Jiao et al. *Boundary Value Problems* 2013, 2013:219
http://www.boundaryvalueproblems.com/content/2013/1/219

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Existence of solutions for a general quasilinear elliptic system via perturbation method

Yujuan Jiao[1*], Shengmao Fu[2] and Yanli Wang[3]

*Correspondence:
jsjyj@xbmu.edu.cn
[1]College of Mathematics and Computer Science, Northwest University for Nationalities, Lanzhou, 730124, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we consider the following quasilinear elliptic system:

$$\begin{cases} -\sum_{i,j=1}^N D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum_{i,j=1}^N D_sa_{ij}(x,u)D_iuD_ju = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^\beta u, & x \in \Omega, \\ -\sum_{i,j=1}^N D_j(b_{ij}(x,v)D_iv) + \frac{1}{2}\sum_{i,j=1}^N D_sb_{ij}(x,v)D_ivD_jv = \frac{2\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}v, & x \in \Omega, \\ u = 0, \quad v = 0, & x \in \partial\Omega, \end{cases}$$

where $D_iu = \frac{\partial u}{\partial x_i}$, $D_sa_{ij}(x,u) = \frac{\partial}{\partial u}a_{ij}(x,u)$, $D_sb_{ij}(x,v) = \frac{\partial}{\partial v}b_{ij}(x,v)$, $\alpha > 2$, $\beta > 2$, $\alpha + \beta < 2 \cdot 2^*$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded smooth domain. By using the perturbation method, we establish the existence of both positive and negative solutions for this system.

MSC: 35J60; 35B33

Keywords: quasilinear elliptic system; positive solution; negative solution; perturbation method

1 Introduction

Let us consider the following quasilinear elliptic system:

$$\begin{cases} -\sum_{i,j=1}^N D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum_{i,j=1}^N D_sa_{ij}(x,u)D_iuD_ju = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^\beta u, & x \in \Omega, \\ -\sum_{i,j=1}^N D_j(b_{ij}(x,v)D_iv) + \frac{1}{2}\sum_{i,j=1}^N D_sb_{ij}(x,v)D_ivD_jv = \frac{2\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}v, & x \in \Omega, \\ u = 0, \quad v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $D_iu = \frac{\partial u}{\partial x_i}$, $D_sa_{ij}(x,u) = \frac{\partial}{\partial u}a_{ij}(x,u)$, $D_sb_{ij}(x,v) = \frac{\partial}{\partial v}b_{ij}(x,v)$, $\alpha > 2$, $\beta > 2$, $\alpha + \beta < 2 \cdot 2^*$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded smooth domain. This system includes the following special class of system with $a_{ij}(x,u) = (1 + u^2)\delta_{ij}$, $b_{ij}(x,v) = (1 + v^2)\delta_{ij}$, *i.e.*,

$$\begin{cases} -\Delta u - \frac{1}{2}u\Delta(u^2) = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^\beta u, & x \in \Omega, \\ -\Delta v - \frac{1}{2}v\Delta(v^2) = \frac{2\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}v, & x \in \Omega, \\ u = 0, \quad v = 0, & x \in \partial\Omega, \end{cases}$$

which is referred to as the so-called modified nonlinear Schrödinger system.

Our assumptions on the functions a_{ij} and b_{ij} are as follows.

(A_1) The functions $a_{ij} \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $b_{ij} \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$, $i, j = 1, 2, \ldots, N$.

(A_2) There exist constants a_0, a_1, b_0, b_1 satisfying $a_1 \geq a_0 > 0$, $b_1 \geq b_0 > 0$, $(\alpha + \beta - 2)a_0 > 2a_1$ and $(\alpha + \beta - 2)b_0 > 2b_1$ such that

$$a_0\bigl(1 + s^2\bigr)|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x,s)\xi_i\xi_j \leq a_1\bigl(1 + s^2\bigr)|\xi|^2,$$

$$b_0\bigl(1 + s^2\bigr)|\xi|^2 \leq \sum_{i,j=1}^{N} b_{ij}(x,s)\xi_i\xi_j \leq b_1\bigl(1 + s^2\bigr)|\xi|^2$$

for $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$.

(A_3)

$$0 \leq \sum_{i,j=1}^{N} D_s a_{ij}(x,s)s\xi_i\xi_j \leq 2\sum_{i,j=1}^{N} a_{ij}(x,s)\xi_i\xi_j,$$

$$0 \leq \sum_{i,j=1}^{N} D_s b_{ij}(x,s)s\xi_i\xi_j \leq 2\sum_{i,j=1}^{N} b_{ij}(x,s)\xi_i\xi_j$$

for $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$.

In recent years, much attention has been devoted to the quasilinear Schrödinger equation of the following form:

$$-\triangle u + \lambda V(x)u - k\triangle\bigl(u^2\bigr)u = |u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{1.2}$$

See, for example, [1] where Poppenberg *et al.* proved the existence of a positive ground state solution by using a constrained minimization argument. Using a change of variables, Liu *et al.* [2] used an Orlicz space to prove the existence of a soliton solution for equation (1.2) via the mountain pass theorem. Colin and Jeanjean [3] also made use of a change of variables but worked in the Sobolev space $H^1(\mathbb{R}^N)$. They proved the existence of a positive solution for equation (1.2) from the classical results given by Berestycki and Lions [4]. Liu *et al.* [5] established the existence of both one-sign and nodal ground states of soliton-type solutions for equation (1.2) by the Nehari method. By using the Nehari manifold method and the concentration compactness principle (see [6]) in the Orlicz space, Guo and Tang [7] considered the following quasilinear Schrödinger system:

$$\begin{cases} -\triangle u + (\lambda a(x) + 1)u - \frac{1}{2}(\triangle|u|^2)u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^{\beta}u, & x \in \mathbb{R}^N, \\ -\triangle u + (\lambda b(x) + 1)u - \frac{1}{2}(\triangle|u|^2)u = \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x \in \mathbb{R}^N, \\ u(x) \to 0, \qquad v(x) \to 0, & |x| \to \infty, \end{cases} \tag{1.3}$$

with $a(x) \geq 0$, $b(x) \geq 0$ having a potential well and $\alpha > 2$, $\beta > 2$, $\alpha + \beta < 2 \cdot 2^*$, and they proved the existence of a ground state solution for system (1.3) which localizes near the potential well int $a^{-1}(0)$ for λ large enough. Guo and Tang [8] considered also ground state solutions of the single quasilinear Schrödinger equation corresponding to system (1.3) by the same methods and obtained similar results. In particular, by the perturbation method, Liu *et al.* [9] considered the existence and multiplicity of solutions for the following quasilinear equation of the form

$$\begin{cases} \sum_{i,j=1}^{N} D_j(a_{ij}(x,u)D_iu) - \frac{1}{2}\sum_{i,j=1}^{N} D_s a_{ij}(x,u)D_iuD_ju + f(x,u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.4}$$

under suitable assumptions.

It is worth pointing out that the existence of one-bump or multi-bump bound state solutions for the related semilinear Schrödinger equation (1.2) for $k = 0$ has been extensively studied. One can see Bartsch and Wang [10], Ambrosetti *et al.* [11], Ambrosetti *et al.* [12], Byeon and Wang [13], Cingolani and Lazzo [14], Cingolani and Nolasco [15], Del Pino and Felmer [16, 17], Floer and Weinstein [18], Oh [19, 20] and the references therein.

Motivated by the single equation (1.4), the purpose of this paper is to study the existence of both positive and negative solutions for the coupled quasilinear system (1.1). We mainly follow the idea of Liu *et al.* [9] to perturb the functional and obtain our main results. We point out that the procedure to system (1.1) is not trivial at all. Since the appearance of the quasilinear terms $\sum_{i,j=1}^{N} D_j(a_{ij}(x,u)D_iu) - \frac{1}{2}\sum_{i,j=1}^{N} D_s a_{ij}(x,u)D_iuD_ju$ and $\sum_{i,j=1}^{N} D_j(b_{ij}(x,v)D_iv) - \frac{1}{2}\sum_{i,j=1}^{N} D_s b_{ij}(x,v)D_ivD_jv$, we need more delicate estimates.

The paper is organized as follows. In Section 2, we introduce a perturbation of the functional and give our main results (Theorem 2.1 and Theorem 2.2). In Section 3, we verify the Palais-Smale condition for the perturbed functional. Section 4 is devoted to some asymptotic behavior of the sequences $\{(u_n, v_n)\} \subset W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$ and $\{\mu_n\} \subset (0,1]$ satisfying some conditions. Finally, our main results will be proved in Section 5.

Throughout this paper, we will use the same C to denote various generic positive constants, and we will use $o(1)$ to denote quantities that tend to 0.

2 Perturbation of the functional and main results

In order to obtain the desired existence of solutions for system (1.1), in this section, we introduce a perturbation of the functional and give our main results.

The weak form of system (1.1) is

$$\begin{aligned} &\int_\Omega \sum_{i,j=1}^{N} a_{ij}(x,u)D_iuD_j\varphi + \frac{1}{2}\int_\Omega \sum_{i,j=1}^{N} D_s a_{ij}(x,u)D_iuD_ju\varphi \\ &\quad + \int_\Omega \sum_{i,j=1}^{N} b_{ij}(x,v)D_ivD_j\psi + \frac{1}{2}\int_\Omega \sum_{i,j=1}^{N} D_s b_{ij}(x,v)D_ivD_jv\psi \\ &\quad - \frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}|v|^\beta u\varphi - \frac{2\beta}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^{\beta-2}v\psi = 0 \end{aligned} \tag{2.1}$$

for all $(\varphi, \psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$, which is formally the variational formulation of the following functional:

$$I_0(u,v) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{N} a_{ij}(x,u)D_iuD_ju + \frac{1}{2}\int_\Omega \sum_{i,j=1}^{N} b_{ij}(x,v)D_ivD_jv - \frac{2}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^\beta. \tag{2.2}$$

We may define the derivative of I_0 at (u,v) in the direction of $(\varphi,\psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ as follows:

$$\begin{aligned}\langle I_0'(u,v),(\varphi,\psi)\rangle &= \int_\Omega \sum_{i,j=1}^N a_{ij}(x,u)D_iuD_j\varphi + \frac{1}{2}\int_\Omega \sum_{i,j=1}^N D_s a_{ij}(x,u)D_iuD_ju\varphi \\ &\quad + \int_\Omega \sum_{i,j=1}^N b_{ij}(x,v)D_ivD_j\psi + \frac{1}{2}\int_\Omega \sum_{i,j=1}^N D_s b_{ij}(x,v)D_ivD_jv\psi \\ &\quad - \frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}|v|^\beta u\varphi - \frac{2\beta}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^{\beta-2}v\psi. \end{aligned} \tag{2.3}$$

We call (u,v) a critical point of I_0 if $(u,v) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, $\int_\Omega u^2|\nabla u|^2 < \infty$, $\int_\Omega v^2|\nabla v|^2 < \infty$ and $\langle I_0'(u,v),(\varphi,\psi)\rangle = 0$ for all $(\varphi,\psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. That is, (u,v) is a weak solution for system (1.1).

When we consider system (1.1) by using the classical critical point theory, we encounter the difficulties due to the lack of an appropriate working space. In general, it seems that there is no suitable space in which the variational functional I_0 possesses both smoothness and compactness properties. For smoothness, one would need to work in a space smaller than $W_0^{1,2}(\Omega)$ to control the term involving the quasilinear term in system (1.1), but it seems impossible to obtain bounds for $(PS)_c$ sequence in this setting. Several ideas and approaches, such as minimizations [1, 21], the Nehari method [5] and change of variables [2, 3], have been used in recent years to overcome the difficulties. In this paper, we consider the perturbed functional

$$\begin{aligned} I_\mu(u,v) &= \frac{1}{4}\mu\int_\Omega \left(|\nabla u|^4 + |\nabla v|^4\right) + I_0(u,v) \\ &= \frac{1}{4}\mu\int_\Omega \left(|\nabla u|^4 + |\nabla v|^4\right) + \frac{1}{2}\int_\Omega \sum_{i,j=1}^N a_{ij}(x,u)D_iuD_ju \\ &\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^N b_{ij}(x,v)D_ivD_jv - \frac{2}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^\beta, \end{aligned} \tag{2.4}$$

where $\mu \in (0,1]$ is a parameter. Then it is easy to see that I_μ is a C^1-functional on $W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$. We can define also the derivative of I_μ at (u,v) in the direction of (φ,ψ) as follows:

$$\begin{aligned}\langle I_\mu'(u,v),(\varphi,\psi)\rangle &= \mu\int_\Omega |\nabla u|^2\nabla u\nabla\varphi + \mu\int_\Omega |\nabla v|^2\nabla v\nabla\psi \\ &\quad + \int_\Omega \sum_{i,j=1}^N a_{ij}(x,u)D_iuD_j\varphi + \frac{1}{2}\int_\Omega \sum_{i,j=1}^N D_s a_{ij}(x,u)D_iuD_ju\varphi \\ &\quad + \int_\Omega \sum_{i,j=1}^N b_{ij}(x,v)D_ivD_j\psi + \frac{1}{2}\int_\Omega \sum_{i,j=1}^N D_s b_{ij}(x,v)D_ivD_jv\psi \\ &\quad - \frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}|v|^\beta u\varphi - \frac{2\beta}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^{\beta-2}v\psi \end{aligned} \tag{2.5}$$

for all $(\varphi, \psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. The idea of this paper is to obtain the existence of the critical points of I_μ for $\mu > 0$ small and establish suitable estimates for the critical points as $\mu \to 0$ so that we may pass to the limit to get the solutions for the original system (1.1).

Our main results are as follows.

Theorem 2.1 *Assume that* (A_1)-(A_3) *hold*, $\alpha > 2$, $\beta > 2$ *and* $\alpha + \beta < 2 \cdot 2^*$. *Let* $\mu_n \to 0$ *and let* $\{(u_n, v_n)\} \subset W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$ *be a sequence of critical points of* I_{μ_n} *satisfying* $I'_{\mu_n}(u_n, v_n) = 0$ *and* $I_{\mu_n}(u_n, v_n) \leq C$ *for some* C *independent of* n. *Then, up to a subsequence,*

$$\begin{aligned}
&u_n \to u, \qquad v_n \to v \quad \text{in } W_0^{1,2}(\Omega),\\
&u_n \nabla u_n \to u \nabla u, \qquad v_n \nabla v_n \to v \nabla v \quad \text{in } L^2(\Omega),\\
&\mu_n \int_\Omega \left(|\nabla u_n|^4 + |\nabla v_n|^4\right) \to 0,\\
&I'_{\mu_n}(u_n, v_n) \to I'_0(u, v)
\end{aligned}$$

as $n \to \infty$, *and* (u, v) *is a critical point of* I_0.

Theorem 2.2 *Assume that* (A_1)-(A_3) *hold*, $\alpha > 2$, $\beta > 2$ *and* $\alpha + \beta < 2 \cdot 2^*$. *Then* I_μ *has a positive critical point* (u_μ, v_μ) *and a negative critical point* $(\widetilde{u}_\mu, \widetilde{v}_\mu)$, *and* (u_μ, v_μ) (*resp.*,$(\widetilde{u}_\mu, \widetilde{v}_\mu)$) *converges to a positive* (*resp., negative*) *solution for system* (1.1) *as* $\mu \to 0$.

Notation We denote by $\|\cdot\|$ the norm of $W_0^{1,4}(\Omega)$ and by $|\cdot|_s$ the norm of $L^s(\Omega)$ $(1 \leq s < +\infty)$.

3 Compactness of the perturbed functional

In this section, we verify the Palais-Smale condition ($(PS)_c$ condition in short) for the perturbed functional $I_\mu(u, v)$. We have the following proposition.

Proposition 3.1 *For* $\mu > 0$ *fixed, the functional* $I_\mu(u, v)$ *satisfies* $(PS)_c$ *condition for all* $c \in \mathbb{R}$. *That is, any sequence* $\{(u_n, v_n)\} \subset W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$ *satisfying, for* $c \in \mathbb{R}$,

$$I_\mu(u_n, v_n) \to c, \tag{3.1}$$

$$I'_\mu(u_n, v_n) \to 0 \quad \text{strongly in } \left(W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)\right)^* \tag{3.2}$$

has a strongly convergent subsequence in $W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$, *where* $(W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega))^*$ *is the dual space of* $W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$.

To give the proof of Proposition 3.1, we need the following lemma firstly.

Lemma 3.2 *Suppose that a sequence* $\{(u_n, v_n)\} \subset W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$ *satisfies* (3.1) *and* (3.2). *Then*

$$\limsup_{n\to\infty} \left\|(u_n, v_n)\right\|^4 \leq \left(\frac{1}{4} - \frac{1}{\alpha + \beta}\right)^{-1} \mu^{-1} c.$$

Proof It follows from (3.1) and (3.2) that

$$
\begin{aligned}
&c + o(1) - \frac{1}{\alpha+\beta} o(1)\big\|(u_n, v_n)\big\| \\
&\quad = I_\mu(u_n, v_n) - \frac{1}{\alpha+\beta}\big\langle I'_\mu(u_n, v_n), (u_n, v_n)\big\rangle \\
&\quad = \left(\frac{1}{4} - \frac{1}{\alpha+\beta}\right)\mu \int_\Omega \big(|\nabla u_n|^4 + |\nabla v_n|^4\big) \\
&\qquad + \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\int_\Omega \sum_{i,j=1}^N a_{ij}(x,u) D_i u D_j u + \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\int_\Omega \sum_{i,j=1}^N b_{ij}(x,v) D_i v D_j v \\
&\qquad - \frac{1}{2(\alpha+\beta)}\int_\Omega \sum_{i,j=1}^N D_s a_{ij}(x,u) D_i u D_j u - \frac{1}{2(\alpha+\beta)}\int_\Omega \sum_{i,j=1}^N D_s b_{ij}(x,v) D_i v D_j v \\
&\quad \ge \left(\frac{1}{4} - \frac{1}{\alpha+\beta}\right)\mu \int_\Omega \big(|\nabla u|^4 + |\nabla v|^4\big) + \frac{(\alpha+\beta-2)a_0 - 2a_1}{2(\alpha+\beta)}\int_\Omega \big(1+u_n^2\big)|\nabla u_n|^2 \\
&\qquad + \frac{(\alpha+\beta-2)b_0 - 2b_1}{2(\alpha+\beta)}\int_\Omega \big(1+v_n^2\big)|\nabla v_n|^2 \\
&\quad \ge \left(\frac{1}{4} - \frac{1}{\alpha+\beta}\right)\mu \int_\Omega \big(|\nabla u|^4 + |\nabla v|^4\big).
\end{aligned}
$$

Thus we have

$$
\limsup_{n\to\infty}\big\|(u_n, v_n)\big\|^4 \le \left(\frac{1}{4} - \frac{1}{\alpha+\beta}\right)^{-1}\mu^{-1}c.
$$

This completes the proof of Lemma 3.2. □

Now we give the proof of Proposition 3.1.

Proof of Proposition 3.1 From Lemma 3.2 , we know that $\{(u_n, v_n)\}$ is bounded in $W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$. So there exists a subsequence of $\{(u_n, v_n)\}$, still denoted by $\{(u_n, v_n)\}$, such that

$$
\begin{aligned}
&(u_n, v_n) \rightharpoonup (u, v) \quad \text{weakly in } W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega) \text{ as } n \to \infty, \\
&u_n \to u, \qquad v_n \to v \quad \text{strongly in } L^s(\Omega) \text{ as } n \to \infty \text{ for any } 2 < s < 2 \cdot 2^*.
\end{aligned}
$$

Now we prove that $(u_n, v_n) \to (u, v)$ in $W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$. In (2.5), choosing $(\varphi, \psi) = (u_n - u_m, v_n - v_m)$, we have

$$
\begin{aligned}
&o(1)\big\|(u_n - u_m, v_n - v_m)\big\| \\
&\quad = \big\langle I'_\mu(u_n, v_n) - I'_\mu(u_m, v_m), (u_n - u_m, v_n - v_m)\big\rangle \\
&\quad = \mu \int_\Omega \big(|\nabla u_n|^2 \nabla u_n - |\nabla u_m|^2 \nabla u_m\big)(\nabla u_n - \nabla u_m) \\
&\qquad + \mu \int_\Omega \big(|\nabla v_n|^2 \nabla v_n - |\nabla v_m|^2 \nabla v_m\big)(\nabla v_n - \nabla v_m) \\
&\qquad + \int_\Omega \sum_{i,j=1}^N \big(a_{ij}(x,u_n) D_i u_n - a_{ij}(x,u_m) D_i u_m\big)(D_j u_n - D_j u_m)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_\Omega \sum_{i,j=1}^N \big(b_{ij}(x,v_n)D_iv_n - b_{ij}(x,v_m)D_iv_m\big)(D_jv_n - D_jv_m) \\
&+\frac{1}{2}\int_\Omega \sum_{i,j=1}^N \big(D_sa_{ij}(x,u_n)D_iu_nD_ju_n - D_sa_{ij}(x,u_m)D_iu_mD_ju_m\big)(u_n-u_m) \\
&+\frac{1}{2}\int_\Omega \sum_{i,j=1}^N \big(D_sb_{ij}(x,v_n)D_iv_nD_jv_n - D_sb_{ij}(x,v_m)D_iv_mD_jv_m\big)(v_n-v_m) \\
&-\frac{2\alpha}{\alpha+\beta}\int_\Omega \big(|u_n|^{\alpha-2}|v_n|^\beta u_n - |u_m|^{\alpha-2}|v_m|^\beta u_m\big)(u_n-u_m) \\
&-\frac{2\beta}{\alpha+\beta}\int_\Omega \big(|u_n|^{\alpha}|v_n|^{\beta-2} v_n - |u_m|^{\alpha}|v_m|^{\beta-2} v_m\big)(v_n-v_m).
\end{aligned}
\tag{3.3}
$$

We may estimate the terms involved as follows:

$$
\begin{aligned}
&\mu\int_\Omega \big(|\nabla u_n|^2\nabla u_n - |\nabla u_m|^2\nabla u_m\big)(\nabla u_n - \nabla u_m) \ge \frac{1}{4}\mu\int_\Omega |\nabla u_n - \nabla u_m|^4, \\
&\mu\int_\Omega \big(|\nabla v_n|^2\nabla v_n - |\nabla v_m|^2\nabla v_m\big)(\nabla v_n - \nabla v_m) \ge \frac{1}{4}\mu\int_\Omega |\nabla v_n - \nabla v_m|^4, \\
&\int_\Omega \sum_{i,j=1}^N \big(a_{ij}(x,u_n)D_iu_n - a_{ij}(x,u_m)D_iu_m\big)(D_ju_n - D_ju_m) \\
&\quad = \int_\Omega \sum_{i,j=1}^N a_{ij}(x,u_n)D_i(u_n-u_m)D_j(u_n-u_m) \\
&\qquad + \int_\Omega \sum_{i,j=1}^N \big(a_{ij}(x,u_n) - a_{ij}(x,u_m)\big)D_iu_mD_j(u_n-u_m) \\
&\quad \ge \int_\Omega \sum_{i,j=1}^N D_sa_{ij}\big(x,tu_n+(1-t)u_m\big)(u_n-u_m)D_iu_mD_j(u_n-u_m) \\
&\quad \ge -C|u_n-u_m|_4\|u_m\|\big(\|u_n\|+\|u_m\|\big) \\
&\quad \to 0 \quad \text{as } m,n\to\infty \text{ for some } t\in(0,1), \\
&\int_\Omega \sum_{i,j=1}^N \big(b_{ij}(x,v_n)D_iv_n - b_{ij}(x,v_m)D_iv_m\big)(D_jv_n - D_jv_m) \\
&\quad = \int_\Omega \sum_{i,j=1}^N b_{ij}(x,v_n)D_i(v_n-v_m)D_j(v_n-v_m) \\
&\qquad + \int_\Omega \sum_{i,j=1}^N \big(b_{ij}(x,v_n) - b_{ij}(x,v_m)\big)D_iv_mD_j(v_n-v_m) \\
&\quad \ge \int_\Omega \sum_{i,j=1}^N D_sb_{ij}\big(x,\tau v_n+(1-\tau)v_m\big)(v_n-v_m)D_iv_mD_j(v_n-v_m) \\
&\quad \ge -C|v_n-v_m|_4\|v_m\|\big(\|v_n\|+\|v_m\|\big) \\
&\quad \to 0 \quad \text{as } m,n\to\infty \text{ for some } \tau\in(0,1),
\end{aligned}
$$

$$\begin{aligned}
&\frac{1}{2}\left|\int_\Omega \sum_{i,j=1}^{N}\big(D_s a_{ij}(x,u_n)D_iu_nD_ju_n - D_s a_{ij}(x,u_m)D_iu_mD_ju_m\big)(u_n-u_m)\right| \\
&\quad\le \frac{1}{2}\int_\Omega \sum_{i,j=1}^{N}\big|D_s a_{ij}(x,u_n)D_iu_nD_ju_n(u_n-u_m)\big| \\
&\qquad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^{N}\big|D_s a_{ij}(x,u_n)D_iu_nD_ju_n(u_n-u_m)\big| \\
&\quad\le C\big(\|u_n\|^2+\|u_m\|^2\big)|u_n-u_m|_4 \\
&\quad\to 0 \quad \text{as } m,n\to\infty, \\
&\frac{1}{2}\left|\int_\Omega \sum_{i,j=1}^{N}\big(D_s b_{ij}(x,v_n)D_iv_nD_jv_n - D_s b_{ij}(x,v_m)D_iv_mD_jv_m\big)(v_n-v_m)\right| \\
&\quad\le \frac{1}{2}\int_\Omega \sum_{i,j=1}^{N}\big|D_s b_{ij}(x,v_n)D_iv_nD_jv_n(v_n-v_m)\big| \\
&\qquad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^{N}\big|D_s b_{ij}(x,v_n)D_iv_nD_jv_n(v_n-v_m)\big| \\
&\quad\le C\big(\|v_n\|^2+\|v_m\|^2\big)|v_n-v_m|_4 \\
&\quad\to 0 \quad \text{as } m,n\to\infty, \\
&\frac{2\alpha}{\alpha+\beta}\left|\int_\Omega \big(|u_n|^{\alpha-2}|v_n|^\beta u_n - |u_m|^{\alpha-2}|v_m|^\beta u_m\big)(u_n-u_m)\right| \\
&\quad\le \frac{2\alpha}{\alpha+\beta}\int_\Omega \big(|u_n|^{\alpha-1}|v_n|^\beta + |u_m|^{\alpha-1}|v_m|^\beta\big)|u_n-u_m| \\
&\quad\le \frac{2\alpha}{\alpha+\beta}\big(|u_n|_{\alpha+\beta}^{\alpha-1}|v_n|_{\alpha+\beta}^{\beta} + |u_m|_{\alpha+\beta}^{\alpha-1}|v_m|_{\alpha+\beta}^{\beta}\big)|u_n-u_m|_{\alpha+\beta} \\
&\quad\to 0 \quad \text{as } m,n\to\infty, \\
&\frac{2\beta}{\alpha+\beta}\left|\int_\Omega \big(|u_n|^{\alpha}|v_n|^{\beta-2} v_n - |u_m|^{\alpha}|v_m|^{\beta-2} v_m\big)(v_n-v_m)\right| \\
&\quad\le \frac{2\beta}{\alpha+\beta}\int_\Omega \big(|u_n|^{\alpha}|v_n|^{\beta-1} + |u_m|^{\alpha}|v_m|^{\beta-1}\big)|v_n-v_m| \\
&\quad\le \frac{2\beta}{\alpha+\beta}\big(|u_n|_{\alpha+\beta}^{\alpha}|v_n|_{\alpha+\beta}^{\beta-1} + |u_m|_{\alpha+\beta}^{\alpha}|v_m|_{\alpha+\beta}^{\beta-1}\big)|v_n-v_m|_{\alpha+\beta} \\
&\quad\to 0 \quad \text{as } m,n\to\infty.
\end{aligned}$$

Returning to (3.3), we have

$$\frac{1}{4}\mu\int_\Omega \big(|\nabla u_n-\nabla u_m|^4 + |\nabla v_n-\nabla v_m|^4\big) \le o(1)\big\|(u_n-u_m,v_n-v_m)\big\| + o(1),$$

which implies that $\|(u_n-u_m,v_n-v_m)\|\to 0$, *i.e.*, $(u_n,u_m)\to(u,v)$ in $W_0^{1,4}(\Omega)\times W_0^{1,4}(\Omega)$. This completes the proof of Proposition 3.1. □

4 Some asymptotic behavior

Proposition 3.1 enables us to apply minimax argument to the functional $I_\mu(u,v)$. In this section, we also study the behavior of the sequences $\{(u_n,v_n)\}\subset W_0^{1,4}(\Omega)\times W_0^{1,4}(\Omega)$ and $\{\mu_n\}\subset(0,1]$ satisfying

$$\mu_n\to 0, \tag{4.1}$$

$$I_{\mu_n}(u_n,v_n)\to c, \tag{4.2}$$

$$\left\|I'_{\mu_n}(u_n,v_n)\right\|^*\to 0. \tag{4.3}$$

The following proposition is the key of this section.

Proposition 4.1 *Assume that the sequences* $\{(u_n,v_n)\}\subset W_0^{1,4}(\Omega)\times W_0^{1,4}(\Omega)$ *and* $\{\mu_n\}\subset(0,1]$ *satisfy* (4.1)-(4.3). *Then, after extracting a sequence, still denoted by n, we have*

$$(u_n,v_n)\rightharpoonup(u,v)\quad\text{in } W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega),$$
$$(u_n\nabla u_n,v_n\nabla v_n)\rightharpoonup(u\nabla u,v\nabla v)\quad\text{in } L^2(\Omega)\times L^2(\Omega)$$

and

$$\bigl(u_n(x),v_n(x)\bigr)\to\bigl(u(x),v(x)\bigr)\quad\text{a.e. } x\in\Omega$$

as $n\to\infty$.

Proof Similar to the proof of Lemma 3.2, by (4.1)-(4.3), we have

$$\begin{aligned}C&\ge I_{\mu_n}(u_n,v_n)-\frac{1}{\alpha+\beta}\bigl\langle I'_{\mu_n}(u_n,v_n),(u_n,v_n)\bigr\rangle\\&\ge\left(\frac14-\frac{1}{\alpha+\beta}\right)\mu_n\int_\Omega\bigl(|\nabla u_n|^4+|\nabla v_n|^4\bigr)\\&\quad+\frac{(\alpha+\beta-2)a_0-2a_1}{2(\alpha+\beta)}\int_\Omega\bigl(1+u_n^2\bigr)|\nabla u_n|^2\\&\quad+\frac{(\alpha+\beta-2)b_0-2b_1}{2(\alpha+\beta)}\int_\Omega\bigl(1+v_n^2\bigr)|\nabla v_n|^2.\end{aligned}\tag{4.4}$$

Thus

$$\mu_n\int_\Omega\bigl(|\nabla u_n|^4+|\nabla v_n|^4\bigr)+\int_\Omega\bigl(|\nabla u_n|^2+|\nabla v_n|^2\bigr)+\int_\Omega\bigl(u_n^2|\nabla u_n|^2+v_n^2|\nabla v_n|^2\bigr)\le C \tag{4.5}$$

for some C independent of n. Then, up to a subsequence, we have

$$(u_n,v_n)\rightharpoonup(u,v)\quad\text{in } W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega),$$
$$(u_n\nabla u_n,v_n\nabla v_n)\rightharpoonup(u\nabla u,v\nabla v)\quad\text{in } L^2(\Omega)\times L^2(\Omega)$$

and

$$\bigl(u_n(x),v_n(x)\bigr)\to\bigl(u(x),v(x)\bigr)\quad\text{a.e. } x\in\Omega$$

as $n\to\infty$. This completes the proof of Proposition 4.1. □

5 Proof of main results

In this section, we give the proof of our main results. Firstly, we prove Theorem 2.1.

Proof of Theorem 2.1 Note that (u_n, v_n) satisfies the following equation:

$$\begin{aligned}
&\mu_n \int_\Omega |\nabla u_n|^2 \nabla u_n \nabla \varphi + \mu_n \int_\Omega |\nabla v_n|^2 \nabla v_n \nabla \psi \\
&\quad + \int_\Omega \sum_{i,j=1}^N a_{ij}(x, u_n) D_i u_n D_j \varphi + \frac{1}{2} \int_\Omega \sum_{i,j=1}^N D_s a_{ij}(x, u_n) D_i u_n D_j u_n \varphi \\
&\quad + \int_\Omega \sum_{i,j=1}^N b_{ij}(x, v_n) D_i v_n D_j \psi + \frac{1}{2} \int_\Omega \sum_{i,j=1}^N D_s b_{ij}(x, v_n) D_i v_n D_j v_n \psi \\
&\quad - \frac{2\alpha}{\alpha+\beta} \int_\Omega |u_n|^{\alpha-2} |v_n|^\beta u_n \varphi - \frac{2\beta}{\alpha+\beta} \int_\Omega |u_n|^\alpha |v_n|^{\beta-2} v_n \psi = 0
\end{aligned} \tag{5.1}$$

for all $(\varphi, \psi) \in W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$. Since

$$\left(\int_\Omega |u_n|^{\frac{4N}{N-2}} \right)^{\frac{N-2}{N}} \le C \int_\Omega \sum_{i,j=1}^N a_{ij}(x, u_n) D_i u_n D_j u_n \le C$$

and

$$\left(\int_\Omega |v_n|^{\frac{4N}{N-2}} \right)^{\frac{N-2}{N}} \le C \int_\Omega \sum_{i,j=1}^N b_{ij}(x, v_n) D_i v_n D_j v_n \le C.$$

By Moser's iteration, we have

$$|u_n|_{L^\infty} \le C, \qquad |v_n|_{L^\infty} \le C. \tag{5.2}$$

Hence

$$|u|_{L^\infty} \le C, \qquad |v|_{L^\infty} \le C \tag{5.3}$$

for some C independent of n. To show that (u, v) is a critical point of I_0, we use some arguments in [22, 23] (see more references therein). In (5.1), we choose $\varphi = \xi \exp(-Mu_n)$, $\psi = \eta \exp(-Mv_n)$, where $\xi \in C_0^\infty(\Omega)$, $\xi \ge 0$, $\eta \in C_0^\infty(\Omega)$, $\eta \ge 0$ and $M > 0$ is a constant. Substituting (φ, ψ) into (5.1), we have

$$\begin{aligned}
0 &= \mu_n \int_\Omega |\nabla u_n|^2 \nabla u_n \bigl(\nabla \xi \exp(-Mu_n) - \xi \nabla u_n \exp(-Mu_n) \bigr) \\
&\quad + \mu_n \int_\Omega |\nabla v_n|^2 \nabla v_n \bigl(\nabla \eta \exp(-Mv_n) - \eta \nabla v_n \exp(-Mv_n) \bigr) \\
&\quad + \int_\Omega \sum_{i,j=1}^N a_{ij}(x, u_n) D_i u_n \bigl(D_j \xi \exp(-Mu_n) - M \xi D_j u_n \exp(-Mu_n) \bigr) \\
&\quad + \int_\Omega \sum_{i,j=1}^N b_{ij}(x, v_n) D_i v_n \bigl(D_j \eta \exp(-Mv_n) - M \eta D_j v_n \exp(-Mv_n) \bigr)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_\Omega \sum_{i,j=1}^N D_s a_{ij}(x,u_n)D_iu_nD_ju_n\xi\exp(-Mu_n)\\
&+\frac{1}{2}\int_\Omega \sum_{i,j=1}^N D_s b_{ij}(x,v_n)D_iv_nD_jv_n\eta\exp(-Mv_n)\\
&-\frac{2\alpha}{\alpha+\beta}\int_\Omega |u_n|^{\alpha-2}|v_n|^\beta u_n\xi\exp(-Mu_n)-\frac{2\beta}{\alpha+\beta}\int_\Omega |u_n|^\alpha|v_n|^{\beta-2}v_n\eta\exp(-Mv_n)\\
\le{}& \mu_n\int_\Omega |\nabla u_n|^2\nabla u_n\nabla\xi\exp(-Mu_n)+\mu_n\int_\Omega |\nabla v_n|^2\nabla v_n\nabla\eta\exp(-Mv_n)\\
&+\int_\Omega\sum_{i,j=1}^N a_{ij}(x,u_n)D_iu_nD_j\xi\exp(-Mu_n)+\int_\Omega\sum_{i,j=1}^N b_{ij}(x,v_n)D_iv_nD_j\eta\exp(-Mv_n)\\
&-\int_\Omega\sum_{i,j=1}^N\Big(Ma_{ij}(x,u_n)-\frac{1}{2}D_sa_{ij}(x,u_n)\Big)D_iu_nD_ju_n\xi\exp(-Mu_n)\\
&-\int_\Omega\sum_{i,j=1}^N\Big(Mb_{ij}(x,v_n)-\frac{1}{2}D_sb_{ij}(x,v_n)\Big)D_iv_nD_jv_n\eta\exp(-Mv_n)\\
&-\frac{2\alpha}{\alpha+\beta}\int_\Omega |u_n|^{\alpha-2}|v_n|^\beta u_n\xi\exp(-Mu_n)\\
&-\frac{2\beta}{\alpha+\beta}\int_\Omega |u_n|^\alpha|v_n|^{\beta-2}v_n\eta\exp(-Mv_n).
\end{aligned}
\tag{5.4}
$$

Note that $Ma_{ij}(x,u_n)-\frac{1}{2}D_sa_{ij}(x,u_n)$, $Mb_{ij}(x,v_n)-\frac{1}{2}D_sb_{ij}(x,v_n)$ are positive for M large enough. By Fatou's lemma, the weak convergence of $\{(u_n,v_n)\}$ and the fact that $\mu_n\int_\Omega(|\nabla u_n|^4+|\nabla v_n|^4)$ is bounded, we have

$$
\begin{aligned}
0\le{}&\int_\Omega\sum_{i,j=1}^N a_{ij}(x,u)D_iuD_j\xi\exp(-Mu)+\int_\Omega\sum_{i,j=1}^N b_{ij}(x,v)D_ivD_j\eta\exp(-Mv)\\
&-\int_\Omega\sum_{i,j=1}^N\Big(Ma_{ij}(x,u)-\frac{1}{2}D_sa_{ij}(x,u)\Big)D_iuD_ju\xi\exp(-Mu)\\
&-\int_\Omega\sum_{i,j=1}^N\Big(Mb_{ij}(x,v)-\frac{1}{2}D_sb_{ij}(x,v)\Big)D_ivD_jv\eta\exp(-Mv)\\
&-\frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}|v|^\beta u\xi\exp(-Mu)-\frac{2\beta}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^{\beta-2}v\eta\exp(-Mv)\\
={}&\int_\Omega\sum_{i,j=1}^N a_{ij}(x,u)D_iuD_j\big(\xi\exp(-Mu)\big)+\int_\Omega\sum_{i,j=1}^N b_{ij}(x,u)D_ivD_j\big(\eta\exp(-Mv)\big)\\
&+\frac{1}{2}\int_\Omega\sum_{i,j=1}^N D_sa_{ij}(x,u)D_iuD_ju\xi\exp(-Mu)\\
&+\frac{1}{2}\int_\Omega\sum_{i,j=1}^N D_sb_{ij}(x,v)D_ivD_jv\eta\exp(-Mv)\\
&-\frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}|v|^\beta u\xi\exp(-Mu)-\frac{2\beta}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^{\beta-2}v\eta\exp(-Mv).
\end{aligned}
\tag{5.5}
$$

Let $(\chi,\omega)\geq(0,0)$, $(\chi,\omega)\in C_0^\infty(\Omega)\times C_0^\infty(\Omega)$. We may choose $\xi=\chi\exp(Mu)$, $\eta=\omega\exp(Mv)$ such that $(\xi,\eta)\in W_0^{1,4}(\Omega)\times W_0^{1,4}(\Omega)$, $|\xi|_{L^\infty(\Omega)}\leq C$ and $|\eta|_{L^\infty(\Omega)}\leq C$. Then we obtain

$$\begin{aligned}&\int_\Omega\sum_{i,j=1}^N a_{ij}(x,u)D_iuD_j\chi+\int_\Omega\sum_{i,j=1}^N b_{ij}(x,v)D_ivD_j\omega\\&\quad+\frac{1}{2}\int_\Omega\sum_{i,j=1}^N D_sa_{ij}(x,u)D_iuD_ju\chi+\frac{1}{2}\int_\Omega\sum_{i,j=1}^N D_sb_{ij}(x,v)D_ivD_jv\omega\\&\quad-\frac{2\alpha}{\alpha+\beta}\int_\Omega|u|^{\alpha-2}|v|^\beta u\chi-\frac{2\beta}{\alpha+\beta}\int_\Omega|u|^\alpha|v|^{\beta-2}v\omega\geq 0\end{aligned}\tag{5.6}$$

for all $(\chi,\omega)\geq(0,0)$, $(\chi,\omega)\in C_0^\infty(\Omega)\times C_0^\infty(\Omega)$.

Similarly, we may obtain an opposite inequality. Thus we have

$$\begin{aligned}&\int_\Omega\sum_{i,j=1}^N a_{ij}(x,u)D_iuD_j\chi+\int_\Omega\sum_{i,j=1}^N b_{ij}(x,v)D_ivD_j\omega\\&\quad+\frac{1}{2}\int_\Omega\sum_{i,j=1}^N D_sa_{ij}(x,u)D_iuD_ju\chi+\frac{1}{2}\int_\Omega\sum_{i,j=1}^N D_sb_{ij}(x,v)D_ivD_jv\omega\\&\quad-\frac{2\alpha}{\alpha+\beta}\int_\Omega|u|^{\alpha-2}|v|^\beta u\chi-\frac{2\beta}{\alpha+\beta}\int_\Omega|u|^\alpha|v|^{\beta-2}v\omega=0\end{aligned}\tag{5.7}$$

for all $(\chi,\omega)\in C_0^\infty(\Omega)\times C_0^\infty(\Omega)$. That is, (u,v) is a critical point of I_0 and a solution for system (1.1). By doing approximations, we have (u,v) in the place of (χ,ω) of (5.7)

$$\begin{aligned}&\int_\Omega\sum_{i,j=1}^N a_{ij}(x,u)D_iuD_ju+\int_\Omega\sum_{i,j=1}^N b_{ij}(x,v)D_ivD_jv\\&\quad+\frac{1}{2}\int_\Omega\sum_{i,j=1}^N D_sa_{ij}(x,u)uD_iuD_ju\\&\quad+\frac{1}{2}\int_\Omega\sum_{i,j=1}^N D_sb_{ij}(x,v)vD_ivD_jv-2\int_\Omega|u|^\alpha|v|^\beta=0.\end{aligned}\tag{5.8}$$

Setting $(\varphi,\psi)=(u_n,v_n)$ in (5.1), we have

$$\begin{aligned}&\mu_n\int_\Omega\left(|\nabla u_n|^4+|\nabla v_n|^4\right)+\int_\Omega\sum_{i,j=1}^N a_{ij}(x,u_n)D_iu_nD_ju_n\\&\quad+\int_\Omega\sum_{i,j=1}^N b_{ij}(x,v_n)D_iv_nD_jv_n+\frac{1}{2}\int_\Omega\sum_{i,j=1}^N D_sa_{ij}(x,u_n)u_nD_iu_nD_ju_n\\&\quad+\frac{1}{2}\int_\Omega\sum_{i,j=1}^N D_sb_{ij}(x,v_n)v_nD_iv_nD_jv_n-2\int_\Omega|u_n|^\alpha|v_n|^\beta=0.\end{aligned}\tag{5.9}$$

Using $\int_\Omega |u_n|^\alpha |v_n|^\beta \to \int_\Omega |u|^\alpha |v|^\beta$ as $n \to \infty$, (5.8), (5.9) and lower semi-continuity, we obtain

$$\begin{aligned}
&\mu_n \int_\Omega \left(|\nabla u_n|^4 + |\nabla v_n|^4\right) \to 0, \\
&\int_\Omega \sum_{i,j=1}^N a_{ij}(x, u_n) D_i u_n D_j u_n \to \int_\Omega \sum_{i,j=1}^N a_{ij}(x, u) D_i u D_j u, \\
&\int_\Omega \sum_{i,j=1}^N b_{ij}(x, v_n) D_i v_n D_j v_n \to \int_\Omega \sum_{i,j=1}^N b_{ij}(x, v) D_i v D_j v
\end{aligned}$$

as $n \to \infty$.

In particular, we have

$$\begin{aligned}
&u_n \to u, \qquad v_n \to v \quad \text{in } W_0^{1,2}(\Omega), \\
&u_n \nabla u_n \to u \nabla u, \qquad v_n \nabla v_n \to v \nabla v \quad \text{in } L^2(\Omega)
\end{aligned}$$

and

$$I'_{\mu_n}(u_n, v_n) \to I'_0(u, v)$$

as $n \to \infty$. This completes the proof of Theorem 2.1. □

Next, we apply the mountain pass theorem to obtain the existence of critical points of I_μ. Set

$$\Sigma_\rho = \left\{ (u, v) \in W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega) \;\middle|\; \int_\Omega \sum_{i,j=1}^N a_{ij}(x, u) D_i u D_j u + \int_\Omega \sum_{i,j=1}^N b_{ij}(x, v) D_i v D_j v \le \rho^2 \right\}$$

for $\rho > 0$.

Let us consider the functional

$$\begin{aligned}
I_\mu^+(u, v) = {}& \frac{1}{4}\mu \int_\Omega \left(|\nabla u|^4 + |\nabla v|^4\right) + \frac{1}{2} \int_\Omega \sum_{i,j=1}^N a_{ij}(x, u) D_i u D_j u \\
&+ \frac{1}{2} \int_\Omega \sum_{i,j=1}^N b_{ij}(x, v) D_i v D_j v - \frac{2}{\alpha + \beta} \int_\Omega \left(u^+\right)^\alpha \left(v^+\right)^\beta.
\end{aligned} \tag{5.10}$$

Here and in what follows, we denote $u^+ = \max\{u, 0\}$. The functional I_μ satisfies $(PS)_c$ condition. Similarly, we may verify that I_μ^+ satisfies $(PS)_c$ condition. By the ε-Young inequality, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left(u^+\right)^\alpha \left(v^+\right)^\beta \le \varepsilon \left(u^+\right)^{\alpha+\beta} + C_\varepsilon \left(v^+\right)^{\alpha+\beta}$$

and

$$\int_\Omega |u|^{\alpha+\beta} \le C\left(\int_\Omega u^2|\nabla u|^2\right)^{\frac{\alpha+\beta}{4}} \le C\left(\int_\Omega \sum_{i,j=1}^N a_{ij}(x,u)D_iuD_ju\right)^{\frac{\alpha+\beta}{4}},$$

$$\int_\Omega |v|^{\alpha+\beta} \le C\left(\int_\Omega v^2|\nabla v|^2\right)^{\frac{\alpha+\beta}{4}} \le C\left(\int_\Omega \sum_{i,j=1}^N b_{ij}(x,v)D_ivD_jv\right)^{\frac{\alpha+\beta}{4}}.$$

Then

$$\begin{aligned}
&-\frac{2}{\alpha+\beta}\int_\Omega (u^+)^\alpha (v^+)^\beta \\
&\quad\ge -\frac{2}{\alpha+\beta}\varepsilon\int_\Omega (u^+)^{\alpha+\beta} - \frac{2}{\alpha+\beta}C_\varepsilon\int_\Omega (u^+)^{\alpha+\beta} \\
&\quad\ge -\frac{2C}{\alpha+\beta}\varepsilon\left(\int_\Omega \sum_{i,j=1}^N a_{ij}(x,u)D_iuD_ju\right)^{\frac{\alpha+\beta}{4}} - \frac{2C_\varepsilon}{\alpha+\beta}\left(\int_\Omega \sum_{i,j=1}^N b_{ij}(x,v)D_ivD_jv\right)^{\frac{\alpha+\beta}{4}} \\
&\quad\ge -\frac{2C}{\alpha+\beta}\varepsilon\rho^{\frac{\alpha+\beta}{2}} - \frac{2C_\varepsilon}{\alpha+\beta}\rho^{\frac{\alpha+\beta}{2}} \\
&\quad\ge -\frac{1}{\alpha+\beta}\rho^2
\end{aligned}$$

for ε, ρ small. Thus we have

$$\begin{aligned}
I_\mu^+(u,v) &\ge \frac{1}{2}\int_\Omega \sum_{i,j=1}^N a_{ij}(x,u)D_iuD_ju + \frac{1}{2}\int_\Omega \sum_{i,j=1}^N b_{ij}(x,v)D_ivD_jv - \frac{2}{\alpha+\beta}\int_\Omega (u^+)^\alpha (v^+)^\beta \\
&\ge \frac{1}{2}\rho^2 - \frac{1}{\alpha+\beta}\rho^2 = \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\rho^2
\end{aligned}$$

for $(u,v) \in \partial\Sigma_\rho$ and for $\rho > 0$ small enough. Choose $(\varphi,\psi) \ge (0,0)$, $(\chi,\omega) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ and $T > 0$. Define a path $(g,h) : [0,1] \to W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$ by $(g(t),h(t)) = (tT\varphi, tT\psi)$. When T is large enough, we have

$$\begin{aligned}
&I_\mu^+\big(g(1),h(1)\big) < 0, \\
&\int_\Omega \sum_{i,j=1}^N a_{ij}\big(x,g(1)\big)D_ig(1)D_jg(1) + \int_\Omega \sum_{i,j=1}^N b_{ij}\big(x,h(1)\big)D_ih(1)D_jh(1) > \rho^2
\end{aligned}$$

and

$$\sup_{t\in[0,1]} I_\mu^+\big(g(t),h(t)\big) \le m$$

for some m independent of $\mu \in (0,1]$.

Define

$$c_\mu = \inf_{(g,h)\in\Gamma}\sup_{t\in[0,1]} I_\mu^+\big(g(t),h(t)\big),$$

where

$$\Gamma = \big\{(g,h) \in C\big([0,1], W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)\big) \big| \\ \big(g(0),h(0)\big) = (0,0), \big(g(1),h(1)\big) = (T\varphi, T\psi)\big\}.$$

From the mountain pass theorem we obtain that

$$c_\mu \geq \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\rho^2$$

is a critical value of I_μ^+.

Let (u_μ, v_μ) be a critical point corresponding to c_μ. We have $(u_\mu, v_\mu) \geq (0,0)$. Thus (u_μ, v_μ) is a positive critical point of I_μ by the strong maximum principle. In summary, we have the following.

Proposition 5.1 *There exist positive constants ρ and m independent of μ such that I_μ has a positive critical point (u_μ, v_μ) satisfying*

$$\left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\rho^2 \leq I_\mu(u_\mu, v_\mu) \leq m.$$

Finally, we give the proof of Theorem 2.2.

Proof of Theorem 2.2 For a positive solution of system (1.1), the proof follows from Proposition 5.1 and Theorem 2.1. A similar argument gives a negative solution of system (1.1). This completes the proof of Theorem 2.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors were involved in carrying out this study. All authors read and approved the final manuscript.

Author details
[1]College of Mathematics and Computer Science, Northwest University for Nationalities, Lanzhou, 730124, P.R. China. [2]College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, P.R. China. [3]School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, P.R. China.

Acknowledgements
This paper was finished while the first author was a visiting fellow at the School of Mathematical Sciences of Beijing Normal University, and the first author would like to express her gratitude for their hospitality during her visit. This work is supported by the National Science Foundation of China (11061031), Fundamental Research Funds for the Central Universities (31920130004) and Fundamental Research Funds for the Gansu University.

Received: 28 May 2013 Accepted: 28 August 2013 Published: 07 Nov 2013

References
1. Poppenberg, M, Schmitt, K, Wang, Z: On the existence of soliton solutions to quasilinear Schrödinger equations. Calc. Var. Partial Differ. Equ. **14**, 329-344 (2002)
2. Liu, J, Wang, Y, Wang, Z: Soliton solutions for quasilinear Schrödinger equation, II. J. Differ. Equ. **187**, 473-493 (2003)
3. Colin, M, Jeanjean, L: Solutions for a quasilinear Schrödinger equation: a dual approach. Nonlinear Anal. **56**, 213-226 (2004)
4. Berestycki, H, Lions, PL: Nonlinear scalar field equations, I. Arch. Ration. Mech. Anal. **82**, 313-346 (1983)
5. Liu, J, Wang, Y, Wang, Z: Solutions for quasilinear Schrödinger equations via the Nehari method. Commun. Partial Differ. Equ. **29**, 879-901 (2004)
6. Lions, PL: The concentration-compactness principle in the calculus of variations. The locally compact case. Part I. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **1**, 109-145 (1984)

7. Guo, Y, Tang, Z: Ground state solutions for the quasilinear Schrödinger systems. J. Math. Anal. Appl. **389**, 322-339 (2012)
8. Guo, Y, Tang, Z: Ground state solutions for the quasilinear Schrödinger equation. Nonlinear Anal. **75**, 3235-3248 (2012)
9. Liu, X, Liu, J, Wang, Z: Quasilinear elliptic equations via perturbation method. Proc. Am. Math. Soc. **141**, 253-263 (2013)
10. Bartsch, T, Wang, Z: Multiple positive solutions for a nonlinear Schrödinger equation. Z. Angew. Math. Phys. **51**, 366-384 (2000)
11. Ambrosetti, A, Badiale, M, Cingolani, S: Semiclassical states of nonlinear Schrödinger equations. Arch. Ration. Mech. Anal. **140**, 285-300 (1997)
12. Ambrosetti, A, Malchiodi, A, Secchi, S: Multiplicity results for some nonlinear Schrödinger equations with potentials. Arch. Ration. Mech. Anal. **159**, 253-271 (2001)
13. Byeon, J, Wang, Z: Standing waves with a critical frequency for nonlinear Schrödinger equations, II. Calc. Var. Partial Differ. Equ. **18**, 207-219 (2003)
14. Cingolani, S, Lazzo, M: Multiple positive solutions to nonlinear Schrödinger equations with competing potential functions. J. Differ. Equ. **160**, 118-138 (2000)
15. Cingolani, S, Nolasco, M: Multi-peaks periodic semiclassical states for a class of nonlinear Schrödinger equations. Proc. R. Soc. Edinb. **128**, 1249-1260 (1998)
16. Del Pino, M, Felmer, P: Semi-classical states for nonlinear Schrödinger equations. Ann. Inst. Henri Poincaré **15**, 127-149 (1998)
17. Del Pino, M, Felmer, P: Multi-peak bound states for nonlinear Schrödinger equations. J. Funct. Anal. **149**, 245-265 (1997)
18. Floer, A, Weinstein, A: Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. J. Funct. Anal. **69**, 397-408 (1986)
19. Oh, YG: On positive multi-bump bound states of nonlinear Schrödinger equations under multiple well potential. Commun. Math. Phys. **131**, 223-253 (1990)
20. Oh, YG: Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of class $(V)_a$. Commun. Partial Differ. Equ. **13**, 1499-1519 (1988)
21. Liu, J, Wang, Z: Soliton solutions for quasilinear Schrödinger equations, I. Proc. Am. Math. Soc. **131**, 441-448 (2003)
22. Canino, A, Degiovanni, M: Nonsmooth critical point theory and quasilinear elliptic equations. In: Topological Methods in Differential Equations and Inclusions (Montreal, PQ, 1994). NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 472, pp. 1-50 (1995)
23. Liu, J, Wang, Z: Bifurcations for quasilinear Schrödinger equations, II. Commun. Contemp. Math. **10**, 723-743 (2008)

10.1186/1687-2770-2013-219
Cite this article as: Jiao et al.: **Existence of solutions for a general quasilinear elliptic system via perturbation method.** *Boundary Value Problems* 2013, 2013:219