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Solutions of perturbed p-Laplacian equation with critical nonlinearity and magnetic fields

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Abstract

In this paper, we consider a perturbed p-Laplacian equation with critical nonlinearity and magnetic fields on $\mathbb{R}^N$. By using the variational method, we establish the existence of nontrivial solutions of the least energy.

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1 Introduction

In this paper, we are concerned with the existence of nontrivial solutions to the following perturbed p-Laplacian equation with critical nonlinearity and magnetic fields of the form

$$-\varepsilon^p \Delta_{p,A} u + V(x)|u|^{p-2}u = K(x)|u|^{p^*-2}u + f(x,|u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\Delta_{p,A} u = \operatorname{div}(|\nabla u + iA(x)u|^{p-2}(\nabla u + iA(x)u))$, i is the imaginary unit, $A(x): \mathbb{R}^N \to \mathbb{R}^N$ is a real vector potential, $1 < p < N$, $p^* = Np/(N-p)$ denotes the Sobolev critical exponent and $N \geq 3$.

This paper is motivated by some works concerning the nonlinear Schrödinger equation with magnetic fields of the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}(\nabla + iA(x))^2 \psi + W(x)\psi$$
$$- K(x)|\psi|^{2^*-2}\psi - h(x,|\psi|^2)\psi \quad \text{for } x \in \mathbb{R}^N, \tag{1.2}$$

where $\hbar$ is Planck's constant, i is the imaginary unit, $2^* = \frac{2N}{N-2}$ ($N \geq 3$) is the critical exponent, $A(x): \mathbb{R}^N \to \mathbb{R}^N$ is a real vector potential, $B = \operatorname{curl} A$ and $W(x)$ is a scalar electric potential.

In physics, we are interested in the standing wave solutions, that is, solutions to (1.2) of the type

$$\psi(x,t) = \exp\left(-\frac{iEt}{\hbar}\right)u(x),$$

where $\hbar$ is a sufficiently small constant, E is a real number, and $u(x)$ is a complex-valued function satisfying

$$-\left(\nabla + iA(x)\right)^2 u(x) + \lambda\left(W(x) - E\right)u(x) = \lambda K(x)|u|^{2^*-2}u + \lambda h\left(x, |u|^2\right)u, \quad x \in \mathbb{R}^N. \quad (1.3)$$

We can conduct the transition from quantum mechanics to classical mechanics by letting $\hbar \to 0$. Thus, the existence of semiclassical solutions has a great charm in physical interest.

Problem (1.3) with $A(x) \equiv 0$ has an extensive literature. Different approaches have been taken to investigate this problem under various hypotheses on the potential and nonlinearity. See for example [1–18] and the references therein. The above-mentioned papers mostly concentrated on the nonlinearities with subcritical conditions. Floer and Weinstein in [11] first studied the existence of single and multiple spike solutions based on the Lyapunov-Schmidt reductions. Subsequently, Oh [16, 17] extended the results in a higher dimension. Kang and Wei [14] established the existence of positive solutions with any prescribed number of spikes, clustering around a given local maximum point of the potential function. In accordance with the Sobolev critical nonlinearities, there have been many papers devoted to studying the existence of solutions to elliptic boundary-valued problems on bounded domains after the pioneering work by Brézis and Nirenberg [4]. Ding and Lin [8] first studied the existence of semi-classical solutions to the problem on the whole space with critical nonlinearities and established the existence of positive solutions, as well as of those that change sign exactly once. They also obtained multiplicity of solutions when the nonlinearity is odd.

As far as problem (1.3) in the case of $A(x) \not\equiv 0$ is concerned, we recall Bartsch [2], Cingolani [5] and Esteban and Lions [10]. This kind of paper first appeared in [10]. The authors obtained the existence results of standing wave solutions for fixed $\hbar > 0$ and special classes of magnetic fields. Cingolani [5] proved that the magnetic potential $A(x)$ only contributes to the phase factor of the solitary solutions for $\hbar > 0$ sufficiently small. For more results, we refer the reader to [19–21] and the references therein.

For general $p \geq 1$, most of the works studied the existence results to equation (1.1) with $A(x) \equiv 0$. See, for example, [22–28] and the references therein. These papers are mostly devoted to the study of the existence of solutions to the problem on bounded domains with the Sobolev subcritical nonlinearities.

However, to our best knowledge, it seems that there is no work on the existence of semiclassical solutions to perturbed p-Laplacian equation on $\mathbb{R}^N$ involving critical nonlinearity and magnetic fields. In this paper, we consider problem (1.1) with magnetic fields. The main difficulty in the case is the lack of compactness of the energy functional associated to equation (1.1) because of unbounded domain $\mathbb{R}^N$ and critical nonlinearity. At the same time, we must consider complex-valued functions for the appearance of electromagnetic potential $A(x)$. To overcome this difficulty, we chiefly follow the ideas of [5]. Notice that although the ideas were used in other problems, the adaption of the procedure to our problem is not trivial at all. We need to make careful and complex estimates and prove that the energy functional possesses a Palais-Smale sequence, which has a strongly convergent sequence.

We make the following assumptions on $V(x)$, $A(x)$, $f(x,s)$ and $K(x)$ throughout the paper:

(V_0) $V \in C(\mathbb{R}^N)$, $V(0) = \inf_{x \in \mathbb{R}^N} V(x) = 0$, and there exists $b > 0$ such that the set $\nu^b := \{x \in \mathbb{R}^N : V(x) < b\}$ has a finite Lebesgue measure;

(A_0) $A \in C(\mathbb{R}^N, \mathbb{R}^N)$ and $A(0) = 0$;

(K_0) $K(x) \in C(\mathbb{R}^N, \mathbb{R}^+)$, $0 < \inf K \le \sup K < \infty$;

(H_1) $f \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R})$ and $f(x,s) = o(|s|)$ uniformly in x as $s \to 0$;

(H_2) there are $c_1 > 0$ and $p < \alpha < p^*$ such that $|f(x,s)| \le c_1(1 + |s|^{\frac{\alpha - p}{p}})$ for all (x,s);

(H_3) there exist $a_0 > 0$, $q > p$ and $\theta \in (p, p^*)$ such that $F(x,s) \ge pa_0|s|^{\frac{q}{p}}$ and $\frac{\theta}{p}F(x,s) \le f(x,s)s$ for all (x,s), where $F(x,s) = \int_0^s f(x,t)\, dt$.

Our main result is the following.

Theorem 1 *Assume that* (V_0), (A_0), (K_0) *and* (H_1)-(H_3) *hold. Then for any* $\sigma > 0$, *there exists* $\varepsilon_\sigma > 0$ *such that if* $\varepsilon \le \varepsilon_\sigma$, *equation* (1.1) *has at least one positive least energy solution* u_ε, *which satisfies*

$$\frac{\theta - p}{p\theta} \int_{\mathbb{R}^N} \left(\varepsilon^p |\nabla u_\varepsilon|^p + V(x)|u_\varepsilon|^p \right) \le \sigma \varepsilon^N.$$

The paper is organized as follows. In Section 2, we give some necessary preliminaries. Section 3 is devoted to the technical lemmas. The proof of Theorem 2 is given in the last section.

2 Preliminaries

Let $\lambda = \varepsilon^{-p}$. Equation (1.1) reads then as

$$-\Delta_{p,A} u + \lambda V(x)|u|^{p-2}u = \lambda K(x)|u|^{p^*-2}u + \lambda f(x, |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{2.1}$$

We are going to prove the following result.

Theorem 2 *Assume that* (V_0), (A_0), (K_0) *and* (H_1)-(H_3) *are satisfied. Then for any* $\sigma > 0$, *there exists* $\lambda_\sigma > 0$ *such that if* $\lambda > \lambda_\sigma$, *then equation* (2.1) *has at least one solution of least energy* u_λ *satisfying*

$$\frac{\theta - p}{p\theta} \int_{\mathbb{R}^N} \left(|\nabla u_\lambda|^p + \lambda V(x)|u_\lambda|^p \right) \le \sigma \lambda^{1 - \frac{N}{p}}. \tag{2.2}$$

In order to prove these theorems, we introduce the space

$$E_{\lambda, A} = \left\{ u \in W^{1,p}(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} \lambda V(x)|u|^p < \infty, \lambda > 0 \right\},$$

equipped with the norm

$$\|u\|_{\lambda, A} = \left(\int_{\mathbb{R}^N} \left(\left| \nabla u + i\lambda^{\frac{1}{p}} A(x)u \right|^p + \lambda V(x)|u|^p \right) \right)^{\frac{1}{p}}.$$

It is known that $E_{\lambda, A}$ is the closure of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$. Similar to the diamagnetic inequality [10], we have the following inequality

$$\left| \nabla |u(x)| \right| \le \left| \nabla u + i\lambda^{\frac{1}{p}} Au \right|.$$

In fact, since $A(x)$ is real-valued, one has

$$\big|\nabla|u(x)|\big| \leq \left|\nabla u \frac{\bar{u}}{|u|}\right| = \left|\mathrm{Re}\big(\nabla u + i\lambda^{\frac{1}{p}} Au\big)\frac{\bar{u}}{|u|}\right| \leq \big|\nabla u + i\lambda^{\frac{1}{p}} Au\big| \tag{2.3}$$

(the bar denotes a complex conjugation). This inequality implies that if $u \in E_{\lambda,A}$, then $|u| \in W^{1,p}(\mathbb{R}^N)$, and, therefore, $u \in L^q(\mathbb{R}^N)$ for any $q \in [p, p^*)$. That is, if $u_n \rightharpoonup u$ in $E_{\lambda,A}$, then $u_n \to u$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for any $q \in [p, p^*)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$.

Solutions of (2.1) will be sought in the Sobolev space $E_{\lambda,A}$ as critical points of the functional

$$I_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N}\big(|\nabla u + i\lambda^{\frac{1}{p}} Au|^p + \lambda V(x)|u|^p\big) - \frac{\lambda}{p^*}\int_{\mathbb{R}^N} K(x)|u|^{p^*} - \frac{\lambda}{p}\int_{\mathbb{R}^N} F\big(x,|u|^p\big)$$

$$= \frac{1}{p}\|u\|^p_{\lambda,A} - \lambda\int_{\mathbb{R}^N} G(x,u),$$

where $G(x,u) = \frac{1}{p^*}K(x)|u|^{p^*} + \frac{1}{p}F(x,|u|^p)$.

It is easy to see that I_λ is a C^1-functional on $E_{\lambda,A}$ [29].

3 Behavior of $(PS)_c$ sequence and a mountain pass structure

In this section, we commence by establishing the necessary results which complete the proof of Theorem 2.

Lemma 3.1 *Let* (V_0), (A_0), (K_0) *and* (H_1)-(H_3) *be satisfied. For the* $(PS)_c$ *sequence* $\{u_n\} \subset E_{\lambda,A}$ *for* I_λ, *we get that* $c \geq 0$ *and* $\{u_n\}$ *is bounded in the space* $E_{\lambda,A}$.

Proof Under assumptions (K_0) and (H_3), we have

$$I_\lambda(u_n) - \frac{1}{\theta}I'_\lambda(u_n)u_n$$

$$= \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p_{\lambda,A} + \left(\frac{1}{\theta} - \frac{1}{p^*}\right)\lambda\int_{\mathbb{R}^N} K(x)|u_n|^{p^*}$$

$$+ \lambda\int_{\mathbb{R}^N}\left(\frac{1}{\theta}f\big(x,|u_n|^p\big)|u_n|^p - \frac{1}{p}F\big(x,|u_n|^p\big)\right).$$

In connection with the facts that $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$, we obtain that the $(PS)_c$ sequence $\{u_n\}$ is bounded in $E_{\lambda,A}$, and the energy level $c \geq 0$. $\qquad\square$

Next, let $\{u_n\}$ denote a $(PS)_c$ sequence. By Lemma 3.1, it is bounded, thus, without loss of generality, we may assume that $u_n \rightharpoonup u$ in $E_{\lambda,A}$. Furthermore, passing to a subsequence, we have $u_n \to u$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for any $q \in [p, p^*)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$.

Lemma 3.2 *For any* $s \in [p, p^*)$, *there is a subsequence* $\{u_{n_j}\}$ *such that for any* $\varepsilon > 0$, *there exists* $r_\varepsilon > 0$ *with*

$$\limsup_{j \to \infty}\int_{B_j \setminus B_r}|u_{n_j}|^s \leq \varepsilon \quad \text{for any } r \geq r_\varepsilon,$$

where $B_r := \{x \in \mathbb{R}^N : |x| \leq r\}$.

Proof It is easily obtained by the similar proof of Lemma 3.2 [8]. □

Let $\eta \in C^\infty(\mathbb{R}^+)$ be a smooth function satisfying $0 \leq \eta(t) \leq 1$, $\eta(t) = 1$ if $t \leq 1$ and $\eta(t) = 0$ if $t \geq 2$. Define $\widetilde{u}_j(x) = \eta(2|x|/j)u(x)$. It is not difficult to see that

$$\|u - \widetilde{u}_j\|_{\lambda,A} \to 0 \quad \text{as } j \to \infty.$$

Lemma 3.3 *One has*

$$\limsup_{j \to \infty} \mathrm{Re} \int_{\mathbb{R}^N} \big(f\big(x, |u_{n_j}|^p\big)|u_{n_j}|^{p-2}u_{n_j}$$
$$- f(x, u_{n_j} - \widetilde{u}_j)|u_{n_j} - \widetilde{u}_j|^{p-2}(u_{n_j} - \widetilde{u}_j) - f\big(x, |\widetilde{u}_j|^p\big)|\widetilde{u}_j|^{p-2}\widetilde{u}_j\big)\varphi = 0$$

uniformly in $\varphi \in E_{\lambda,A}$ with $\|\varphi\|_{\lambda,A} \leq 1$.

Proof By direct computation, we easily obtain $\widetilde{u}_j \to u$ in $E_{\lambda,A}$. The local compactness of the Sobolev embedding implies that, for any $r \geq 0$, we have

$$\limsup_{j \to \infty} \mathrm{Re} \int_{B_r} \big(f\big(x, |u_{n_j}|^p\big)|u_{n_j}|^{p-2}u_{n_j}$$
$$- f(x, u_{n_j} - \widetilde{u}_j)|u_{n_j} - \widetilde{u}_j|^{p-2}(u_{n_j} - \widetilde{u}_j) - f\big(x, |\widetilde{u}_j|^p\big)|\widetilde{u}_j|^{p-2}\widetilde{u}_j\big)\bar{\varphi} = 0$$

uniformly in $\|\varphi\|_{\lambda,A} \leq 1$. For any $\varepsilon > 0$, there is $r_\varepsilon \geq 0$ such that

$$\limsup_{j \to \infty} \int_{B_j \setminus B_r} |\widetilde{u}_j|^s \leq \int_{\mathbb{R}^N \setminus B_r} |u|^s \leq \varepsilon$$

for all $r \geq r_\varepsilon$. By the assumptions and the Hölder inequality, we have

$$\limsup_{j \to \infty} \mathrm{Re} \int_{\mathbb{R}^N} \big(f\big(x, |u_{n_j}|^p\big)|u_{n_j}|^{p-2}u_{n_j}$$
$$- f(x, u_{n_j} - \widetilde{u}_j)|u_{n_j} - \widetilde{u}_j|^{p-2}(u_{n_j} - \widetilde{u}_j) - f\big(x, |\widetilde{u}_j|^p\big)|\widetilde{u}_j|^{p-2}\widetilde{u}_j\big)\varphi$$
$$= \limsup_{j \to \infty} \mathrm{Re} \int_{B_j \setminus B_r} \big(f\big(x, |u_{n_j}|^p\big)|u_{n_j}|^{p-2}u_{n_j}$$
$$- f(x, u_{n_j} - \widetilde{u}_j)|u_{n_j} - \widetilde{u}_j|^{p-2}(u_{n_j} - \widetilde{u}_j) - f\big(x, |\widetilde{u}_j|^p\big)|\widetilde{u}_j|^{p-2}\widetilde{u}_j\big)\varphi$$
$$\leq c_1 \limsup_{j \to \infty} \int_{B_j \setminus B_r} \big(|u_{n_j}|^{p-1} + |\widetilde{u}_j|^{p-1}\big)|\varphi|$$
$$+ c_2 \limsup_{j \to \infty} \int_{B_j \setminus B_r} \big(|u_{n_j}|^{\alpha-1} + |\widetilde{u}_j|^{\alpha-1}\big)|\varphi|$$
$$\leq c_1 \limsup_{j \to \infty} \big(\|u_{n_j}\|_{L^{\frac{p}{p-1}}(B_j \setminus B_r)} + \|\widetilde{u}_j\|_{L^{\frac{p}{p-1}}(B_j \setminus B_r)}\big)\|\varphi\|_{L^p(B_j \setminus B_r)}$$
$$+ c_2 \limsup_{j \to \infty} \big(\|u_{n_j}\|_{L^\alpha(B_j \setminus B_r)}^{\alpha-1} + \|\widetilde{u}_j\|_{L^\alpha(B_j \setminus B_r)}^{\alpha-1}\big)\|\varphi\|_{L^\alpha(B_j \setminus B_r)}$$
$$\leq c_3 \varepsilon^{\frac{p-1}{p}} + c_4 \varepsilon^{\frac{\alpha-1}{\alpha}}.$$

This proof is completed. □

Lemma 3.4 *One has along a subsequence*

$$I_\lambda(u_n - \widetilde{u}_n) \to c - I_\lambda(u)$$

and

$$I'_\lambda(u_n - \widetilde{u}_n) \to 0 \quad \text{in } E_\lambda^{-1} \text{ (the dual space of } E_\lambda\text{)}.$$

Proof Combining Lemma 2.1 of [30] and the arguments of [31], one has

$$
\begin{aligned}
I_\lambda(&u_n - \widetilde{u}_n) \\
&= I_\lambda(u_n) - I_\lambda(\widetilde{u}_n) \\
&\quad + \frac{\lambda}{p^*} \int_{\mathbb{R}^N} K(x) \big(|u_n|^{p^*} - |u_n - \widetilde{u}_n|^{p^*} - |\widetilde{u}_n|^{p^*} \big) \\
&\quad + \frac{\lambda}{p} \int_{\mathbb{R}^N} \big(F(x, |u_n|^p) - F(x, |u_n - \widetilde{u}_n|^p) - F(x, |\widetilde{u}_n|^p) \big) + o(1).
\end{aligned}
$$

By the Brézis-Lieb lemma [32], we get

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x) \big(|u_n|^{p^*} - |u_n - \widetilde{u}_n|^{p^*} - |\widetilde{u}_n|^{p^*} \big) = 0$$

and

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \big(F(x, |u_n|^p) - F(x, |u_n - \widetilde{u}_n|^p) - F(x, |\widetilde{u}_n|^p) \big) = 0.$$

We now observe that $I_\lambda(u_n) \to c$ and $I_\lambda(\widetilde{u}_n) \to I_\lambda(u)$, which gives

$$I_\lambda(u_n - \widetilde{u}_n) \to c - I_\lambda(u).$$

Moreover, by direct computation, we get

$$
\begin{aligned}
I'_\lambda(u_n - \widetilde{u}_n)\varphi = {}& I'_\lambda(u_n)\varphi - I'_\lambda(\widetilde{u}_n)\varphi \\
& + \lambda \operatorname{Re} \int_{\mathbb{R}^N} K(x) \big(|u_n|^{p^*-2} u_n - |u_n - \widetilde{u}_n|^{p^*-2}(u_n - \widetilde{u}_n) - |\widetilde{u}_n|^{p^*-2}\widetilde{u}_n \big) \varphi \\
& + \lambda \operatorname{Re} \int_{\mathbb{R}^N} \big(f(x, |u_n|^p) |u_n|^{p-2} u_n \\
& - f(x, |u_n - \widetilde{u}_n|^p) |u_n - \widetilde{u}_n|^{p-2}(u_n - \widetilde{u}_n) - f(x, |\widetilde{u}_n|^p) |\widetilde{u}_n|^{p-2}\widetilde{u}_n \big) \varphi + o(1).
\end{aligned}
$$

It then follows from the standard arguments that

$$\lim_{n \to \infty} \operatorname{Re} \int_{\mathbb{R}^N} K(x) \big(|u_n|^{p^*-2} u_n - |u_n - \widetilde{u}_n|^{p^*-2}(u_n - \widetilde{u}_n) - |\widetilde{u}_n|^{p^*-2}\widetilde{u}_n \big) \varphi = 0$$

uniformly in $\|\varphi\|_{\lambda,A} \le 1$. Combining Lemma 3.3, we get $I'_\lambda(u_n - \widetilde{u}_n) \to 0$. The proof is completed. $\qquad\square$

Let $u_n^1 = u_n - \tilde{u}_n$, then $u_n - u = u_n^1 + (\tilde{u}_n - u)$. Therefore, $u_n \to u$ in $E_{\lambda,A}$ if and only if $u_n^1 \to 0$ in $E_{\lambda,A}$.

Note that

$$I_\lambda\left(u_n^1\right) - \frac{1}{p}I_\lambda'\left(u_n^1\right)u_n^1$$

$$= \left(\frac{1}{p} - \frac{1}{p^*}\right)\lambda\int_{\mathbb{R}^N} K(x)\left|u_n^1\right|^{p^*} + \lambda\int_{\mathbb{R}^N}\frac{1}{p}\left(\left|u_n^1\right|^p f\left(x,\left|u_n^1\right|^p\right) - F\left(x,\left|u_n^1\right|^p\right)\right)$$

$$\geq \frac{\lambda}{N}\int_{\mathbb{R}^N} K(x)\left|u_n^1\right|^{p^*}$$

$$\geq \frac{\lambda}{N}K_{\min}\left\|u_n^1\right\|_{p^*}^{p^*},$$

where $K_{\min} = \inf_{x\in\mathbb{R}^N} K(x) > 0$. Together with Lemma 3.4, one has

$$\left\|u_n^1\right\|_{p^*}^{p^*} \leq \frac{N(c - I_\lambda(u))}{\lambda K_{\min}} + o(1). \tag{3.1}$$

In the following, we consider the energy level of the functional I_λ below which the $(PS)_c$ condition holds.

Denote $V_b(x) := \max\{V(x), b\}$, where b is the positive constant in assumption (V_0). Since the set ν_b has a finite measure, combining the fact that $u_n^1 \to 0$ in $L_{\mathrm{loc}}^p(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N} V(x)\left|u_n^1\right|^p = \int_{\mathbb{R}^N} V_b(x)\left|u_n^1\right|^p + o(1). \tag{3.2}$$

Furthermore, by (K_0) and (H_1)-(H_3), there exists $C_b > 0$ such that

$$\int_{\mathbb{R}^N}\left(K(x)\left|u_n^1\right|^{p^*} + \left|u_n^1\right|^p f\left(x,\left|u_n^1\right|^p\right)\right) \leq b\left\|u_n^1\right\|_p^p + C_b\left\|u_n^1\right\|_{p^*}^{p^*}. \tag{3.3}$$

Let S be the best Sobolev constant of the immersion

$$S\|u\|_{p^*}^p \leq \int_{\mathbb{R}^N} |\nabla u|^p \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N).$$

Lemma 3.5 *There exists $\alpha_0 > 0$ (independent of λ) such that, for any $(PS)_c$ sequence $\{u_n\} \subset E_{\lambda,A}$ for I_λ with $u_n \rightharpoonup u$, either $u_n \to u$ in $E_{\lambda,A}$ or $c - I_\lambda(u) \geq \alpha_0 \lambda^{1-\frac{N}{p}}$.*

Proof Arguing by contradiction, assume that $u_n \nrightarrow u$, then

$$\liminf_{n\to\infty}\left\|u_n^1\right\|_{\lambda,A} > 0.$$

Combining the Sobolev inequality, (3.2) and (3.3), we get

$$S\left\|u_n^1\right\|_{p^*}^p \leq \int_{\mathbb{R}^N}\left|\nabla u_n^1\right|^p$$

$$\leq \int_{\mathbb{R}^N}\left(\left|\nabla u_n^1 + i\lambda^{\frac{1}{p}}A(x)u_n^1\right|^p + \lambda V(x)\left|u_n^1\right|^p\right) - \lambda\int_{\mathbb{R}^N} V(x)\left|u_n^1\right|^p$$

$$= \lambda\int_{\mathbb{R}^N}\left(K(x)\left|u_n^1\right|^{p^*} + \left|u_n^1\right|^p f\left(x,\left|u_n^1\right|^p\right)\right) - \lambda\int_{\mathbb{R}^N} V_b(x)\left|u_n^1\right|^p + o(1)$$

$$\leq \lambda b \|u_n^1\|_p^p + \lambda C_b \|u_n^1\|_{p^*}^{p^*} - \lambda b \|u_n^1\|_p^p + o(1)$$

$$= \lambda C_b \|u_n^1\|_{p^*}^{p^*} + o(1),$$

which further gives

$$S \leq \lambda C_b \|u_n^1\|_{p^*}^{p^*-p} + o(1)$$

$$\leq \lambda C_b \left(\frac{N(c - I_\lambda(u))}{\lambda K_{\min}} \right)^{\frac{p}{N}} + o(1)$$

$$= \lambda^{1-\frac{p}{N}} C_b \left(\frac{N}{K_{\min}} \right)^{\frac{p}{N}} \left(c - I_\lambda(u) \right)^{\frac{p}{N}} + o(1).$$

Denote $\alpha_0 = S^{\frac{N}{p}} C_b^{-\frac{N}{p}} N^{-1} K_{\min}$, then

$$\alpha_0 \lambda^{1-\frac{N}{p}} \leq c - I_\lambda(u) + o(1).$$

We obtain the desired conclusion. □

Lemma 3.6 *There exists a constant $\alpha_0 > 0$ (independent of λ) such that if a $(PS)_c$ sequence $\{u_n\} \subset E_{\lambda,A}$ for I_λ satisfies $c \leq \alpha_0 \lambda^{1-\frac{N}{p}}$, the sequence $\{u_n\}$ has a strongly convergent subsequence in $E_{\lambda,A}$.*

Proof By the fact that $I_\lambda(u) \geq 0$ and Lemma 3.5, we easily get the required conclusion. □

Now, we consider $\lambda \geq 1$ and prove that the energy functional I_λ possesses the mountain pass structure.

Lemma 3.7 *Under the assumptions of Theorem 2, there exist $\alpha_\lambda, \rho_\lambda > 0$ such that*

$$I_\lambda(u) > 0 \quad if\ 0 < \|u\|_{\lambda,A} < \rho_\lambda \quad and \quad I_\lambda(u) \geq \alpha_\lambda \quad if\ \|u\|_{\lambda,A} = \rho_\lambda.$$

Proof The proof of Lemma 3.7 is similar to the one of Lemma 4.1 in [8]. □

Lemma 3.8 *For any finite dimensional subspace $F \subset E_{\lambda,A}$, we have*

$$I_\lambda(u) \to -\infty, \quad u \in F \ as\ \|u\|_{\lambda,A} \to \infty.$$

Proof By assumptions (K_0) and (H_3), one has

$$I_\lambda(u) \leq \frac{1}{p} \|u\|_{\lambda,A}^p - \lambda a_0 \|u\|_q^q \quad for\ all\ u \in E_{\lambda,A}.$$

Since all norms in a finite-dimensional space are equivalent, in connection with $q > p$, we obtain the desired conclusion. □

For λ large enough and c_λ small sufficiently, I_λ satisfies $(PS)_{c_\lambda}$ condition by Lemma 3.6. Furthermore, we will find special finite-dimensional subspace, by which we establish sufficiently small minimax levels.

Define the functional

$$\Phi_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N}\left(\left|\nabla u + i\lambda^{\frac{1}{p}}A(x)u\right|^p + \lambda V(x)|u|^p\right) - \lambda a_0\int_{\mathbb{R}^N}|u|^q.$$

It is easy to see that $\Phi_\lambda \in C^1(E_{\lambda,A})$ and $I_\lambda(u) \le \Phi_\lambda(u)$ for all $u \in E_{\lambda,A}$. Note that

$$\inf\left\{\int_{\mathbb{R}^N}|\nabla\phi|^p : \phi \in C_0^\infty(\mathbb{R}^N,\mathbb{R}), \|\phi\|_{L^q(\mathbb{R}^N)} = 1\right\} = 0.$$

For any $\delta > 0$, there is $\phi_\delta \in C_0^\infty(\mathbb{R}^N,\mathbb{R})$ with $\|\phi_\delta\|_{L^q(\mathbb{R}^N)} = 1$ and $\operatorname{supp}\phi_\delta \subset B_{r_\delta}(0)$ such that $\|\nabla\phi_\delta\|_p^p < \delta$. Let $e_\lambda(x) = \phi_\delta(\sqrt[p]{\lambda}x)$, then $\operatorname{supp}e_\lambda \subset B_{\lambda^{-\frac{1}{p}}r_\delta}(0)$. For any $t \ge 0$, we have

$$\begin{aligned}
\Phi_\lambda(te_\lambda) &= \frac{t^p}{p}\|e_\lambda\|_{\lambda,A}^p - a_0\lambda t^q\int_{\mathbb{R}^N}\left|\phi_\delta(\sqrt[p]{\lambda}x)\right|^q \\
&= \lambda^{1-\frac{N}{p}}J_\lambda(t\phi_\delta),
\end{aligned}$$

where

$$J_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N}\left(\left|\nabla u + i\lambda^{\frac{1}{p}}A(x)u\right|^p + V\left(\lambda^{-\frac{1}{p}}x\right)|u|^p\right) - a_0\int_{\mathbb{R}^N}|u|^q.$$

We derive that

$$\max_{t\ge 0}J_\lambda(t\phi_\delta) \le \frac{q-p}{pq(qa_0)^{\frac{p}{q-p}}}\left(\int_{\mathbb{R}^N}\left(|\nabla\phi_\delta|^p + \left(A\left(\lambda^{-\frac{1}{p}}x\right) + V\left(\lambda^{-\frac{1}{p}}x\right)\right)|\phi_\delta|^p\right)\right)^{\frac{q}{q-p}}.$$

Observe that $A(0) = 0$, $V(0) = 0$ and $\|\nabla\phi_\delta\|_p^p < \delta$. Therefore, there exists $\Lambda_\delta > 0$ such that for all $\lambda \ge \Lambda_\delta$, we have

$$\max_{t\ge 0}I_\lambda(t\phi_\delta) \le \left(\frac{q-p}{pq(qa_0)^{\frac{p}{q-p}}}(5\delta)^{\frac{q}{q-p}}\right)\lambda^{1-\frac{N}{p}}. \tag{3.4}$$

Lemma 3.9 *Under the assumptions of Theorem 2, for any $\sigma > 0$, there is $\Lambda_\sigma > 0$ such that for each $\lambda \ge \Lambda_\sigma$, there exists $\bar{e}_\lambda \in E_{\lambda,A}$ with $\|\bar{e}_\lambda\|_{\lambda,A} > \rho_\lambda$, $I_\lambda(\bar{e}_\lambda) \le 0$ and*

$$\max_{t\ge 0}I_\lambda(t\bar{e}_\lambda) \le \sigma\lambda^{1-\frac{N}{p}},$$

where ρ_λ is defined in Lemma 3.7.

Proof For any $\sigma > 0$, we can choose $\delta < 0$ so small that

$$\frac{q-p}{pq(qa_0)^{\frac{p}{q-p}}}(5\delta)^{\frac{q}{q-p}} \le \sigma.$$

Denote $e_\lambda(x) = \phi_\delta(\sqrt[p]{\lambda}x)$ and $\Lambda_\sigma = \Lambda_\delta$. Let $\bar{t}_\lambda > 0$ be such that $\bar{t}_\lambda\|e_\lambda\|_{\lambda,A} > \rho_\lambda$ and $I_\lambda(te_\lambda) \le 0$ for all $t \ge \bar{t}_\lambda$. Then, combining (3.4), $\bar{e}_\lambda = \bar{t}_\lambda e_\lambda$ meets the requirements. □

4 Proof of Theorem 2

In this section, we give the proof of Theorem 2.

Proof By Lemma 3.9, for any $\sigma > 0$ with $0 < \sigma < \alpha_0$, we choose $\Lambda_\sigma > 0$ and define the minimax value

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda\big(\gamma(t)\big) \quad \text{with } c_\lambda \leq \sigma \lambda^{1-\frac{N}{p}} \text{ for each } \lambda \geq \Lambda_\sigma,$$

where $\Gamma_\lambda = \{\gamma \in C([0,1], E_{\lambda,A}) : \gamma(0) = 0, \gamma(1) = \bar{e}_\lambda\}$.

Lemma 3.6 shows that I_λ satisfies $(PS)_{c_\lambda}$ condition. Therefore, by the mountain pass theorem, there exists $u_\lambda \in E_{\lambda,A}$, which satisfies $I_\lambda(u_\lambda) = c_\lambda$ and $I'_\lambda(u_\lambda) = 0$. That is, u_λ is a weak solution of (2.1). Furthermore, it is well known that u_λ is the least energy solution of equation (2.1).

Moreover, together with $I_\lambda(u_\lambda) \leq \sigma \lambda^{1-\frac{N}{p}}$ and $I'_\lambda(u_\lambda) = 0$, we have

$$
\begin{aligned}
I_\lambda(u_\lambda) &= I_\lambda(u_\lambda) - \frac{1}{\theta} I'_\lambda(u_\lambda)(u_\lambda) \\
&= \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_\lambda\|_{\lambda,A}^p + \left(\frac{1}{\theta} - \frac{1}{p^*}\right)\lambda \int_{\mathbb{R}^N} K(x)|u_\lambda|^{p^*} \\
&\quad + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\theta}|u_\lambda|^p f\big(x, |u_\lambda|^p\big) - \frac{1}{p} F\big(x, |u_\lambda|^p\big)\right) \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_\lambda\|_{\lambda,A}^p.
\end{aligned}
$$

By inequality (2.3), we obtain

$$\frac{\theta - p}{p\theta} \int_{\mathbb{R}^N} \big(|\nabla u_\lambda|^p + \lambda V(x)|u_\lambda|^p\big) \leq \sigma \lambda^{1-\frac{N}{p}}.$$

The proof is complete. □

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