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Dynamic phase transition for the Taylor problem in the wide-gap case

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Abstract

The main objective of this paper is to study the stability and the type of transition of the Taylor problem in the wide-gap case by using the averaging method, and we conclude that the stability of the Taylor problem in the wide-gap case is essentially the same with that in the case of the narrow-gap. The main technical tools are the spectral theory for linear and completely continuous fields, the dynamic bifurcation theory and the transition theory for incompressible flows, both developed by Ma and Wang (Bifurcation Theory and Applications, 2005; Stability and Bifurcation of Nonlinear Evolution Equations, 2007).

Keywords: Taylor problem; spectral theory; dynamic bifurcation; transition theory

1 Introduction

In 1921, Taylor [1] observed and studied the stability of laminar flow, which is known as the Couette flow. Taylor studied the case, where the gap between two cylinders is smaller in comparison with the mean radius, and both of them rotate in the same direction. He found that when the Taylor number T is smaller than a critical value $T_c > 0$, called the critical Taylor number, the Couette flow is stable, and when the Taylor number crosses T_c, the Couette flow breaks out into a cellular pattern which is radially symmetric.

Since Taylor's work, there have been many studies on this kind of problem. Such as Couette [2] and Mallock [3] did a lot of experiments. Cloes [4] published the most comprehensive experimental results associated with the Taylor vortex and the secondary instability. Chandrasekhar [5], Walowit *et al.* [6], Drazin and Reid [7] studied the linear theories. Velte [8], Kirchgässner [9], Kirchgässner and Sorger [10], Yudovich [11], Ma and Wang [12–14] studied the nonlinear theories. Especially, Ma and Wang established a new notion of bifurcation, called an attractor bifurcation, which was applied to the Taylor problem and obtained a series of fine results. This paper focuses on the Taylor problem in the wide-gap case. In this case, the radius of inner cylinder is small, while the radius of the outer cylinder is big. In addition to the same direction, the two cylinders could rot in the converse direction.

The main objective is to study the stability and the type of transition of the Taylor problem in the wide-gap case by using the averaging method and to compare with the Taylor problem in the narrow-gap case.

The main technical tools are the spectral theory for linear and completely continuous fields, the dynamic bifurcation theory and the transition theory for incompressible flows. These theories are directly applied to the Taylor problem in the wide-gap case.

The main conclusion is that the stability of the Taylor problem in the wide-gap case is essentially the same with that in the case of a narrow-gap. The main theorems are presented in Section 4, through which we can give pictures depicting the Couette flow stability in the wide-gap case, and compare with the Taylor problem in the narrow-gap case. In the later research, we intend to simulate the Taylor problem in the wide-gap case by using computer.

This paper is organized as follows. Section 2 introduces the governing equations for the Taylor problem. Section 3 studies the Taylor problem in the wide-gap case by using the averaging method, and establishes its mathematical frame. All the main theorems and the proofs are presented in Section 4.

2 Governing equations for the Taylor problem

The governing equations for the Taylor problem are the Navier-Stocks equations in the cylindrical coordinates (z, r, θ), which have the form

$$
\begin{cases}
\frac{\partial u_z}{\partial t} + (u \cdot \nabla) u_z = -\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right) + \nu \Delta u_z, \\
\frac{\partial u_r}{\partial t} + (u \cdot \nabla) u_r - \frac{u_\theta^2}{r} = -\frac{\partial}{\partial r}\left(\frac{p}{\rho}\right) + \nu\left(\Delta u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right), \\
\frac{\partial u_\theta}{\partial t} + (u \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} = -\frac{\partial}{\partial \theta}\left(\frac{p}{\rho}\right) + \nu\left(\Delta u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right), \\
\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta} + \frac{\partial(ru_z)}{\partial z} = 0,
\end{cases}
\tag{2.1}
$$

where ν is the kinematic viscosity, ρ the density, $u = (u_z, u_r, u_\theta)$ the velocity field, p the pressure function, and

$$
u \cdot \nabla = u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta},
$$

$$
\Delta = \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.
$$

The basic flow for (2.1) is the Couette flow, namely a steady state solution defined by

$$
u_z = u_r = 0, \qquad u_\theta = V(r), \qquad p = \rho \int \frac{1}{r} V^2(r)\, dr,
$$

$$
V(r) = ar + \frac{b}{r},
\tag{2.2}
$$

where

$$
a = -\Omega_1 \eta^2 \frac{1 - \mu/\eta^2}{1 - \eta^2}, \qquad b = \Omega_1 \frac{r_1^2(1 - \mu)}{1 - \eta^2},
$$

$$
\mu = \frac{\Omega_2}{\Omega_1}, \qquad \eta = \frac{r_1}{r_2}.
$$

In order to investigate the stability of the Couette flow defined by (2.1), we need to consider the perturbed state

$$
u_z, \qquad u_r, \qquad u_\theta + V(r), \qquad p + \rho \int \frac{1}{r} V^2(r)\, dr.
$$

Assume that the perturbations are axisymmetric and independent of θ, from (2.1), we have:

$$
\begin{cases}
\frac{\partial u_z}{\partial t} + (\tilde{u} \cdot \nabla)u_z = \nu \Delta u_z - \frac{\partial}{\partial z}\left(\frac{p}{\rho}\right), \\
\frac{\partial u_r}{\partial t} + (\tilde{u} \cdot \nabla)u_r - \frac{u_\theta^2}{r} = \nu\left(\Delta u_r - \frac{u_r}{r^2}\right) - \frac{\partial}{\partial r}\left(\frac{p}{\rho}\right) + \frac{2V}{r}u_\theta, \\
\frac{\partial u_\theta}{\partial t} + (\tilde{u} \cdot \nabla)u_\theta + \frac{u_r u_\theta}{r} = \nu\left(\Delta u_\theta - \frac{u_\theta}{r^2}\right) - \left(V' + \frac{V}{r}\right)u_r, \\
\frac{\partial(ru_r)}{\partial r} + \frac{\partial(ru_z)}{\partial z} = 0,
\end{cases}
\tag{2.3}
$$

where

$$
\tilde{u} \cdot \nabla = u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r},
$$

$$
\Delta = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}.
$$

The spatial domain for (2.3) is $M = (r_1, r_2) \times (0, L) \subset R^2$, where L is the height of the field between the two cylinders. There are different physically sound boundary conditions.

In the radial direction, there are two kinds of boundary conditions:

Free boundary condition:

$$
\frac{\partial u_z}{\partial r} = 0, \qquad u_r = u_\theta = 0, \quad \text{at } r = r_1, r_2.
$$

Rigid boundary condition:

$$
u_z = u_r = u_\theta = 0, \quad \text{at } r = r_1, r_2.
$$

In the z direction, there are four kinds of boundary conditions:

Free slip boundary condition:

$$
u_z = 0, \qquad \frac{\partial u_r}{\partial z} = \frac{\partial U_\theta}{\partial z} = 0, \quad \text{at } z = 0, L.
$$

Dirichlet boundary condition (or rigid condition):

$$
u_z = u_r = u_\theta = 0, \quad \text{at } z = 0, L.
$$

Free rigid boundary condition:

$$
u_z = 0, \qquad \frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = 0, \quad \text{at } z = L,
$$

$$
u_z = u_r = u_\theta = 0, \quad \text{at } z = 0.
$$

Periodic condition:

$$
u = (u_z, u_r, u_\theta) \text{ is periodic in the } z \text{ direction.}
$$

Using the transform:

$$
\begin{cases}
x = lx', \quad x = (z, r, \theta), \\
t = l^2 t'/\nu, \\
u = \nu u'/l, \quad u = (u_z, u_r, u_\theta), \\
p = \rho \nu^2 p'/l^2,
\end{cases}
$$

where l is a certain length unit in (2.3).

We get the dimensionless form:

$$
\begin{cases}
\frac{\partial u_z}{\partial t} = \Delta u_z - \frac{\partial p}{\partial z} - (\tilde{u} \cdot \nabla) u_z, \\
\frac{\partial u_r}{\partial t} = (\Delta u_r - \frac{u_r}{r^2}) - \frac{\partial p}{\partial r} + 2\Omega_1 \frac{l^2}{\nu} (\frac{r_1^2}{r^2} \frac{1-\mu}{1-\eta^2} - \frac{\eta^2 - \mu}{1-\eta^2} u_\theta) + \frac{u_\theta^2}{r} - (\tilde{u} \cdot \nabla) u_r, \\
\frac{\partial u_\theta}{\partial t} = (\Delta u_\theta - \frac{u_\theta}{r^2}) + 2\Omega_1 \frac{l^2}{\nu} \frac{\eta^2 - \mu}{1-\eta^2} u_r - \frac{u_r u_\theta}{r} - (\tilde{u} \cdot \nabla) u_\theta, \\
\frac{\partial (r u_r)}{\partial r} + \frac{\partial (r u_z)}{\partial z} = 0.
\end{cases}
\tag{2.4}
$$

3 Averaging for the Taylor problem in the wide-gap case

In the case of a wide-gap, r_1 is small, while r_2 is big. The averaging method is that we let $l = r_1 = 1$, and $r_0 = (r_1 + r_2)/2$ replace variable r approximately. Since r_x is big, r_1 and ν are small, we can ignore r^{-n} ($n \geq 1$), which are associated with the inner friction of fluid. In this paper, moreover, our analysis will be conducted using free boundary conditions in z and radial directions. Then from (2.4), we obtain the averaged governing equation for the case of the wide-gap:

$$
\begin{cases}
\frac{\partial u_z}{\partial t} = \Delta u_z - \frac{\partial p}{\partial z} - (\tilde{u} \cdot \nabla) u_z, \\
\frac{\partial u_r}{\partial t} = \Delta u_r - \frac{\partial p}{\partial r} + \lambda(\frac{1}{r_0^2} - \kappa) u_\theta + \frac{u_\theta^2}{r_0} - (\tilde{u} \cdot \nabla) u_r, \\
\frac{\partial u_\theta}{\partial t} = \Delta u_\theta + \lambda \kappa u_r - \frac{u_r u_\theta}{r_0} - (\tilde{u} \cdot \nabla) u_\theta, \\
\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0,
\end{cases}
\tag{3.1}
$$

$$
\begin{cases}
\frac{\partial u_z}{\partial r} = 0, \qquad u_r = u_\theta = 0, \quad \text{at } r = 1, r_2, \\
u_z = 0, \qquad \frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = 0, \quad \text{at } z = 0, L,
\end{cases}
\tag{3.2}
$$

where $\lambda = \sqrt{T}$, T is the Taylor number, T and κ are defined by

$$
T = \frac{4\Omega_1^2 (1-\mu^2)}{\nu^2 (1-\eta^2)}, \qquad \kappa = \frac{\eta^2 - \mu}{1-\mu},
$$

$$
\tilde{u} \cdot \nabla = u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r},
$$

$$
\Delta = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2},
$$

spatial domain for (3.1) is $M = (1, r_2) \times (0, L) \subset R^2$.

Let $\tilde{u} = (u_z, u_r)$,

$$
H = \{ u = (\tilde{u}, u_\theta) \in L^2(M, R^3) \mid \operatorname{div} \tilde{u} = 0, \tilde{u} \cdot n|_{\partial M} = 0 \},
$$

$$
H_1 = \{ u = (\tilde{u}, u_\theta) \in H^2(M, R^3) \cap H \mid u \text{ satisfies } (3.2) \}.
$$

By the Hodge decomposition theorem, $L^2(M, R^3)$ can be decomposed as follows.

$$L^2(M, R^3) = H \oplus G,$$
$$H = \{u = (\tilde{u}, u_\theta) \in L^2(M, R^3) | \operatorname{div} \tilde{u} = 0, \tilde{u} \cdot n|_{\partial M} = 0\},$$
$$G = \{\nabla_\varphi \in L^2(M, R^3) | \varphi \in H^1(M)\},$$
$$G \perp H.$$

Here, $H^1(M)$ and $H^2(M, R^3)$ are the usual Sobolev spaces.

Then we can define an orthogonal projection, called Leray projection

$$P : L^2(M, R^3) \to H.$$

Let

$$L_\lambda = -A + \lambda B : H_1 \to H,$$
$$G : H_1 \to H$$

be a mapping defined by

$$A(u) = -P(\Delta u_z, \Delta u_r, \Delta u_\theta),$$
$$B(u) = P\left(0, \left(\frac{1}{r_0^2} - \kappa\right)u_\theta, \kappa u_r\right), \tag{3.3}$$
$$G(u) = -P\left((\tilde{u} \cdot \nabla)u_z, (\tilde{u} \cdot \nabla)u_r - \frac{u_\theta^2}{r_0}, (\tilde{u} \cdot \nabla)u_\theta + \frac{u_\theta u_r}{r_0}\right),$$

where $u = (u_z, u_r, u_\theta) \in H_1$, and the mapping P is the Leray projection.

Thus, (3.1) can be written in the abstract form:

$$\frac{du}{dt} = L_\lambda u + Gu, \quad u \in H_1. \tag{3.4}$$

So far, the stability of the Couette flows of equation (2.1) equals to the stability of $(u_z, u_r, u_\theta) = 0$ of equation (3.4).

4 Main results and proofs

Let X_1, X be two Banach spaces, $X_1 \subset X$ be a dense and compact inclusion. Consider the following nonlinear evolution equations

$$\begin{cases} \frac{du}{dt} = Lu + Gu, \\ u(0) = \varphi, \end{cases} \tag{4.1}$$

where $L : X_1 \to X$ is bounded linear operator, $G : X_1 \to X$ is C^r ($r \geq 1$) mapping.

Definition 4.1 [14] Let $\Omega \subset X$ be a bounded open set, we say that (4.1) is Lyapunov stable in Ω, if the solution of (4.1) $u(t, \varphi) \in \Omega$, $\forall t \geq 0$, for any initial point $\varphi \in \Omega$.

Definition 4.2 [14] We say (4.1) is asymptotically Lyapunov stable in Ω, or u_0 is an asymptotically stable equilibrium point of (4.1) in Ω, if there exists a steady solution $u_0 \in \Omega$, satisfying

$$\lim_{t \to \infty} \left\| u(t, \varphi) - u_0 \right\|_X = 0 \quad \text{for any } \varphi \in \Omega.$$

Lemma 4.3 [14] *Let* $L : X_1 \to x$ *be a sectorial operator,* $G : X_\alpha \to X$ *be a* C^r ($r \geq 1$) *mapping for certain* $0 \leq \alpha < 1$, v *be a steady solution of* (4.1). *If the spectral* $\beta(\lambda)$ *of* $L + DG(v)$ *satisfy* $\operatorname{Re} \beta(\lambda) < -\varepsilon$, *for certain* $\varepsilon > 0$, *then* v *is a locally asymptotically stable equilibrium point of* (4.1), *and it decays exponentially. Namely, there exist* $M > 0$, $\sigma > 0$, $\delta > 0$, *for the solution of* (4.1) $u(t, \varphi)$, $\|u(t, \varphi) - v\|_X \leq Me^{-\sigma t}$ ($\forall t \geq 0$) *hold true, as* $\|\varphi - v\| < \delta$.

Let

$$\delta = \frac{3r_2^2 - 2r_2 - 1}{r_2^2(r_2 - 1)(r_2 + 3)} > 0, \quad \text{as } r_2 > r_1 = 1,$$

and as above

$$\mu = \frac{\Omega_2}{\Omega_1}, \qquad \eta = \frac{r_1}{r_2}.$$

Theorem 4.4 *If* $\mu \leq -\delta$, *or* $\eta^2 \leq \mu$, *then* $(u_z, u_r, u_\theta) = 0$ *is a locally asymptotically stable equilibrium point of* (3.4), *and it decays exponentially.*

Theorem 4.5 *If* $-\delta < \mu < \eta^2$ ($\mu \neq 0$), *we have the following conclusions for* (3.4):

(1) *Equation* (3.4) *happens with a continuous transition at* $(u, \lambda) = (0, \sqrt{T_c})$, *namely there is an attractor bifurcation, and it bifurcates exactly into two singular points* v_i ($i = 1, 2$), *which attract two open subsets of* U *separately.* U *is the neighborhood of* $u = 0$.

(2) *The two singular points* v_i ($i = 1, 2$) *can be written by*:

$$v_{1,2}(\lambda) = \pm |\beta_{n_1 1}/\sigma|^{\frac{1}{2}} \psi_{n_1 1} + o\left(|\beta_{n_1 1}/\sigma|^{\frac{1}{2}}\right),$$

where $\sigma = \frac{-1}{4\beta_{2n_1 2}} a_1^4 b_1^2$, $\beta_{n_1 1}$ *is the first eigenvalue.*

Remark 4.6 From the point of view of mathematics, Theorem 4.4 explains that there exists an open set $\Omega \subset H_1$, which guarantees that if the initial point $u_0 \in \Omega$, then the solution of (3.4) satisfies $\|u(t, u_0)\|_H \to 0$ ($t \to \infty$).

Remark 4.7 From the point of view of physics, Theorem 4.4 explains that in the wide-gap case, the Couette flow of the Taylor problem is metastable. Namely, if the initial perturbation is in a certain range, the disturbed fluid will become the Couette flow in a short time. But if the initial perturbation is beyond that range above, the disturbed fluid will become another steady flow. The explanation is the same as that for Theorem 4.5.

Remark 4.8 Theorems 4.4-4.5 give the entire results of stability of the Taylor problem in the wide-gap case; see Figure 1. Σ_1 represents the unstable area, Σ_2 represents the stable

Figure 1 Wide-gap Case.

Figure 2 Narrow-gap Case.
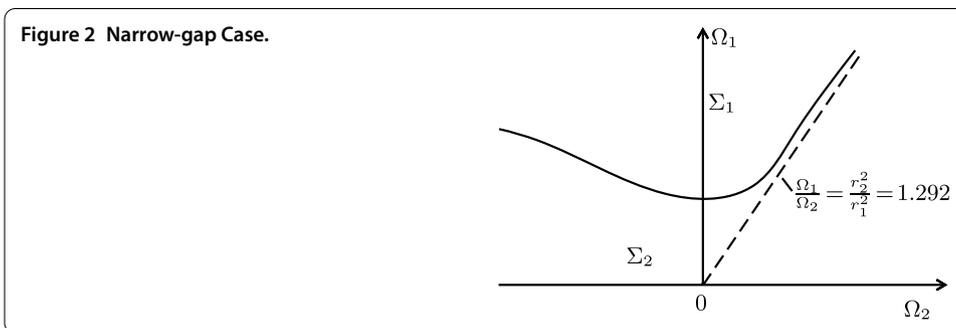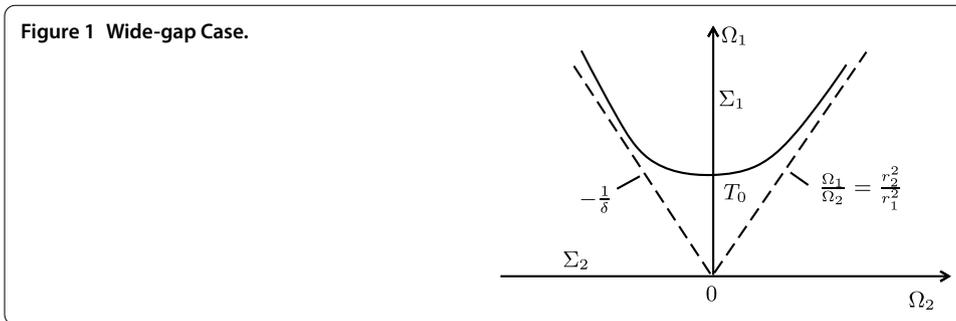
area, $T_0 \neq 0$ and

$$T_0 = \min_{(n,l)} \frac{\alpha^3 \nu^2 (1-\eta^2)^2}{4a^2 \eta^2 (\frac{1}{r_0^2} - \eta^2)}.$$

Remark 4.9 [15] In the narrow-gap case, let $r_1 = 3.55$ cm and $r_2 = 4.035$ cm, the results can be seen in Figure 2. Σ_1 represents the unstable area, Σ_2 represents the stable area.

Remark 4.10 By Figure 1 and Figure 2, we can conclude that the results of the stability of the Taylor problem in the wide-gap case, obtained by using the averaging method, are essentially the same as those of the Taylor problem in the narrow-gap case.

Proof of Theorem 4.4 Obviously, A and B are linear operators. According to [16], A is a homeomorphism, then it is a sectorial operator. According to the Sobolev compact embedding theorem [17] and the Leray projection, P is bounded, B is compact. So,

$$L_\lambda = -A + \lambda B : H_1 \to H$$

is a sectorial operator, and also a linear completely continuous field.

Now,

$$\int_\Omega (u \cdot \nabla) u \cdot v + u^2 \cdot v \, dx \le \|u\|_{C^0} \|\nabla u\|_{L^2} \|v\|_{L^2} + \|u\|_{C^0} \|u\|_{L^2} \|v\|_{L^2}$$

$$= \|u\|_{C^0} \|u\|_{W^{1,2}} \|v\|_{L^2},$$

we have

$$\|Gu\|_H \le \|u\|_{C^0} \|u\|_{W^{1,2}}.$$

According to [18], Theorem 7.2.2, the fraction spaces of $L_\lambda, H_\alpha = D(L_\lambda^\alpha)$, satisfy

$$\|u\|_{C^0} + \|u\|_{W^{1,2}} \le C\|u\|_{H_\alpha},$$

where $\alpha > \frac{3}{4}$. Then

$$G : H_\alpha \to H \quad \left(\frac{3}{4} < \alpha < 1\right) \text{ is a bounded analytic mapping.}$$

Consider the eigenvalue problem:

$$L_\lambda \psi = \beta(\lambda)\psi, \quad \psi = (u_z, u_r, u_\theta) \in H_1.$$

By (3.3), the abstract form can be referred to the following eigenvalue equations in H_1:

$$\Delta u_z - \frac{\partial p}{\partial z} = \beta(\lambda) u_z, \tag{4.2}$$

$$\Delta u_r - \frac{\partial p}{\partial r} + \lambda \left(\frac{1}{r_0^2} - \kappa\right) u_\theta = \beta(\lambda) u_r, \tag{4.3}$$

$$\Delta u_\theta + \lambda \kappa u_r = \beta(\lambda) u_\theta, \tag{4.4}$$

$$\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0, \tag{4.5}$$

$$\frac{\partial u_z}{\partial r} = 0, \qquad u_r = u_\theta = 0, \quad \text{at } r = 1, r_2, \tag{4.6}$$

$$u_z = 0, \qquad \frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = 0, \quad \text{at } z = 0, L. \tag{4.7}$$

According to [5], we take the separation of variables as follows:

$$u_z = \frac{dh(z)}{dt} \frac{dR(r)}{dr},$$
$$u_r = a^2 h(z) R(r), \tag{4.8}$$
$$u_\theta = h(z)\varphi(r).$$

We utilize (4.8) in (4.5) and (4.7) to obtain:

$$\begin{cases} h''R' + a^2 hR' = 0, \\ h'(0) = h'(L) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} h = \cos az, \\ a = \frac{n\pi}{L}, \quad n = 1, 2, \dots. \end{cases} \tag{4.9}$$

We utilize (4.8), (4.9) in (4.4) to obtain:

$$h''\varphi + h\varphi'' + \lambda \kappa a^2 hR = \beta(\lambda)h\varphi$$

that equals to

$$\left(\frac{d^2}{dr^2} - a^2\right)\varphi = -\lambda\kappa a^2 R + \beta(\lambda)\varphi. \tag{4.10}$$

We utilize (4.8), (4.9) in (4.2), then differentiate with respect to r,

$$h'''R'' + h'R'''' - \frac{\partial^2 p}{\partial r \partial z} = \beta(\lambda)h'R'',$$

and utilize (4.8), (4.9) in (4.3), then differentiate with respect to z,

$$a^2 h'''R + a^2 h'R'' + \lambda\left(\frac{1}{r_0^2} - \kappa\right)h'\varphi - \frac{\partial^2 p}{\partial z \partial r} = \beta(\lambda)a^2 h'R,$$

the two equations above subtract to obtain:

$$\left(\frac{d^2}{dr^2} - a^2\right)^2 R = \lambda\left(\frac{1}{r_0^2} - \kappa\right)\varphi + \beta(\lambda)\left(\frac{d^2}{dr^2} - a^2\right)R. \tag{4.11}$$

Utilizing (4.8) in (4.6), we have

$$\varphi(1) = \varphi(r_2) = R(1) = R(r_2) = R''(1) = R''(r_2) = 0,$$

then let

$$R = \sin l\pi\left(\frac{r-1}{r_2-1}\right), \qquad \varphi = C_{nl}\sin l\pi\left(\frac{r-1}{r_2-1}\right), \quad l = 1, 2, \ldots.$$

Combine (4.10), (4.11) to obtain:

$$\begin{cases} \alpha C_{nl} = a^2\lambda\kappa - C_{nl}\beta(\lambda), \\ \alpha^2 = C_{nl}(\frac{1}{r_0^2} - \kappa)\lambda - \alpha\beta(\lambda), \end{cases} \qquad \alpha = a^2 + \left(\frac{l\pi}{r_2-1}\right)^2, \tag{4.12}$$

therefore, we get

$$\begin{cases} C_{nl} = \frac{2\alpha\lambda\kappa a^2}{\pm\sqrt{\Delta}}, \\ \beta_{nl} = -\alpha \pm \frac{\sqrt{\Delta}}{2\alpha}, \end{cases} \tag{4.13}$$

where

$$\Delta = 4\alpha a^2\lambda^2\kappa\left(\frac{1}{r_0^2} - \kappa\right).$$

The corresponding eigenvectors of (4.2)-(4.7) are as follows:

$$\psi_{nl} = \begin{cases} \frac{nl\pi^2}{L(r_2-1)}\sin\frac{n\pi z}{L}\cos\frac{l\pi(r-1)}{(r_2-1)}, \\ (\frac{n\pi}{L})^2\cos\frac{n\pi z}{L}\sin\frac{l\pi(r-1)}{(r_2-1)}, \\ C_{nl}\cos\frac{n\pi z}{L}\sin\frac{l\pi(r-1)}{(r_2-1)}. \end{cases} \tag{4.14}$$

Now, as $(n, l) \neq (0, 0)$,

$$L_\lambda \psi_{nl} = \beta_{nl}(\lambda)\psi_{nl},$$

$$E_1 = \left\{ \psi_{nl} | \psi_{nl} \text{ satisfies } (4.14), (n, l) \in N \times N \right\},$$

$$B_1 = \left\{ \beta_{nl}(\lambda) | \beta_{nl}(\lambda) \text{ satisfies } (4.13), (n, l) \in N \times N \right\},$$

as $n = 0$, from (4.2)-(4.8),

$$L_\lambda \psi_{0l} = \beta_{0l}(\lambda)\psi_{0l},$$

where

$$\psi_{0l} \in E_2 = \left\{ \psi_{0l} \Big| \psi_{0l} = \left(0, 0, \sin \frac{l\pi(r-1)}{(r_2-1)} \right)^T, l = 1, 2, \dots \right\},$$

$$\beta_{0l} \in B_2 = \left\{ \beta_{0l} \Big| \beta_{0l} = -\left(\frac{l\pi}{r_2-1} \right)^2, l = 1, 2, \dots \right\},$$

as $l = 0$, $\psi_{n0} = 0$.

Now, according to the Fourier expansion, $B_1 \cup B_2$ are all the eigenvalues of (4.2)-(4.7), the corresponding eigenvectors $E_1 \cup E_2$ form a complete basis in H_1.

In view of the condition $\mu < -\delta$, by simple calculation, we know that $\kappa > 0$, $\kappa r_0^2 - 1 > 0$, then

$$\Delta \leq 0,$$

thus,

$$\text{Re } \beta_{nl} < -\varepsilon \quad (0 < \varepsilon < \alpha),$$

and

$$\beta_{0l} = -\left(\frac{l\pi}{r_2-1} \right)^2 < 0.$$

Now, by Lemma 4.3, the proof of Theorem 4.4 is completed. □

Proof of Theorem 4.5 Utilizing the method in Theorem 4.4, we have the following results for the eigenvalue problem of L_λ^*, here L_λ^* is adjoint operator of L_λ.

As $(n, l) \neq (0, 0)$,

$$\begin{cases} C_{nl}^* = \dfrac{2\alpha(\frac{1}{r_0^2} - \kappa)\alpha^2\lambda}{\pm\sqrt{\Delta}}, \\ \beta_{nl}(\lambda) = -\alpha \pm \dfrac{\sqrt{\Delta}}{2\alpha}, \end{cases} \tag{4.15}$$

where

$$\Delta = 4\alpha a^2 \lambda^2 \kappa \left(\frac{1}{r_0^2} - \kappa \right),$$

the corresponding eigenvectors are as follows:

$$
\psi_{nl}^* = \begin{cases}
\frac{nl\pi^2}{L(r_2-1)} \sin \frac{n\pi z}{L} \cos \frac{l\pi(r-1)}{(r_2-1)}, \\
(\frac{n\pi}{L})^2 \cos \frac{n\pi z}{L} \sin \frac{l\pi(r-1)}{(r_2-1)}, \\
C_{nl}^* \cos \frac{n\pi z}{L} \sin \frac{l\pi(r-1)}{(r_2-1)},
\end{cases} \tag{4.16}
$$

satisfying

$$
E_1^* = \left\{ \psi_{nl}^* \middle| \psi_{nl}^* \text{ satisfies } (4.16), (n,l) \in N \times N \right\},
$$

$$
B_1 = \left\{ \beta_{nl}(\lambda) \middle| \beta_{nl}(\lambda) \text{ satisfies } (4.15), (n,l) \in N \times N \right\}.
$$

As $n = 0$,

$$
L_\lambda^* \psi_{0l}^* = \beta_{0l}(\lambda)\psi_{0l}^*,
$$

where

$$
\psi_{0l}^* \in E_2^* = \left\{ \psi_{0l}^* \middle| \psi_{0l}^* = \left(0, 0, \sin \frac{l\pi(r-1)}{(r_2-1)} \right)^T, l = 1, 2, \dots \right\},
$$

$$
\beta_{0l} \in B_2 = \left\{ \beta_{0l} \middle| \beta_{0l} = -\left(\frac{l\pi}{r_2-1} \right)^2, l = 1, 2, \dots \right\}.
$$

As $l = 0$, $\psi_{n0}^* = 0$.

Consider equation $\beta_{nl}(\lambda) = 0$, by (4.15), we have

$$
\alpha \left(\alpha + \frac{1}{r_0^2} \right)^2 - \lambda^2 a^2 \kappa \left(\frac{1}{r_0^2 - \kappa} \right) = 0,
$$

$$
\lambda^2 = \frac{\alpha^3}{a^2 \kappa (\frac{1}{r_0^2} - \kappa)}.
$$

Now, we have the critical Taylor number

$$
T_c = \min_{(n,l) \in N^2} \frac{\alpha^3}{a^2 \kappa (\frac{1}{r_0^2} - \kappa)}, \tag{4.17}
$$

through sample calculation, as $(n,l) = (\frac{L}{\sqrt{2}(r_2-1)}, 1)$, $\frac{\alpha^3}{a^2\kappa(\frac{1}{r_0^2}-\kappa)}$ reaches its minimal value. We assume that $n + \frac{1}{2} \neq \frac{L}{\sqrt{2}(r_2-1)}$ for $\forall n > 0$, then there exists only one couple (n_1, l), making $\frac{\alpha^3}{a^2\kappa(\frac{1}{r_0^2}-\kappa)}$ reach the minimal value.

Therefore, we have

$$
\beta_{n_1 l}(\lambda) = \begin{cases}
< 0, & \text{if } \lambda < T_c, \\
= 0, & \text{if } \lambda = T_c, \\
> 0, & \text{if } \lambda > T_c,
\end{cases} \tag{4.18}
$$

$$
\beta_{nl}(T_c) < 0, \quad \forall (n,l) \neq (n_1, 1).
$$

The reduced form of (3.4) into its center manifold in H is:

$$\frac{dx}{dt} = \beta_{n_1 1} x + \frac{1}{\langle \psi_{n_1 1}, \psi_{n_1 1}^* \rangle_H} \langle G(\psi, \psi), \psi_{n_1 1}^* \rangle_H, \tag{4.19}$$

where $\psi \in H_1$ can be written by

$$\psi = x \psi_{n_1 1} + \Phi, \tag{4.20}$$

Φ is a center manifold function. $G(u, v)$ is a bilinear operator, satisfying

$$G(u, v) = -P \left((\tilde{u} \cdot \nabla) v_z, (\tilde{u} \cdot \nabla) v_r - \frac{u_\theta v_\theta}{r_0}, (\tilde{u} \cdot \nabla) v_\theta + \frac{u_\theta v_r}{r_0} \right). \tag{4.21}$$

Through direct calculation, we have

$$\langle G(\psi_{n_1 1}, \psi_{n_1 1}), \psi_j^* \rangle_H = \begin{cases} 0, & \psi_j^* \neq \psi_{2n_1 2}^*, \\ -a_1^2 b_1 c_{11} c_{11}^* L(r_2 - 1), & \psi_j^* = \psi_{2n_1 2}^*, \end{cases} \tag{4.22}$$

where,

$$a_1 = \frac{n_1 \pi}{L}, \qquad b_1 = \frac{\pi}{r_2 - 1}, \qquad c_{11} = C_{n_1 1}, \qquad c_{11}^* = C_{n_1 1}^*.$$

Here, we use the mark as follows:

$$o(2) = o(x^2) + O(|\beta_{n_1 l}(\lambda) x^2|), \qquad o(3) = o(x^3) + O(|\beta_{n_1 l}(\lambda) x^3|).$$

Then the center manifold function can be written by

$$\Phi = -\frac{x^2 \langle G(\psi_{n_1 1}, \psi_{n_1 1}), \psi_{2n_1 2}^* \rangle_H}{\beta_{2n_1 2} \langle \psi_{2n_1 2}, \psi_{2n_1 2}^* \rangle_H} \psi_{2n_1 2} + o(2). \tag{4.23}$$

Direct computation yields that

$$\langle G(\psi_{n_1 1}, \psi_{2n_1 2}), \psi_{n_1 1}^* \rangle_H = 2 a_1^2 b_1 c_{11} c_{11}^* L(r_2 - 1),$$

$$\langle G(\psi_{2n_1 2}, \psi_{n_1 1}), \psi_{n_1 1}^* \rangle_H = -a_1^2 b_1 c_{11} c_{11}^* L(r_2 - 1), \tag{4.24}$$

$$\langle \psi_{n_1 1}, \psi_{n_1 1}^* \rangle_H = \frac{1}{2} c_{11} c_{11}^* L(r_2 - 1), \qquad \langle \psi_{2n_1 2}, \psi_{2n_1 2}^* \rangle_H = 8 c_{11} c_{11}^* L(r_2 - 1).$$

Finally, we utilize (4.20), (4.22)-(4.24) in (4.19) to obtain

$$\frac{dx}{dt} = \beta_{n_1 1} x - \sigma x^3 + o(3), \tag{4.25}$$

where

$$\sigma = \frac{-1}{4 \beta_{2n_1 2} a_1^4 b_1^2} > 0.$$

Now, according to Theorem 6.9 in [14], the proof of Theorem 4.5 is completed. □

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