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Global existence and uniform decay for the one-dimensional model of thermodiffusion with second sound

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Abstract

In this paper, we investigate an initial boundary value problem for the one-dimensional linear model of thermodiffusion with second sound in a bounded region. Using the semigroup approach, boundary control and the multiplier method, we obtain the existence of global solutions and the uniform decay estimates for the energy.

MSC: 35B40; 35M13; 35Q79

Keywords: thermodiffusion; second sound; global existence; exponential decay

1 Introduction

In this paper, we investigate the global existence and uniform decay rate of the energy for solutions for the one-dimensional model of thermodiffusion with second sound:

$$\rho u_{tt} - (\lambda + 2\mu)u_{xx} + \gamma_1\theta_{1x} + \gamma_2\theta_{2x} = 0 \quad \text{in } (0, 1) \times (0, +\infty), \quad (1.1)$$

$$c\theta_{1t} + \sqrt{k}q_{1x} + \gamma_1u_{tx} + d\theta_{2t} = 0 \quad \text{in } (0, 1) \times (0, +\infty), \quad (1.2)$$

$$n\theta_{2t} + \sqrt{D}q_{2x} + \gamma_2u_{tx} + d\theta_{1t} = 0 \quad \text{in } (0, 1) \times (0, +\infty), \quad (1.3)$$

$$\tau_1q_{1t} + q_1 + \sqrt{k}\theta_{1x} = 0 \quad \text{in } (0, 1) \times (0, +\infty), \quad (1.4)$$

$$\tau_2q_{2t} + q_2 + \sqrt{D}\theta_{2x} = 0 \quad \text{in } (0, 1) \times (0, +\infty), \quad (1.5)$$

together with the initial conditions

$$\begin{cases} u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), \\ \theta_1(x, 0) = \theta_1^0(x), & \theta_2(x, 0) = \theta_2^0(x), \\ q_1(x, 0) = q_1^0(x), & q_2(x, 0) = q_2^0(x), \end{cases} \quad (1.6)$$

and the boundary conditions

$$u(x, t)|_{x=0,1} = \theta_1(x, t)|_{x=0,1} = \theta_2(x, t)|_{x=0,1} = 0, \quad (1.7)$$

where u , θ_1 , and q_1 are the displacement, temperature, and heat flux, θ_2 , and q_2 are the chemical potentials and the associated flux. The boundary conditions (1.7) model a rigidly

clamped medium with temperature and chemical potentials held constant on the boundary.

Here, we denote by λ, μ the material constants, ρ the density, γ_1, γ_2 the coefficients of thermal and diffusion dilatation, k, D the coefficients of thermal conductivity, n, c, d the coefficients of thermodiffusion, and τ_1, τ_2 the (in general very small) relaxation time. All the coefficients above are positive constants and satisfy the condition

$$nc - d^2 > 0. \tag{1.8}$$

The classical thermodiffusion equations were first given by Nowacki [1, 2] in 1971. The equations describe the process of thermodiffusion in a solid body (see, e.g., [1–5]):

$$\begin{cases} \rho u_{tt} - (\lambda + 2\mu)u_{xx} + \gamma_1\theta_{1x} + \gamma_2\theta_{2x} = 0, \\ c\theta_{1t} - k\theta_{1xx} + \gamma_1u_{tx} + d\theta_{2t} = 0, \\ n\theta_{2t} - D\theta_{2xx} + \gamma_2u_{tx} + d\theta_{1t} = 0. \end{cases} \tag{1.9}$$

There are many results about the classical thermodiffusion equations. By the method of integral transformations and integral equations, Nowacki [2], Podstrigach [6] and Fichera [7] investigated the initial boundary value problem for the linear homogeneous system. Gawinecki [8] proved the existence, uniqueness and regularity of solutions to an initial boundary value problem for the linear system of thermodiffusion in a solid body. Szymaniec [5] proved the L^p - L^q time decay estimates along the conjugate line for the solutions of the linear thermodiffusion system. Using the results from [5], Szymaniec [9] obtained the global existence and uniqueness of small data solutions to the Cauchy problem of non-linear thermodiffusion equations in a solid body. Using the semigroup approach and the multiplier method, Qin *et al.* [4] obtained the global existence and exponential stability of solutions for homogeneous, nonhomogeneous and semilinear thermodiffusion equations subject to various boundary conditions. Liu and Reissig [3] studied the Cauchy problem for one-dimensional models of thermodiffusion and explained qualitative properties of solutions and showed which part of the model has a dominant influence on wellposedness, propagation of singularities, L^p - L^q decay estimates on the conjugate line and the diffusion phenomenon.

If we neglect the diffusion in (1.9), then we obtain the classical thermo-elasticity equations. Today models of type I (classical model of thermo-elasticity), of type II (thermal wave), of type III (visco-elastic damping) or second sound present some classification of models of thermo-elasticity (see, e.g., [3, 10, 11]). By considerations of the total energy equation and comparisons with the models of classical thermo-elasticity and thermodiffusion, we shall propose the linear one-dimensional model of thermodiffusion with second sound as mentioned above. Due to our knowledge, there exist no results for thermodiffusion models with second sound.

Our paper is organized as follows. In Section 2, we present some notations and the main result. Section 3 is devoted to the proof of the main result.

2 Notations and main result

Let $\Omega = (0, 1)$ and

$$\begin{aligned} L_*^2(\Omega) &= \left\{ y \in L^2(\Omega) \mid \int_0^1 y(x) \, dx = 0 \right\}, \\ H_*^1(\Omega) &= \left\{ y \in H^1(\Omega) \mid \int_0^1 y(x) \, dx = 0 \right\}. \end{aligned} \tag{2.1}$$

The associated first-order and second-order energy is defined by

$$\begin{aligned} E_1(t) &\equiv E_1(t; u_x, u_t, \theta_1, \theta_2, q_1, q_2) \\ &:= \frac{1}{2} \int_0^1 ((\lambda + 2\mu)u_x^2 + \rho u_t^2 + c\theta_1^2 + n\theta_2^2 + \tau_1 q_1^2 + \tau_2 q_2^2 + 2d\theta_1\theta_2) \, dx, \end{aligned} \tag{2.2}$$

$$E_2(t) := E_1(t; u_{xt}, u_{tt}, \theta_{1t}, \theta_{2t}, q_{1t}, q_{2t}). \tag{2.3}$$

The energy $E(t)$ is defined by

$$E(t) := E_1(t) + E_2(t). \tag{2.4}$$

Our main result reads as follows.

Theorem 2.1 *Assume that $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, $\theta_1^0, \theta_2^0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $q_1^0, q_2^0 \in H^2(\Omega) \cap H_*^1(\Omega)$. Then problem (1.1)-(1.7) has a unique global solution such that*

$$\begin{aligned} u(t) &\in C^2([0, +\infty), L^2(\Omega)) \cap C^1([0, +\infty), H_0^1(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H_0^1(\Omega)), \\ \theta_1(t), \theta_2(t) &\in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H_0^1(\Omega)), \\ q_1(t), q_2(t) &\in C^1([0, +\infty), H_*^1(\Omega)) \cap C([0, +\infty), H^2(\Omega) \cap H_*^1(\Omega)). \end{aligned}$$

Moreover, the associated energy $E(t)$ defined by (2.4) decays exponentially, i.e., there exist positive constants c_0 and C_0 such that

$$E(t) \leq C_0 e^{-c_0 t} E(0), \quad \forall t > 0. \tag{2.5}$$

Remark 2.1 If the initial value $(u_0, u_1, \theta_1^0, \theta_2^0, q_1^0, q_2^0) \in D(A^n)$, $n \in \mathbb{N}$, ($D(A^n)$ will be defined later), then the solution $(u, u_t, \theta_1, \theta_2, q_1, q_2) \in C([0, +\infty), D(A^n))$, and problem (1.1)-(1.7) yields higher regularity in t .

3 Proof of the main result

We shall divide the proof into two steps: in Step 1, we shall use the semigroup approach to prove the existence of global solutions and the Remark 2.1; Step 2 is devoted to proving the uniform decay of the energy by the boundary control and the multiplier method.

Step 1. Existence of global solutions.

The proof is based on the semigroup approach (see [4, 12]) that can be used to reduce problem (1.1)-(1.7) to an abstract initial value problem for a first-order evolution equation. In order to choose proper space for (1.1)-(1.7), we shall consider the static system associ-

ated with them (see [4]). Considering the energy and the property of operator A , we can choose the following state space and the domain of operator A for problem (1.1)-(1.7):

$$H = H_0^1(\Omega) \times (L^2(\Omega))^3 \times (H_*^1(\Omega))^2, \tag{3.1}$$

$$D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))^2 \times (H^2(\Omega) \cap H_*^1(\Omega))^2. \tag{3.2}$$

Using the same method as in [4, 12], we can prove that the operator A generates a C_0 -semigroup of contractions on the Hilbert space H . Define

$$D(A^k) = \{y|y \in D(A^{k-1}), Au \in D(A^{k-1}), k \in \mathbb{N}\}.$$

Then by Theorem 2.3.1 of [13] about the existence and regularities of solutions, we can complete the proof.

Step 2. Uniform decay of the energy.

In this section, we shall assume the existence of solutions in the Sobolev spaces that we need for our computations. The proof of uniform decay is difficult. It is necessary to construct a suitable Lyapunov function and to combine various techniques from energy method, multiplier approaches and boundary control (see [10, 11]). We mainly refer to Racke [11] for the approaches of thermo-elastic models with second sound.

Multiplying (1.1) by u_t , (1.2) by θ_1 , (1.3) by θ_2 , (1.4) by q_1 , and (1.5) by q_2 in L^2 , respectively, and summing up the results, yields

$$\frac{d}{dt} E_1(t) = - \int_0^1 (q_1^2 + q_2^2) dx. \tag{3.3}$$

Similarly, we can get

$$\frac{d}{dt} E_2(t) = - \int_0^1 (q_{1t}^2 + q_{2t}^2) dx. \tag{3.4}$$

Multiplying (1.1) by $\frac{1}{\lambda+2\mu} u_{xx}$ in L^2 , we get

$$\begin{aligned} & \frac{2}{3} \int_0^1 u_{xx}^2 dx + \frac{\rho}{\lambda+2\mu} \frac{d}{dt} \left(\int_0^1 u_{tx} u_x dx \right) \\ & \leq \frac{\rho}{\lambda+2\mu} \int_0^1 u_{tx}^2 dx + \frac{3 \int_0^1 (\gamma_1^2 \theta_{1x}^2 + \gamma_2^2 \theta_{2x}^2) dx}{2(\lambda+2\mu)^2}. \end{aligned} \tag{3.5}$$

Multiplying (1.2) by $\frac{3\rho}{2(\lambda+2\mu)\gamma_1} u_{tx}$, (1.3) by $\frac{3\rho}{2(\lambda+2\mu)\gamma_2} u_{tx}$ in L^2 and summing them up, yields

$$\begin{aligned} & \frac{3\rho}{\lambda+2\mu} \int_0^1 u_{tx}^2 dx \\ & \leq \frac{d}{dt} G_1(t) + \frac{1}{6} \int_0^1 u_{xx}^2 dx + h_1 \int_0^1 \theta_{1x}^2 dx + h_2 \int_0^1 \theta_{2x}^2 dx + h_3 \int_0^1 \theta_{1x} \theta_{2x} dx \\ & \quad + \frac{81\rho^2 k}{4(\lambda+2\mu)^2 \gamma_1^2} \int_0^1 q_{1t}^2 dx + \frac{81\rho^2 D}{4(\lambda+2\mu)^2 \gamma_2^2} \int_0^1 q_{2t}^2 dx \\ & \quad - \frac{3\rho\sqrt{k}}{2(\lambda+2\mu)\gamma_1} [q_1 u_{tx}]_{x=0}^{x=1} - \frac{3\rho\sqrt{D}}{2(\lambda+2\mu)\gamma_2} [q_2 u_{tx}]_{x=0}^{x=1}, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
 G_1(t) &:= \frac{3\rho\sqrt{k}}{2(\lambda+2\mu)^2\gamma_1} \int_0^1 \left(\frac{\rho\sqrt{k}}{\gamma_1} q_1 u_{tt} + \frac{\rho\sqrt{D}}{\gamma_2} q_2 u_{tt} - \tau_1 q_1 q_{1t} - q_1^2 - q_2^2 - \frac{\rho k + \rho D}{\sqrt{kD}} q_1 q_2 \right. \\
 &\quad \left. - \tau_2 q_2 q_{2t} - \frac{\sqrt{k}\tau_2}{\sqrt{D}} q_1 q_{2t} - \frac{\sqrt{D}\tau_1}{\sqrt{k}} q_2 q_{1t} \right) dx - \frac{3\rho}{2(\lambda+2\mu)\gamma_1\gamma_2} \\
 &\quad \times \int_0^1 \left[\frac{c\gamma_2 + d\gamma_1}{\sqrt{k}} (q_1 u_t + \tau_1 q_{1t} u_t) + \frac{d\gamma_2 + n\gamma_1}{\sqrt{D}} (q_2 u_t + \tau_2 q_{2t} u_t) \right] dx, \\
 h_3 &:= \frac{3}{\lambda+2\mu} \left(\frac{c\gamma_2}{2\gamma_1} + \frac{n\gamma_1}{2\gamma_2} + d \right), \quad h_1 := \frac{3}{2} \left(\frac{c}{\lambda+2\mu} + \frac{d\gamma_1}{\gamma_2} \right) + \frac{81}{4} \left(\frac{c^2}{\gamma_1^2} + \frac{d^2}{\gamma_2^2} \right), \\
 h_2 &:= \frac{3}{2} \left(\frac{n}{\lambda+2\mu} + \frac{d\gamma_2}{\gamma_1} \right) + \frac{81}{4} \left(\frac{d^2}{\gamma_1^2} + \frac{n^2}{\gamma_2^2} \right).
 \end{aligned}$$

Combining (3.5) with (3.6), we get

$$\begin{aligned}
 &\frac{2\rho}{\lambda+2\mu} \int_0^1 u_{tx}^2 dx + \frac{1}{2} \int_0^1 u_{xx}^2 dx + \frac{d}{dt} \left(-G_1(t) + \frac{\rho}{\lambda+2\mu} \int_0^1 u_{tx} u_x dx \right) \\
 &\leq \frac{81\rho^2 k}{4(\lambda+2\mu)^2 \gamma_1^2} \int_0^1 q_{1t}^2 dx + \left(h_1 + \frac{h_3}{2} + \frac{3\gamma_1^2}{2(\lambda+2\mu)^2} \right) \int_0^1 \theta_{1x}^2 dx \\
 &\quad + \frac{81\rho^2 D}{4(\lambda+2\mu)^2 \gamma_2^2} \int_0^1 q_{2t}^2 dx + \left(h_2 + \frac{h_3}{2} + \frac{3\gamma_2^2}{2(\lambda+2\mu)^2} \right) \int_0^1 \theta_{2x}^2 dx \\
 &\quad - \frac{3\rho\sqrt{k}}{2(\lambda+2\mu)\gamma_1} [q_1 u_{tx}]_{x=0}^{x=1} - \frac{3\rho\sqrt{D}}{2(\lambda+2\mu)\gamma_2} [q_2 u_{tx}]_{x=0}^{x=1}. \tag{3.7}
 \end{aligned}$$

Now, we conclude from (1.1), (1.4), (1.5) and Poincaré inequality

$$\begin{aligned}
 &\int_0^1 (u_t^2 + u_{tt}^2 + \theta_1^2 + \theta_2^2) dx \\
 &\leq \frac{2(\lambda+2\mu)^2}{\rho^2} \int_0^1 u_{xx}^2 dx + \frac{2}{D} \left(\frac{2\gamma_2^2}{\rho^2} + \frac{1}{\pi^2} \right) \int_0^1 (q_2^2 + \tau_2^2 q_{2t}^2) dx \\
 &\quad + \frac{1}{\pi^2} \int_0^1 u_{tx}^2 dx + \frac{2}{k} \left(\frac{2\gamma_1^2}{\rho^2} + \frac{1}{\pi^2} \right) \int_0^1 (q_1^2 + \tau_1^2 q_{1t}^2) dx. \tag{3.8}
 \end{aligned}$$

Multiplying (1.1) by u in L^2 , we obtain

$$\frac{\lambda+2\mu}{2} \int_0^1 u_x^2 dx \leq \frac{3}{2\pi^2(\lambda+2\mu)} \int_0^1 (\rho^2 u_{tt}^2 + \gamma_1^2 \theta_{1x}^2 + \gamma_2^2 \theta_{2x}^2) dx. \tag{3.9}$$

From (1.2) and (1.3), we get

$$\theta_{1t} = -\frac{1}{nc-d^2} (n\sqrt{k}q_{1x} - d\sqrt{D}q_{2x} + (n\gamma_1 - d\gamma_2)u_{tx}), \tag{3.10}$$

$$\theta_{2t} = -\frac{1}{nc-d^2} (c\sqrt{D}q_{2x} - d\sqrt{k}q_{1x} + (c\gamma_2 - d\gamma_1)u_{tx}). \tag{3.11}$$

Multiplying (3.10) by θ_{1t} , (3.11) by θ_{2t} in L^2 , and summing up the results, we get

$$\begin{aligned} & \int_0^1 (\theta_{1t}^2 + \theta_{2t}^2) dx - \frac{d}{dt} G_2(t) \\ & \leq \frac{2k(n^2 + d^2)}{(nc - d^2)^2} \int_0^1 q_{1t}^2 dx + \frac{(n\gamma_1 - d\gamma_2)^2 + (c\gamma_2 - d\gamma_1)^2}{(nc - d^2)^2} \int_0^1 u_{tx}^2 dx \\ & \quad + \int_0^1 (\theta_{1x}^2 + \theta_{2x}^2) dx + \frac{2D(c^2 + d^2)}{(nc - d^2)^2} \int_0^1 q_{2t}^2 dx, \end{aligned} \tag{3.12}$$

where $G_2(t) := \frac{2}{nc-d^2} \int_0^1 (n\sqrt{k}q_1\theta_{1x} - d\sqrt{D}q_2\theta_{1x} + c\sqrt{D}q_2\theta_{2x} - d\sqrt{k}q_1\theta_{2x}) dx$.

The boundary terms are estimated as follows.

$$\left| \left[\frac{3\rho\sqrt{k}}{2(\lambda + 2\mu)\gamma_1} q_1 u_{tx} \right]_{x=0}^{x=1} \right| \leq \frac{9\rho^2 k (|q_1(1)|^2 + |q_1(0)|^2)}{8(\lambda + 2\mu)^2 \gamma_1^2 \hat{\varepsilon}} + \frac{\hat{\varepsilon} (|u_{tx}(1)|^2 + |u_{tx}(0)|^2)}{2}, \tag{3.13}$$

$$\left| \left[\frac{3\rho\sqrt{D}}{2(\lambda + 2\mu)\gamma_2} q_2 u_{tx} \right]_{x=0}^{x=1} \right| \leq \frac{9\rho^2 D (|q_2(1)|^2 + |q_2(0)|^2)}{8(\lambda + 2\mu)^2 \gamma_2^2 \hat{\varepsilon}} + \frac{\hat{\varepsilon} (|u_{tx}(1)|^2 + |u_{tx}(0)|^2)}{2}, \tag{3.14}$$

for some $\hat{\varepsilon} > 0$,

$$|q_1(1)|^2 + |q_1(0)|^2 \leq 2 \left(1 + \frac{1}{\hat{\varepsilon}^2} \right) \int_0^1 q_1^2 dx + \frac{4\hat{\varepsilon}^2}{k} \int_0^1 (c^2 \theta_{1t}^2 + d^2 \theta_{2t}^2 + \gamma_1^2 u_{tx}^2) dx, \tag{3.15}$$

$$|q_2(1)|^2 + |q_2(0)|^2 \leq 2 \left(1 + \frac{1}{\hat{\varepsilon}^2} \right) \int_0^1 q_2^2 dx + \frac{4\hat{\varepsilon}^2}{D} \int_0^1 (d^2 \theta_{1t}^2 + n^2 \theta_{2t}^2 + \gamma_2^2 u_{tx}^2) dx. \tag{3.16}$$

Combining (3.13)-(3.16), we get

$$\begin{aligned} & \left| \left[\frac{3\rho\sqrt{k}}{2(\lambda + 2\mu)\gamma_1} q_1 u_{tx} \right]_{x=0}^{x=1} \right| + \left| \left[\frac{3\rho\sqrt{D}}{2(\lambda + 2\mu)\gamma_2} q_2 u_{tx} \right]_{x=0}^{x=1} \right| \\ & \leq \frac{9\rho^2(1 + \hat{\varepsilon}^2)}{4(\lambda + 2\mu)^2 \hat{\varepsilon}^3} \int_0^1 \left(\frac{k}{\gamma_1^2} q_1^2 + \frac{D}{\gamma_2^2} q_2^2 \right) dx + \hat{\varepsilon} (|u_{tx}(1)|^2 + |u_{tx}(0)|^2) \\ & \quad + \frac{9\rho^2 \hat{\varepsilon}}{2(\lambda + 2\mu)^2} \int_0^1 \left[\left(\frac{c^2}{\gamma_1^2} + \frac{d^2}{\gamma_2^2} \right) \theta_{1t}^2 + \left(\frac{d^2}{\gamma_1^2} + \frac{n^2}{\gamma_2^2} \right) \theta_{2t}^2 + 2u_{tx}^2 \right] dx. \end{aligned} \tag{3.17}$$

Differentiating (1.1) with respect to t and multiplying by $\varphi(x)u_{tx}$, where

$$\varphi(x) := 1 - 2x, \tag{3.18}$$

we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\int_0^1 \rho u_{tt} \varphi u_{tx} dx \right) - \rho \int_0^1 u_{tt}^2 dx + \frac{(\lambda + 2\mu)}{2} (u_{tx}^2(1) + u_{tx}^2(0)) - (\lambda + 2\mu) \int_0^1 u_{tx}^2 dx \\ & \quad + \gamma_1 \int_0^1 \theta_{1tx} \varphi u_{tx} dx + \gamma_2 \int_0^1 \theta_{2tx} \varphi u_{tx} dx = 0. \end{aligned} \tag{3.19}$$

Multiplications of (1.2) by $\varphi\theta_{1tx}$ and (1.3) by $\varphi\theta_{2tx}$ yield

$$\begin{aligned} & -c \int_0^1 \theta_{1t}^2 dx - \sqrt{k} \frac{d}{dt} \left(\int_0^1 q_{1x} \varphi \theta_{1x} dx \right) + \sqrt{k} \int_0^1 q_{1tx} \varphi \theta_{1x} dx \\ & = \gamma_1 \int_0^1 u_{tx} \varphi \theta_{1tx} dx + d \int_0^1 \theta_{2t} \varphi \theta_{1tx} dx, \\ & -n \int_0^1 \theta_{2t}^2 dx - \sqrt{D} \frac{d}{dt} \left(\int_0^1 q_{2x} \varphi \theta_{2x} dx \right) + \sqrt{D} \int_0^1 q_{2tx} \varphi \theta_{2x} dx \\ & = \gamma_2 \int_0^1 u_{tx} \varphi \theta_{2tx} dx + d \int_0^1 \theta_{1t} \varphi \theta_{2tx} dx, \end{aligned}$$

which implies, using (1.4) and (1.5),

$$\begin{aligned} & -c \int_0^1 \theta_{1t}^2 dx - \sqrt{k} \frac{d}{dt} \left(\int_0^1 q_{1x} \varphi \theta_{1x} dx \right) \\ & \quad - \frac{\sqrt{k}}{\tau_1} \int_0^1 q_{1x} \varphi \theta_{1x} dx + \frac{k}{2\tau_1} (\theta_{1x}^2(1) + \theta_{1x}^2(0)) \\ & = \frac{k}{\tau_1} \int_0^1 \theta_{1x}^2 dx + \gamma_1 \int_0^1 u_{tx} \varphi \theta_{1tx} dx + d \int_0^1 \theta_{2t} \varphi \theta_{1tx} dx, \end{aligned} \tag{3.20}$$

$$\begin{aligned} & -n \int_0^1 \theta_{2t}^2 dx - \sqrt{D} \frac{d}{dt} \left(\int_0^1 q_{2x} \varphi \theta_{2x} dx \right) \\ & \quad - \frac{\sqrt{D}}{\tau_2} \int_0^1 q_{2x} \varphi \theta_{2x} dx + \frac{D}{2\tau_2} (\theta_{2x}^2(1) + \theta_{2x}^2(0)) \\ & = \frac{D}{\tau_2} \int_0^1 \theta_{2x}^2 dx + \gamma_2 \int_0^1 u_{tx} \varphi \theta_{2tx} dx + d \int_0^1 \theta_{1t} \varphi \theta_{2tx} dx = 0. \end{aligned} \tag{3.21}$$

Combining (3.19)-(3.21), we conclude

$$\begin{aligned} & \frac{d}{dt} \left(\int_0^1 \rho u_{tt} \varphi u_{tx} - \sqrt{k} q_{1x} \varphi \theta_{1x} - \sqrt{D} q_{2x} \varphi \theta_{2x} dx \right) + \frac{\lambda + 2\mu}{2} (u_{tx}^2(1) + u_{tx}^2(0)) \\ & \leq \rho \int_0^1 u_{tt}^2 dx + (\lambda + 2\mu) \int_0^1 u_{tx}^2 dx + c \int_0^1 \theta_{1t}^2 dx + n \int_0^1 \theta_{2t}^2 dx + \int_0^1 q_{1x}^2 dx \\ & \quad + \frac{k}{\tau_1} \int_0^1 \theta_{1x}^2 dx + \frac{D}{\tau_2} \int_0^1 \theta_{2x}^2 dx + \frac{k}{4\tau_1^2} \int_0^1 \theta_{1x}^2 dx \\ & \quad + \frac{D}{4\tau_2^2} \int_0^1 \theta_{2x}^2 dx + \int_0^1 q_{2x}^2 dx. \end{aligned} \tag{3.22}$$

Using (1.2) and (1.3), we get

$$\begin{aligned} & \frac{d}{dt} \left(\frac{2\hat{\varepsilon}}{\lambda + 2\mu} \int_0^1 \rho u_{tt} \varphi u_{tx} - \sqrt{k} q_{1x} \varphi \theta_{1x} - \sqrt{D} q_{2x} \varphi \theta_{2x} dx \right) + \hat{\varepsilon} (u_{tx}^2(1) + u_{tx}^2(0)) \\ & \leq \left(2\hat{\varepsilon} + \frac{4\hat{\varepsilon}\gamma_1^2}{(\lambda + 2\mu)k} + \frac{4\hat{\varepsilon}\gamma_2^2}{(\lambda + 2\mu)D} \right) \int_0^1 u_{tx}^2 dx \\ & \quad + \left(\frac{2\hat{\varepsilon}c}{\lambda + 2\mu} + \frac{4\hat{\varepsilon}c^2}{(\lambda + 2\mu)k} + \frac{4\hat{\varepsilon}d^2}{(\lambda + 2\mu)D} \right) \int_0^1 \theta_{1t}^2 dx \end{aligned}$$

$$\begin{aligned}
 & + \frac{2\hat{\varepsilon}\rho}{\lambda + 2\mu} \int_0^1 u_{tt}^2 dx + \left(\frac{2\hat{\varepsilon}n}{\lambda + 2\mu} + \frac{4\hat{\varepsilon}d^2}{(\lambda + 2\mu)k} + \frac{4\hat{\varepsilon}n^2}{(\lambda + 2\mu)D} \right) \int_0^1 \theta_{2t}^2 dx \\
 & + \left(\frac{2\hat{\varepsilon}k}{(\lambda + 2\mu)\tau_1} + \frac{\hat{\varepsilon}k}{2(\lambda + 2\mu)\tau_1^2} \right) \int_0^1 \theta_{1x}^2 dx \\
 & + \left(\frac{2\hat{\varepsilon}D}{(\lambda + 2\mu)\tau_2} + \frac{\hat{\varepsilon}D}{2(\lambda + 2\mu)\tau_2^2} \right) \int_0^1 \theta_{2x}^2 dx.
 \end{aligned} \tag{3.23}$$

With (3.17) and (3.23), we can estimate

$$\begin{aligned}
 & \left| \left[\frac{3\rho\sqrt{k}}{2(\lambda + 2\mu)\gamma_1} q_1 u_{tx} \right]_{x=0}^{x=1} \right| + \left| \left[\frac{3\rho\sqrt{D}}{2(\lambda + 2\mu)\gamma_2} q_2 u_{tx} \right]_{x=0}^{x=1} \right| \\
 & \leq \frac{9\rho^2(1 + \hat{\varepsilon}^2)}{4(\lambda + 2\mu)^2 \hat{\varepsilon}^3} \int_0^1 \left(\frac{k}{\gamma_1^2} q_1^2 + \frac{D}{\gamma_2^2} q_2^2 \right) dx + \frac{2\hat{\varepsilon}\rho}{\lambda + 2\mu} \int_0^1 u_{tt}^2 dx - \frac{d}{dt} G_3(t) \\
 & + \hat{\varepsilon} \left(\frac{9\rho^2}{2(\lambda + 2\mu)^2} \left(\frac{c^2}{\gamma_1^2} + \frac{d^2}{\gamma_2^2} \right) + \frac{2c}{\lambda + 2\mu} + \frac{4c^2}{(\lambda + 2\mu)k} + \frac{4d^2}{(\lambda + 2\mu)D} \right) \int_0^1 \theta_{1t}^2 dx \\
 & + \hat{\varepsilon} \left(\frac{9\rho^2}{2(\lambda + 2\mu)^2} \left(\frac{d^2}{\gamma_1^2} + \frac{n^2}{\gamma_2^2} \right) + \frac{2n}{\lambda + 2\mu} + \frac{4d^2}{(\lambda + 2\mu)k} + \frac{4n^2}{(\lambda + 2\mu)D} \right) \int_0^1 \theta_{2t}^2 dx \\
 & + \hat{\varepsilon} \left(\frac{2k}{(\lambda + 2\mu)\tau_1} + \frac{k}{2(\lambda + 2\mu)\tau_1^2} \right) \int_0^1 \theta_{1x}^2 dx \\
 & + \hat{\varepsilon} \left(\frac{2D}{(\lambda + 2\mu)\tau_2} + \frac{D}{2(\lambda + 2\mu)\tau_2^2} \right) \int_0^1 \theta_{2x}^2 dx \\
 & + \hat{\varepsilon} \left(\frac{9\rho^2}{(\lambda + 2\mu)^2} + 2 + \frac{4\gamma_1^2}{(\lambda + 2\mu)k} + \frac{4\gamma_2^2}{(\lambda + 2\mu)D} \right) \int_0^1 u_{tx}^2 dx,
 \end{aligned} \tag{3.24}$$

where $G_3 := \frac{2\hat{\varepsilon}}{\lambda + 2\mu} \int_0^1 \rho u_{tt} \varphi u_{tx} - \sqrt{k} q_1 x \varphi \theta_{1x} - \sqrt{D} q_2 x \varphi \theta_{2x} dx$.

Define $\xi := \frac{\rho(nc-d^2)^2}{4(\lambda+2\mu)[(n\gamma_1-d\gamma_2)^2+(c\gamma_2-d\gamma_1)^2]}$. Multiplying both sides of (3.12) by ξ and combining the result with (3.7) and (3.24), we obtain for sufficiently small $\hat{\varepsilon}$ the estimate

$$\begin{aligned}
 & \frac{\rho}{\lambda + 2\mu} \int_0^1 u_{tx}^2 dx + \frac{1}{4} \int_0^1 u_{xx}^2 dx + \frac{\xi}{2} \int_0^1 (\theta_{1t}^2 + \theta_{2t}^2) dx + \frac{dG(t)}{dt} \\
 & \leq C_1 \int_0^1 (q_1^2 + q_2^2 + q_{1t}^2 + q_{2t}^2) dx,
 \end{aligned} \tag{3.25}$$

where $G(t) = -G_1(t) + \frac{\rho}{\lambda + 2\mu} \int_0^1 u_{tx} u_x dx - \xi G_2(t) + G_3(t)$, and $C_1 = C_1(\hat{\varepsilon}, \lambda, \mu, \gamma_1, \gamma_2, k, D, n, c, d, \tau_1, \tau_2)$. Now, we can define the desired Lyapunov functional $F(t)$. For $\varepsilon > 0$, to be determined later on, let

$$F(t) := \frac{1}{\varepsilon} E(t) + G(t). \tag{3.26}$$

Then we conclude from (3.3), (3.4), and (3.15)

$$\begin{aligned}
 \frac{d}{dt} F(t) & \leq -\frac{1}{\varepsilon} \int_0^1 (q_1^2 + q_2^2 + q_{1t}^2 + q_{2t}^2) dx - \frac{\rho}{\lambda + 2\mu} \int_0^1 u_{tx}^2 dx - \frac{1}{4} \int_0^1 u_{xx}^2 dx \\
 & \quad - \frac{\xi}{2} \int_0^1 (\theta_{1t}^2 + \theta_{2t}^2) dx + C_1 \int_0^1 (q_1^2 + q_2^2 + q_{1t}^2 + q_{2t}^2) dx.
 \end{aligned} \tag{3.27}$$

By using (3.8), we arrive at

$$\begin{aligned}
 & -C_2\varepsilon \int_0^1 (q_1^2 + q_2^2 + q_{1t}^2 + q_{2t}^2) dx - \frac{\varepsilon}{\pi^2} \int_0^1 u_{tx}^2 dx - \frac{2\varepsilon(\lambda + 2\mu)^2}{\rho^2} \int_0^1 u_{xx}^2 dx \\
 & \leq -\varepsilon \int_0^1 (u_t^2 + u_{tt}^2 + \theta_1^2 + \theta_2^2) dx,
 \end{aligned} \tag{3.28}$$

while (3.9) yields

$$-C_3\varepsilon^2 \int_0^1 u_{xx}^2 dx - C_4\varepsilon^2 \int_0^1 (q_1^2 + q_2^2 + q_{1t}^2 + q_{2t}^2) dx \leq -\varepsilon^2 \frac{\lambda + 2\mu}{2} \int_0^1 u_x^2 dx. \tag{3.29}$$

Combining (3.27)-(3.29), we conclude

$$\begin{aligned}
 \frac{d}{dt}F(t) & \leq -\left(\frac{1}{\varepsilon} - C_1 - C_2\varepsilon - C_4\varepsilon^2\right) \int_0^1 (q_1^2 + q_2^2 + q_{1t}^2 + q_{2t}^2) dx - \varepsilon^2 \frac{\lambda + 2\mu}{2} \int_0^1 u_x^2 dx \\
 & \quad - \left(\frac{2\rho}{\lambda + 2\mu} - \frac{\varepsilon}{\pi^2}\right) \int_0^1 u_{tx}^2 dx - \left[\frac{1}{4} - \frac{2\varepsilon(\lambda + 2\mu)^2}{\rho^2} - C_3\varepsilon^2\right] \int_0^1 u_{xx}^2 dx \\
 & \quad - \frac{\xi}{2} \int_0^1 (\theta_{1t}^2 + \theta_{2t}^2) dx - \varepsilon \int_0^1 (u_t^2 + u_{tt}^2 + \theta_1^2 + \theta_2^2) dx.
 \end{aligned} \tag{3.30}$$

We choose $0 < \varepsilon \leq 1$ such that all terms on the right-hand side of (3.30) become negative,

$$\text{i.e., } \varepsilon \leq \varepsilon_1 := \min \left\{ \frac{\rho\pi^2}{\lambda + 2\mu}, \frac{\rho^2}{32(\lambda + 2\mu)^2}, \frac{1}{4} \sqrt{\frac{1}{C_3}}, \frac{1}{2(C_1 + C_2 + C_4)} \right\}. \tag{3.31}$$

Choosing ε as in (3.31), we obtain from (3.30)

$$\frac{d}{dt}F(t) \leq -d_1 \int_0^1 (u_x^2 + u_t^2 + u_{tx}^2 + u_{tt}^2 + \theta_1^2 + \theta_2^2 + \theta_{1t}^2 + \theta_{2t}^2 + q_1^2 + q_2^2 + q_{1t}^2 + q_{2t}^2) dx$$

with

$$d_1 := \min \left\{ \frac{\varepsilon}{2}, \frac{1}{8}, \frac{\rho}{\lambda + 2\mu}, \frac{(\lambda + 2\mu)\varepsilon^2}{4}, \frac{\xi}{4}, \frac{1}{2\varepsilon} \right\}, \tag{3.32}$$

which implies

$$\frac{d}{dt}F(t) \leq -d_2E(t), \quad d_2 := \frac{d_1}{2} \min \left\{ \frac{1}{\rho}, \frac{1}{\lambda + 2\mu}, \frac{1}{c}, \frac{1}{n}, \frac{1}{\tau_1}, \frac{1}{\tau_2} \right\}. \tag{3.33}$$

There exist positive constants C_5, C_6 and ε_2 such that for any $\varepsilon \leq \varepsilon_2$ and $t \geq 0$, it holds

$$C_5E(t) \leq F(t) \leq C_6E(t), \tag{3.34}$$

where C_5, C_6 are determined later on. In fact,

$$|G(t)| \leq C_5E(t) \tag{3.35}$$

with $C_5 := \max\{\frac{2i_1}{\tau_1}, \frac{2i_2}{\tau_2}, \frac{2i_3}{\tau_1}, \frac{2i_4}{\tau_2}, \frac{2i_5}{\rho}, \frac{2i_6}{\lambda+2\mu}, \frac{2i_7}{\rho}, \frac{2i_8}{c}, \frac{2i_9}{n}, \frac{\rho}{(\lambda+2\mu)^2}\}$,

$$\begin{aligned}
 i_1 &:= \frac{3\rho}{4(\lambda+2\mu)^2} \left[\frac{\rho\sqrt{k}}{\gamma_1} + (\tau_1+2) + \frac{k+D}{\sqrt{kD}} + \frac{\sqrt{k}\tau_2}{\sqrt{D}} \right] + \frac{3\rho(c\gamma_2+d\gamma_1)}{4(\lambda+2\mu)\gamma_1\gamma_2\sqrt{k}} \\
 &\quad + \frac{(n+d)\sqrt{k}}{nc-d^2} + \left[\frac{n\sqrt{k}+d\sqrt{D}}{nc-d^2} + \frac{\hat{\varepsilon}\sqrt{k}}{\lambda+2\mu} \right] \frac{2}{k}, \\
 i_2 &:= \frac{3\rho}{4(\lambda+2\mu)^2} \left[\frac{\rho\sqrt{D}}{\gamma_2} + (\tau_2+2) + \frac{k+D}{\sqrt{kD}} + \frac{\sqrt{D}\tau_1}{\sqrt{k}} \right] + \frac{3\rho(n\gamma_1+d\gamma_2)}{4(\lambda+2\mu)\gamma_1\gamma_2\sqrt{D}} \\
 &\quad + \frac{(c+d)\sqrt{D}}{nc-d^2} + \left[\frac{c\sqrt{D}+d\sqrt{k}}{nc-d^2} + \frac{\hat{\varepsilon}\sqrt{D}}{\lambda+2\mu} \right] \frac{2}{D}, \\
 i_3 &:= \frac{3\rho\tau_1}{4(\lambda+2\mu)^2} \left(1 + \frac{\sqrt{D}}{\sqrt{k}} \right) + \frac{3\rho\tau_1(c\gamma_2+d\gamma_1)}{4(\lambda+2\mu)\gamma_1\gamma_2\sqrt{k}} + \left[\frac{n\sqrt{k}+d\sqrt{D}}{nc-d^2} + \frac{\hat{\varepsilon}\sqrt{k}}{\lambda+2\mu} \right] \frac{2\tau_1^2}{k}, \\
 i_4 &:= \frac{3\rho\tau_2}{4(\lambda+2\mu)^2} \left(1 + \frac{\sqrt{k}}{\sqrt{D}} \right) + \frac{3\rho\tau_2(d\gamma_2+n\gamma_1)}{4(\lambda+2\mu)\gamma_1\gamma_2\sqrt{D}} + \left[\frac{c\sqrt{D}+d\sqrt{k}}{nc-d^2} + \frac{\hat{\varepsilon}\sqrt{D}}{\lambda+2\mu} \right] \frac{2\tau_2^2}{D}, \\
 i_5 &:= \frac{3\rho(1+\tau_1)(d\gamma_1+c\gamma_2)}{4(\lambda+2\mu)\gamma_1\gamma_2\sqrt{k}} + \frac{3\rho(1+\tau_2)(d\gamma_2+n\gamma_1)}{4(\lambda+2\mu)\gamma_1\gamma_2\sqrt{D}}, \\
 i_6 &:= \frac{1}{\lambda+2\mu} \left[\frac{\rho(1+2\hat{\varepsilon})}{2} + \frac{2\hat{\varepsilon}\gamma_1^2}{\sqrt{k}} + \frac{2\hat{\varepsilon}\gamma_2^2}{\sqrt{D}} \right], \\
 i_7 &:= \frac{\rho}{\lambda+2\mu} \left[\frac{3\sqrt{k}}{4(\lambda+2\mu)\gamma_1} + \frac{3\sqrt{D}}{4(\lambda+2\mu)\gamma_2} + \hat{\varepsilon} \right], \\
 i_8 &:= \frac{2\hat{\varepsilon}c^2}{(\lambda+2\mu)\sqrt{k}} + \frac{2\hat{\varepsilon}d^2}{(\lambda+2\mu)\sqrt{D}}, \\
 i_9 &:= \frac{2\hat{\varepsilon}d^2}{(\lambda+2\mu)\sqrt{k}} + \frac{2\hat{\varepsilon}n^2}{(\lambda+2\mu)\sqrt{D}}.
 \end{aligned}$$

At this point, we choose $\varepsilon \leq \varepsilon_2 := \frac{1}{2C_5}$, $C_6 := \frac{1}{\varepsilon} + C_5$. Finally, we choose $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. Thus, we have the validity of (3.34). Combining (3.33) with (3.34), we get

$$\frac{d}{dt}F(t) \leq -d_0F(t), \quad d_0 := \frac{d_2}{C_6}. \tag{3.36}$$

Hence, it follows from (3.36), $F(t) \leq e^{-d_0t}F(0)$. Applying (3.34) again, we can conclude (2.5) with $C_0 := \frac{C_6}{C_5}$. The proof is complete.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The paper is a joint work of all authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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