_{2k} = \lambda - \mu k^4 - \alpha k^2, & k = 1, 2, \ldots, \\ e_{2k-1} = \frac{\sin kx}{\sqrt{\pi}}, \qquad e_{2k} = \frac{\cos kx}{\sqrt{\pi}}, \end{cases} \tag{4.2}$$

under condition that we get the principle of exchange of stabilities

$$\beta_1(\lambda) = \beta_2(\lambda) \begin{cases} < 0, & \lambda < \mu + \alpha, \\ = 0, & \lambda = \mu + \alpha, \\ > 0, & \lambda > \mu + \alpha, \end{cases}$$

$$\beta_j(\mu + \alpha) < 0, \quad j \geq 3.$$

Step 2. We verify that for any $\lambda \in \mathbb{R}$, operator $L_\lambda + G$ satisfies conditions (2.2) and (2.3).

Thanks to the results in [9, 18, 19], we know that the operator $L_\lambda : H_1 \to H$ is a sectorial operator which implies that condition (2.2) holds true.

It is easy to get the following inequality:

$$\begin{aligned} \|G(u)\|_H^2 &= \int_0^{2\pi} |g(u)|^2 \, dx \\ &\leq C \int_0^{2\pi} \left(\sum_{k=2}^{p} |u|^{2k} \right) dx \\ &\leq C \sum_{k=2}^{p} \|u\|_{L^{2k}(0,2\pi)}^{2k} \\ &\leq C \sum_{k=2}^{p} \|u\|_{H_{\frac{1}{2}}}^{2k}, \end{aligned}$$

which implies that $G(u) = o(\|u\|_{H_{\frac{1}{2}}})$, where

$$H_{\frac{1}{2}} = \left\{ u \in H^2(0, 2\pi) \,\middle|\, \int_0^{2\pi} u \, dx = 0, u(x + 2\pi) = u(x) \right\},$$

then condition (2.3) holds true.

Step 3. In this part, we prove the existence of attractor bifurcation and analyze the topological structure of the attractor Σ_λ.

Let $E_1^\lambda = E_0 = \operatorname{span}\{e_1, e_2\}$, $E_2^\lambda = E_0^\perp$. Let Φ be the center manifold function, in the neighborhood of $(u, \lambda) = (0, \mu + \alpha)$, we have

$$u = y + \Phi(y),$$

where $y = x_1 e_1 + x_2 e_2$.

Then the reduction equations of (3.2) are as follows:

$$\begin{cases} \frac{dx_1}{dt} = (\lambda - \mu - \alpha)x_1 + \langle G(u), e_1\rangle, \\ \frac{dx_2}{dt} = (\lambda - \mu - \alpha)x_2 + \langle G(u), e_2\rangle. \end{cases} \tag{4.3}$$

To get the exact form of the reduction equations, we need to obtain the expression of $\langle G(u), e_1\rangle$ and $\langle G(u), e_2\rangle$.

Let $G_2 : H_1 \times H_1 \to H$ and $G_3 : H_1 \times H_1 \times H_1 \to H$ be the bilinear and trilinear operators of G respectively, *i.e.*,

$$G_2(u_1, u_2) = a_2 u_1 u_2,$$
$$G_3(u_1, u_2, u_3) = a_3 u_1 u_2 u_3.$$

Since

$$\langle G_2(y, y), e_1\rangle = \langle G_2(y, y), e_2\rangle = 0,$$

the first order approximation of (4.3) does not work. Now, we shall find out the second order approximation of (4.3). And the most important of all is to obtain the approximation expression of the center manifold function.

By direct calculation, we have

$$\langle G_2(y, y), e_k\rangle = \begin{cases} \frac{a_2}{\sqrt{\pi}} x_1 x_2, & k = 3, \\ \frac{a_2}{2\sqrt{\pi}} x_2^2 - \frac{a_2}{2\sqrt{\pi}} x_1^2, & k = 4, \\ 0, & k \neq 3, 4. \end{cases}$$

According to the formula of Theorem 3.8 in [15] (or Remark 4.1), the center manifold function Φ, in the neighborhood of $(u, \lambda) = (0, \mu + \alpha)$, can be expressed as

$$\begin{aligned} \Phi(y) &= -\sum_{k=3}^{\infty} \beta_k^{-1} \langle G_2(y, y), e_k\rangle e_k + O\big((|\beta_1|^2 + |\beta_2|^2)|y|^2\big) + o\big(|y|^2\big) \\ &= -(\lambda - 16\mu - 4\alpha)^{-1} \frac{a_2}{2\sqrt{\pi}} \big(2x_1 x_2 e_3 + x_2^2 e_4 - x_1^2 e_4\big) \\ &\quad + O\big(|\lambda - \mu - \alpha|^2 \big(|x_1|^2 + |x_2|^2\big)\big) + o\big(|x_1|^2 + |x_2|^2\big). \end{aligned}$$

In the following, we calculate $\langle G(u), e_j\rangle$, $j = 1, 2$.

$$\begin{aligned} \langle G(u), e_j\rangle &= \langle G_2\big(y, \Phi(y)\big), e_j\rangle + \langle G_2\big(\Phi(y), y\big), e_j\rangle + \langle G_3(y, y, y), e_j\rangle \\ &\quad + O\big(|\lambda - \mu - \alpha|^2 \big(|x_1|^3 + |x_2|^3\big)\big) + o\big(|x_1|^3 + |x_2|^3\big), \quad j = 1, 2. \end{aligned}$$

By direct calculation, we have

$$\begin{aligned} &\langle G_2\big(y, \Phi(y)\big), e_1\rangle \\ &\quad = \langle G_2\big(\Phi(y), y\big), e_1\rangle \end{aligned}$$

$$
\begin{aligned}
&= -(\lambda - 16\mu - 4\alpha)^{-1}\frac{a_2^2}{4\pi}x_1^3 - (\lambda - 16\mu - 4\alpha)^{-1}\frac{a_2^2}{4\pi}x_1x_2^2 \\
&\quad + O\bigl(|\lambda - \mu - \alpha|^2\bigl(|x_1|^3 + |x_2|^3\bigr)\bigr) + o\bigl(|x_1|^3 + |x_2|^3\bigr), \\
&\bigl\langle G_2\bigl(y, \Phi(y)\bigr), e_2\bigr\rangle \\
&= \bigl\langle G_2\bigl(\Phi(y), y\bigr), e_2\bigr\rangle \\
&= -(\lambda - 16\mu - 4\alpha)^{-1}\frac{a_2^2}{4\pi}x_2^3 - (\lambda - 16\mu - 4\alpha)^{-1}\frac{a_2^2}{4\pi}x_1^2x_2 \\
&\quad + O\bigl(|\lambda - \mu - \alpha|^2\bigl(|x_1|^3 + |x_2|^3\bigr)\bigr) + o\bigl(|x_1|^3 + |x_2|^3\bigr), \\
&\bigl\langle G_3(y,y,y), e_1\bigr\rangle = \frac{3a_3}{4\pi}x_1^3 + \frac{3a_3}{4\pi}x_1x_2^2, \\
&\bigl\langle G_3(y,y,y), e_2\bigr\rangle = \frac{3a_3}{4\pi}x_2^3 + \frac{3a_3}{4\pi}x_1^2x_2,
\end{aligned}
$$

then we obtain the expression of $\langle G(u), e_j\rangle$, $j = 1, 2$.

$$
\begin{aligned}
&\bigl\langle G(u), e_1\bigr\rangle = Ax_1^3 + Ax_1x_2^2 + O\bigl(|\lambda - \mu - \alpha|^2\bigl(|x_1|^3 + |x_2|^3\bigr)\bigr) + o\bigl(|x_1|^3 + |x_2|^3\bigr), \\
&\bigl\langle G(u), e_2\bigr\rangle = Ax_1^2x_2 + Ax_2^3 + O\bigl(|\lambda - \mu - \alpha|^2\bigl(|x_1|^3 + |x_2|^3\bigr)\bigr) + o\bigl(|x_1|^3 + |x_2|^3\bigr),
\end{aligned}
\tag{4.4}
$$

where $A = -(\lambda - 16\mu - 4\alpha)^{-1}\frac{a_2^2}{2\pi} + \frac{3a_3}{4\pi}$.

Putting (4.4) into (4.3), we have the reduction equations

$$
\begin{cases}
\frac{dx_1}{dt} = (\lambda - \mu - \alpha)x_1 + Ax_1^3 + Ax_1x_2^2 \\
\qquad + O\bigl(|\lambda - \mu - \alpha|^2(|x_1|^3 + |x_2|^3)\bigr) + o(|x_1|^3 + |x_2|^3), \\
\frac{dx_2}{dt} = (\lambda - \mu - \alpha)x_2 + Ax_1^2x_2 + Ax_2^3 \\
\qquad + O\bigl(|\lambda - \mu - \alpha|^2(|x_1|^3 + |x_2|^3)\bigr) + o(|x_1|^3 + |x_2|^3).
\end{cases}
\tag{4.5}
$$

For the case of $\lambda < \mu + \alpha$, it is obvious that $u = 0$ is locally asymptotically stable. For the case of $\lambda = \mu + \alpha$, if $2a_2^2 + (45\mu + 9\alpha)a_3 < 0$, which implies that $A < 0$, then $u = 0$ is also locally asymptotically stable. Assertion (1) of Theorem 4.1 is proved.

Since the following equality holds true:

$$
x_1\bigl(Ax_1^3 + Ax_1x_2^2\bigr) + x_2\bigl(Ax_1^2x_2 + Ax_2^3\bigr) = A\bigl(x_1^2 + x_2^2\bigr)^2,
$$

according to Theorems 2.1, 2.2 and Remark 2.1, we can conclude that if $\lambda > \mu + \alpha$, equation (1.1) bifurcates from $u = 0$ to an attractor Σ_λ, which is homeomorphic to S^1.

Step 4. In the last step, we show that the bifurcated attractor of (3.2) consists of a singularity cycle.

Since the even function space is an invariant subspace of $L_\lambda + G$ defined by (3.1), we shall consider the bifurcation problem in the even function space and prove that system (1.1) bifurcates from $(u, \lambda) = (0, \mu + \alpha)$ to two steady solutions. For any function v in the even function space can be expressed as follows:

$$
v = \sum_{k\geq 1} x_{2k}e_{2k},
$$

by the Lyapunov-Schmidt reduction method used in Step 3, we can deduce that the reduction equation of (1.1) is as follows:

$$\frac{dx_2}{dt} = (\lambda - \mu - \alpha)x_2 + Ax_2^3 + O\big(|\lambda - \mu - \alpha|^2|x_2|^3\big) + o\big(|x_2|^3\big), \tag{4.6}$$

which implies that (1.1) bifurcates from $(u,\lambda) = (0,\mu+\alpha)$ to two steady solutions $V_\lambda^{\pm}(x,t) = \pm\sqrt{\frac{4(16\mu+4\alpha-\lambda)(\mu+\alpha-\lambda)}{3a_3(16\mu+4\alpha-\lambda)+2a_2^2}}\cos x$ + h.o.t. in the space of even functions.

Since the solutions of (2.1) are translation invariant,

$$V_\lambda^+(x,t) \to V_\lambda^+(x+\theta,t), \quad \forall\theta \in \mathbb{R},$$

the set

$$T = \big\{V_\lambda^+(x+\theta,t) | \theta \in \mathbb{R}\big\}$$

represents S^1 in H_1, which implies that $\sum_\lambda$ consists of exactly one circle of steady solutions of (1.1). This completes the proof of Theorem 4.1. □

Remark 4.1 Suppose that $\{e_i\}$, the generalized eigenvectors of L_λ, form a basis of H with the dual basis $\{e_i^*\}$ such that

$$\langle e_i, e_j^*\rangle_H \begin{cases} = 0, & \text{if } i \neq j, \\ \neq 0, & \text{if } i = j. \end{cases}$$

We have

$$v = x + y \in E_1^\lambda \oplus E_2^\lambda,$$
$$x = \sum_{i=1}^m x_i e_i \in E_1^\lambda,$$
$$y = \sum_{i=m+1}^\infty x_i e_i \in E_2^\lambda.$$

Then near $\lambda = \lambda_0$, the center manifold function $\phi(x,\lambda)$ in Theorem 3.8 in [15] can be expressed as follows:

$$\phi(x,\lambda) = \sum_{j=m+1}^\infty \phi_j(x,\lambda)e_j + O\big(\big|\mathrm{Re}\,\beta(\lambda)\big| \cdot \|x\|^k\big) + o\big(\|x\|^k\big), \tag{4.7}$$

where

$$\phi_j(x,\lambda) = -\frac{1}{\beta_j(\lambda)}\big\langle G_k(x,\ldots,x), e_j^*\big\rangle_H.$$

Remark 4.2 If $g(s)$ in (1.1) is not a polynomial but a C^ω with Taylor's expansion in $s = 0$ as $g(s) = \sum_{k=2}^\infty a_k s^k$; if $2a_2^2 + 45\mu a_3 + 9\alpha a_3 < 0$ is satisfied, then the conclusions of Theorem 4.1 also hold true.

Remark 4.3 If the higher order terms $\sum_{k=4}^{p} a_k u^k$ in $g(u)$ are omitted, from the proof of Theorem 4.1, it is easy to see that the conclusions of Theorem 4.1 also hold true.

5 Applications

In this section, we apply Theorem 4.1 to a model of the population density for single-species as follows:

$$\begin{cases} \frac{\partial v}{\partial t} = -\mu \frac{\partial^4}{\partial x^4} v + \alpha \frac{\partial^2}{\partial x^2} v + b_1 v + b_2 v^2 + a_3 v^3 + b_0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ \int_0^{2\pi} v(x,t)\,dx = \frac{a_2}{2a_3}\pi, & t \geq 0, \\ v(x,t) = v(x+2k\pi,t), & \forall k \in \mathbb{Z}, \\ v(x,0) = u_0 + v_0, & x \in \mathbb{R}, \end{cases} \tag{5.1}$$

where μ, α are the diffusion coefficients, v is the population density for single-species, and $a_2 < 0$, $a_3 < 0$, $b_0 = -\lambda \frac{a_2}{4a_3} + \frac{3}{64}\frac{a_2^3}{a_3^2}$, $b_1 = \lambda - \frac{5}{16}\frac{a_2^2}{a_3}$, $b_2 = \frac{a_2}{4}$. It is easy to see that $b_0 < 0$, $b_1 > 0$ and $b_2 < 0$. Inspired by the work of Murray [20], b_1 represents the birth rate, $b_2 v^2 + a_3 v^3$ describes the intra specific competition, and b_0 stands for the emigration which arises from disease.

It is not difficult to verify that $v_0 = \frac{a_2}{4a_3}$ is a positive steady solution of system (5.1). From the translation

$$u(x,t) = v(x,t) - v_0, \tag{5.2}$$

we derive the following system:

$$\begin{cases} \frac{\partial u}{\partial t} = -\mu \frac{\partial^4}{\partial x^4} u + \alpha \frac{\partial^2}{\partial x^2} u + \lambda u + a_2 u^2 + a_3 u^3, & (x,t) \in \mathbb{R} \times (0,\infty), \\ \int_0^{2\pi} u(x,t)\,dx = 0, & t \geq 0, \\ u(x,t) = u(x+2k\pi,t), & \forall k \in \mathbb{Z}, \\ u(x,0) = u_0, & x \in \mathbb{R}. \end{cases} \tag{5.3}$$

According to Remark 4.3, if the condition $2a_2^2 + 45\mu a_3 + 9\alpha a_3 < 0$ is satisfied, the conclusions of Theorem 4.1 for system (5.3) also hold true. Consequently, from the translation (5.2), we have the following results for (5.1).

Theorem 5.1 *For problem* (5.1), *if* $2a_2^2 + 45\mu a_3 + 9\alpha a_3 < 0$ *is satisfied, then the following assertions hold true*:

(1) *If* $b_1 \leq \mu + \alpha - \frac{5}{16}\frac{a_2^2}{a_3}$, *the steady state* $v_0 = \frac{a_2}{4a_3}$ *is locally asymptotically stable* (*Figure* 1).

(2) *If* $b_1 > \mu + \alpha - \frac{5}{16}\frac{a_2^2}{a_3}$, *system* (5.1) *bifurcates from the solution* v_0 *to an attractor* Σ_{b_1}. *This implies that the stability will switch from the original state* (*i.e.*, v_0) *to a new one* (*i.e.*, Σ_{b_1}) (*Figure* 1).

(3) Σ_{b_1} *is homeomorphic to* S^1 *and consists of exactly one cycle of steady solutions of* (5.1) (*Figure* 1).

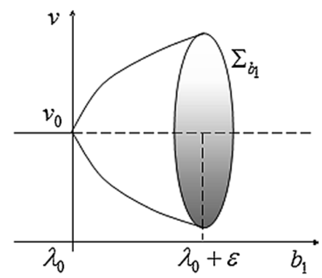

Figure 1 Bifurcation diagram for the model of the population density for single-species. (1) Bifurcation appears at $\lambda_0 = \mu + \alpha - \frac{5}{16}\frac{a_2^2}{a_3}$. (2) Bifurcated attractor Σ_{b_1} is the boundary of the shaded region. (3) The first horizontal solid line from above denotes that the solution $v = v_0$ is stable, and the horizontal dotted line means this solution is unstable.

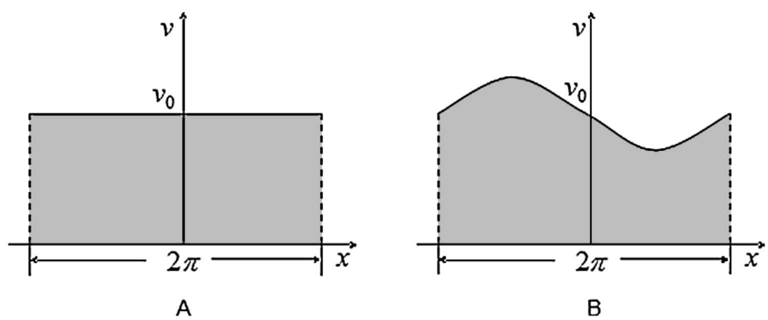

Figure 2 The spatial distribution of the population density. (1) Figure 2(A) shows that the population density keeps a uniform spatial distribution when the birth rate is low. (2) Figure 2(B) shows that the population density changes periodically with space when the birth rate becomes high enough. (3) The area of the shaded regions stands for the population of this single-species. And the area of the shaded region in Figure 2(A) is equal to the area of the shaded region in Figure 2(B).

(4) Σ_{b_1} *can be expressed as*

$$\Sigma_{b_1} = \{v_0 + \tilde{x}\cos(x+\theta) + o(|\tilde{x}|) | \theta \in \mathbb{R}\},$$

where $\tilde{x} = \sqrt{\frac{4(16\mu+4\alpha-\lambda)(\mu+\alpha-\lambda)}{3a_3(16\mu+4\alpha-\lambda)+2a_2^2}}$, *and* $\mu + \alpha < \lambda < \mu + \alpha + \epsilon$, ϵ *is sufficiently small.*

Furthermore, Theorem 5.1 and the equality

$$\int_0^{2\pi} v(x,t)\,dx = \frac{a_2}{2a_3}\pi, \quad t \geq 0,$$

yield the following biological results.

Biological results For the model (5.1), if $2a_2^2 + 45\mu a_3 + 9\alpha a_3 < 0$ is satisfied, we have the following assertions:

(1) The population of this single-species is a conservative quantity.
(2) If the birth rate is low, then the population density will keep a uniform spatial distribution (Figure 2(A)).
(3) If the birth rate becomes high enough, then the spatial distribution of the population density will not keep uniform but change periodically with space (Figure 2(B)).

6 Discussion

Taking $\alpha = 1$, $\lambda = 1$, $g(u) = -u^3$ in (1.1), Peletier and Troy [6] analyzed stationary antisymmetric single-bump periodic solutions. They found that the coefficient of the fourth-order derivative term μ played a role of system parameter. If $\mu \leq \frac{1}{8}$, the family of periodic solutions is still very similar to that of the Fisher-Kolmogorov equations. However, if $\mu > \frac{1}{8}$, different families of periodic solutions emerged.

Taking $\mu = 1$, $\lambda = 1$ in (1.1), and under hypothesis that $g(1) = -1$, $g'(1) < -1$, $g'(u) < 0$ for $0 < u < 1$, Rottschäer and Wayne [8] showed that for every positive wavespeed, there exists a traveling wave. And they also found that there exists a critical wavespeed c^*. If $c \geq c^*$, the solution is monotonic; otherwise, the solution is oscillatory.

Unlike the work mentioned above, which focuses on the structure of solutions varying with the system parameter (μ or c), the manuscript presented here investigates the topological structure and the stability of solutions varying with the system parameter, *i.e.*, λ. Firstly, if $\lambda \leq \mu + \alpha$, the bifurcated attractor consists of the trivial solution; if $\lambda > \mu + \alpha$, the bifurcated attractor consists of only one cycle of steady state solutions and is homeomorphic to S^1. Secondly, if $\lambda \leq \mu + \alpha$, the trivial solution is locally asymptotically stable. However, if $\lambda > \mu + \alpha$, the stability switches from the trivial solution to the bifurcated attractor.

Since the increment of dimension of spatial domain may lead to much richer bifurcated behavior, further investigation on higher dimension of spatial domain is necessary in the future.

7 Conclusions

In this article, we first prove the existence of attractor bifurcation when the system parameter crosses critical number $\mu+\alpha$, which is the first eigenvalue of the eigenvalue problem of the linearized equation of (1.1). Second, we show that the stability of solutions varies with the system parameter λ. If $\lambda \leq \mu+\alpha$, the trivial solution $u = 0$ is locally asymptotically stable. However, if $\lambda > \mu + \alpha$, the stability switches from $u = 0$ to Σ_λ. Third, the topological structure of the attractor is investigated. We prove that the attractor Σ_λ consists of only one cycle of steady state solutions and is homeomorphic to S^1. At last, the expression of bifurcated solution is also obtained.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

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