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Existence of three solutions for a Navier boundary value problem involving the $(p(x), q(x))$-biharmonic

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Abstract

In this paper, we study $(p(x), q(x))$-biharmonic systems with Navier boundary condition on a bounded domain and obtain three solutions under appropriate hypotheses. The technical approach is mainly based on the general three critical points theorem obtained by Ricceri.

Keywords: three solutions; $(p(x), q(x))$-biharmonic; Navier condition; Ricceri's three critical points theorem

1 Introduction and main results

In this paper, we consider the Navier boundary value problem involving the $(p(x), q(x))$-biharmonic systems

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda F_u(x,u,v) + \mu G_u(x,u,v), & \text{in } \Omega, \\ \Delta(|\Delta u|^{q(x)-2}\Delta u) = \lambda F_v(x,u,v) + \mu G_v(x,u,v), & \text{in } \Omega, \\ u = \Delta u = v = \Delta v = 0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\lambda, \mu \in [0, +\infty)$, $\Omega \subset R^N$ $(N \geq 1)$ is a nonempty bounded open set with a boundary $\partial\Omega$ of class C^1, $F, G : \Omega \times R \times R \to R$ are functions such that $F(\cdot, s, t)$, $G(\cdot, s, t)$ are measurable in Ω for all $(s,t) \in R \times R$ and $F(x, \cdot, \cdot)$ is C^1 in $R \times R$ for a.e. $x \in \Omega$, F_i denotes the partial derivative of F with respect to i, $i = u, v$, so does G_i. And $p, q \in C(\overline{\Omega})$, $1 < p^- = \inf_{x\in\overline{\Omega}} p(x) \leq p^+ = \sup_{x\in\overline{\Omega}} p(x) < +\infty$, $1 < q^- = \inf_{x\in\overline{\Omega}} q(x) \leq q^+ = \sup_{x\in\overline{\Omega}} q(x) < +\infty$. Moreover,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N, \end{cases}$$

is the critical exponent just as in many papers. Obviously, $p(x) < p^*(x)$, $q(x) < q^*(x)$ for all $x \in \Omega$.

In what follows, E denotes the Cartesian product of two Sobolev spaces $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ and $W^{2,q(x)}(\Omega) \cap W_0^{1,q(x)}(\Omega)$, *i.e.*, $E = (W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)) \times (W^{2,q(x)}(\Omega) \cap W_0^{1,q(x)}(\Omega))$, and X denotes the Sobolev space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$.

In recent years, the study of differential equations and variational problems with $p(x)$-growth conditions has been an interesting topic resulting from nonlinear electrorheological fluids (see [1]) and elastic mechanics (see [2]).

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Some authors considered elliptic systems (see [3–16]) which have been used in a wide range of applications. Existence and multiplicity results for elliptic systems involving variational structure have been extensively investigated.

In [3], Boccardo and Figueiredo studied the following boundary value problem involving the (p,q)-Laplacian:

$$\begin{cases} -\Delta_{\mathrm{p}} u = F_u(x,u,v), \\ -\Delta_{\mathrm{q}} u = F_v(x,u,v), \end{cases}$$

where p and q are real numbers larger than 1.

In [4], applying the fibering method established by Pohozaev, Bozhkova and Mitidieri, the authors studied the existence of multiple solutions for quasilinear system involving a pair of (p,q)-Laplacian operators.

In [5], Chun Li and Chun-Lei Tang ensured the existence of three solutions for the problem

$$\begin{cases} -\Delta_p u = \lambda F_u(x,u,v), & \text{in } \Omega, \\ -\Delta_q u = \lambda F_v(x,u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $p > N$, $q > N$ and F satisfies suitable assumptions.

In [6], Afrouzi and Heidarkhani studied the existence of three solutions for a class of Dirichlet quasilinear elliptic systems involving the $(p_1,\ldots,p_n)$-Laplacian. In [7], Jing-Jing Liu and Xia-Yang Shi proved the existence of multiple solutions for a quasilinear system involving a pair of $(p(x),q(x))$-Laplacian operators. In [8], Bin Ge and Ji-Hong Shen obtained multiple solutions for a class of differential inclusion systems involving the $(p(x),q(x))$-Laplacian.

Recently, Lin Li and Chun-Lei Tang (see [9]) considered the Navier boundary value problem involving the (p,q)-biharmonic systems

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda F_u(x,u,v) + \mu G_u(x,u,v), & \text{in } \Omega, \\ \Delta(|\Delta u|^{q-2}\Delta u) = \lambda F_v(x,u,v) + \mu G_v(x,u,v), & \text{in } \Omega, \\ u = \Delta u = v = \Delta v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $p > \max\{1, \frac{N}{2}\}$, $q > \max\{1, \frac{N}{2}\}$, and F, G satisfy suitable assumptions.

The main result of this paper is the following theorem.

Theorem 1.1 *Suppose that there exist two positive constants C, d and two functions $\gamma(x), \beta(x) \in C(\overline{\Omega})$ with $1 < \gamma^- < \gamma^+ < p^-$, $1 < \beta^- < \beta^+ < q^-$ such that*

(j1) *$F(x,s,t) \geq 0$ for a.e. $x \in \Omega$ and all $(s,t) \in [0,d] \times [0,d]$;*

(j2) *$\exists p_1(x), q_1(x) \in C(\overline{\Omega})$ and $p^+ < p_1^- \leq p_1(x) < p^*(x)$, $q^+ < q_1^- \leq q_1(x) < q^*(x)$ such that*

$$\limsup_{(s,t)\to(0,0)} \sup_{x\in\Omega} \frac{F(x,s,t)}{|s|^{p_1(x)} + |t|^{q_1(x)}} < +\infty;$$

(j3) *$|F(x,s,t)| \leq C(1 + |s|^{\gamma(x)} + |t|^{\beta(x)})$ for a.e. $x \in \Omega$ and all $(s,t) \in R \times R$;*

(j4) $F(x,0,0)=0$ *for a.e.* $x\in\Omega$. *Then there exist an open interval* $\Lambda\subseteq[0,+\infty)$ *and a positive real number* r *with the following property*: *for each* $\lambda\in\Lambda$ *and each function* $G:\Omega\times R\times R\mapsto R$, *measurable in* Ω, C^1 *in* $R\times R$ *and satisfying*

$$\sup_{(x,s,t)\to(\Omega\times R\times R)}\frac{|G(x,s,t)|}{1+|s|^{p_2(x)}+|t|^{q_2(x)}}<\infty,$$

where $p_2,q_2\in C(\overline{\Omega})$ *and* $p_2(x)<p^*(x)$, $q_2(x)<q^*(x)$ *for all* $x\in\overline{\Omega}$, *there exists* $\delta>0$ *such that*, *for each* $\mu\in[0,\delta]$, *problem* (P) *has at least three weak solutions whose norms in* $(W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega))\times(W^{2,q(x)}(\Omega)\cap W_0^{1,q(x)}(\Omega))$ *are less than* r.

The paper is organized as follows. In Section 2, we present some necessary preliminary knowledge about the Lebesgue and Sobolev spaces with variable exponents, and present Ricceri's three-critical-points theorem. In Section 3, we prove the main result.

2 Preliminaries

Assume that Ω is a bounded domain of R^N $(N\ge 1)$ with a smooth boundary $\partial\Omega$. Let

$$C_+(\overline{\Omega})=\{h|h\in C(\overline{\Omega}),h(x)>1\text{ for all }x\in\overline{\Omega}\},$$
$$L_+^\infty(\Omega)=\Big\{p\in L^\infty(\Omega):\operatorname*{ess\,inf}_{x\in\Omega}p(x)>1\Big\}.$$

For $p\in L_+^\infty(\Omega)$, set

$$p^-=p^-(\Omega)=\operatorname*{ess\,inf}_{x\in\Omega}p(x),\qquad p^+=p^+(\Omega)=\operatorname*{ess\,sup}_{x\in\Omega}p(x).$$

For $p\in L_+^\infty(\Omega)$, define

$$L^{p(x)}(\Omega)=\Big\{u|u:\Omega\to R\text{ is measurable and }\int_\Omega|u|^{p(x)}\,dx<\infty\Big\}$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega)}=|u|_{p(x)}=\inf\Big\{\lambda:\int_\Omega\Big|\frac{u}{\lambda}\Big|^{p(x)}\,dx\le 1\Big\}$$

and

$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega):|\nabla u|\in L^{p(x)}(\Omega)\}$$

endowed with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)}=|u|_{p(x)}+|\nabla u|_{p(x)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

For the basic properties of the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, please refer to [17–20]. Now we recite some known results which will be used later.

Proposition 2.1 (see [17])

(i) *The spaces* $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces*;

(ii) *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *then the imbedding from* $W^{1,p(x)}(\Omega)$ *to* $L^{q(x)}(\Omega)$ *is compact and continuous*;

(iii) *There is a constant* $C > 0$ *such that* $|u|_{p(x)} \leq C|\nabla u|_{p(x)}$ $\forall u \in W_0^{1,p(x)}(\Omega)$.

By (iii) of Proposition 2.1, we know that $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. We use $|\nabla u|_{p(x)}$ to replace $\|u\|$ in the following discussion.

Proposition 2.2 (see [18]) *Set* $\rho(u) = \int_\Omega |u|^{p(x)}\,dx$ *for* $u, u_k \in L^{p(x)}(\Omega)$, *we obtain*

(1) *For* $u \neq 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$;

(2) $|u|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho(\frac{u}{\lambda}) < 1\ (= 1; > 1)$;

(3) *If* $|u|_{p(x)} > 1$, *then* $|u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;

(4) *If* $|u|_{p(x)} < 1$, *then* $|u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;

(5) $\lim_{k\to\infty} |u_k|_{p(x)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = 0$;

(6) $|u_k|_{p(x)} \to \infty \Leftrightarrow \rho(u_k) \to \infty$.

In this paper, the space E is endowed with the following equivalent norm:

$$\|(u,v)\| = \|u\| + \|v\|,$$

where

$$\|u\| = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\Delta u}{\lambda}\right|^{p(x)} dx \leq 1\right\}, \qquad \|v\| = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{\Delta v}{\mu}\right|^{q(x)} dx \leq 1\right\}.$$

Similar to Proposition 2.2, we obtain the following.

Proposition 2.3 *Let* $\phi(u) = \int_\Omega |\Delta u|^{p(x)}\,dx$ *for* $u, u_k \in W^{2,p(x)}(\Omega)$, *we obtain*:

(1) *For* $u \neq 0$, $\|u\| = \lambda \Leftrightarrow \phi(\frac{u}{\lambda}) = 1$;

(2) $\|u\| < 1\ (= 1; > 1) \Leftrightarrow \phi(\frac{u}{\lambda}) < 1\ (= 1; > 1)$;

(3) *If* $\|u\| > 1$, *then* $\|u\|^{p^-} \leq \phi(u) \leq \|u\|^{p^+}$;

(4) *If* $\|u\| < 1$, *then* $\|u\|^{p^+} \leq \phi(u) \leq \|u\|^{p^-}$;

(5) $\lim_{k\to\infty} \|u_k\| = 0 \Leftrightarrow \lim_{k\to\infty} \phi(u_k) = 0$;

(6) $\|u_k\| \to \infty \Leftrightarrow \phi(u_k) \to \infty$.

Let $G(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\,dx$, $u \in X$ and denote $L = G' : X \to X^*$, then

$$\bigl(L(u), v\bigr) = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v\,dx \quad \forall u, v \in X.$$

Proposition 2.4 (see [17])

(i) $L : X \to X^*$ *is a continuous, bounded and strictly monotone operator*;

(ii) L *is a mapping of type* (S_+), *i.e., if* $u_n \to u$ *in* X *and* $\overline{\lim}_{n\to\infty}((L(u_n) - L(u), u_n - u)) \leq 0$, *then* $u_n \to u$ *in* X;

(iii) $L : X \to X^*$ *is a homeomorphism*.

Proposition 2.5 (see [21]) *Let X be a separable and reflexive real Banach space; $I \subseteq R$; let $\Phi : X \to R$ be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $J : X \to R$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. In addition, Φ is bounded on each bounded subset of X. Assume that*

$$\lim_{\|u\|\to+\infty} \bigl(\Phi(u) + \lambda J(u)\bigr) = +\infty \tag{2.1}$$

for all $\lambda \in I \subseteq [0, \infty[$, and that there exists $\rho \in R$ such that

$$\sup_{\lambda\in I} \inf_{u\in X} \bigl(\Phi(u) + \lambda\bigl(J(u) + \rho\bigr)\bigr) < \inf_{u\in X} \sup_{\lambda\in I} \bigl(\Phi(u) + \lambda\bigl(J(u) + \rho\bigr)\bigr). \tag{2.2}$$

Then there exist a nonempty open set $A \subseteq I$ and a positive real number r with the following property: for every $\lambda \in A$ and every C^1 functional $\Psi : X \to R$ with a compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(u) + \lambda J'(u) + \mu \Psi'(u) = 0$$

has at least three solutions in X whose norms are less than r.

Proposition 2.6 (see [22]) *Let X be a nonempty set, and let Φ, J be two real functionals on X. Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that*

$$\begin{aligned} &\Phi(x_0) + J(x_0) = 0, \qquad \Phi(x_1) > r, \\ &\sup_{x\in\Phi^{-1}(]-\infty,r])} J(x) < r\frac{J(x_1)}{\Phi(x_1)}. \end{aligned} \tag{2.3}$$

Then, for each ρ satisfying

$$\sup_{x\in\Phi^{-1}(]-\infty,r])} J(x) < \rho < r\frac{J(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda\geq 0} \inf_{x\in X} \bigl(\Phi(x) + \lambda\bigl(\rho - J(x)\bigr)\bigr) < \inf_{x\in X} \sup_{\lambda\geq 0} \bigl(\Phi(x) + \lambda\bigl(\rho - J(x)\bigr)\bigr).$$

3 Proof of the main result

Definition 3.1 A weak solution of problem (P) is any $(u, v) \in E$ such that

$$\begin{aligned} &\int_\Omega \bigl(|\Delta u|^{p(x)-2} \Delta u \Delta \xi + |\Delta v|^{q(x)-2} \Delta v \Delta \eta\bigr)\, dx \\ &\quad - \lambda \int_\Omega (F_u \xi + F_v \eta)\, dx - \mu \int_\Omega (G_u \xi + G_v \eta)\, dx = 0 \end{aligned}$$

for all $\forall(\xi,\eta)\in E$. We define the corresponding energy functional of problem (P) as

$$\begin{aligned}H(u,v)&=\Phi(u,v)+\lambda J(u,v)+\mu\Psi(u,v)\\&=\int_\Omega\left(\frac{1}{p(x)}|\Delta u|^{p(x)}+\frac{1}{q(x)}|\Delta v|^{q(x)}\right)dx-\lambda\int_\Omega F(x,u,v)\,dx-\mu\int_\Omega G(x,u,v)\,dx,\end{aligned}$$

where

$$\begin{aligned}&\Phi(u,v)=\int_\Omega\left(\frac{1}{p(x)}|\Delta u|^{p(x)}+\frac{1}{q(x)}|\Delta v|^{q(x)}\right)dx,\\&J(u,v)=-\int_\Omega F(x,u,v)\,dx;\qquad\Psi(u,v)=-\int_\Omega G(x,u,v)\,dx.\end{aligned}$$

Then $H(u,v)$ is a C^1 functional and the critical points of it are weak solutions of problem (P).

Proof of Theorem 1.1 Let Φ, J, Ψ as above. So, for each $u,v,\xi,\eta\in E$, one has

$$\begin{aligned}&\Phi'(u,v)(\xi,\eta)=\int_\Omega\left(|\Delta u|^{p(x)-2}\Delta u\Delta\xi+|\Delta v|^{q(x)-2}\Delta v\Delta\eta\right)dx,\\&J'(u,v)(\xi,\eta)=-\int_\Omega F_u(x,u,v)\xi\,dx-\int_\Omega F_v(x,u,v)\eta\,dx,\\&\Psi'(u,v)(\xi,\eta)=-\int_\Omega G_u(x,u,v)\xi\,dx-\int_\Omega G_v(x,u,v)\eta\,dx.\end{aligned}$$

Therefore, the weak solutions of problem (P) are exactly the solutions of the equation

$$\Phi'(u,v)+\lambda J'(u,v)+\mu\Psi'(u,v)=0.$$

In view of Proposition 2.4 (or [17] for details), certainly, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on E^*. Moreover, J and Ψ are continuously Gâteaux differentiable functionals whose Gâteaux derivatives are compact. Obviously, Φ is bounded on each bounded subset of X under our assumptions.

By Proposition 2.3, set $G(u)=\int_\Omega\frac{1}{p(x)}|\Delta u|^{p(x)}\,dx$ just as before, then for $\|u\|\geq 1$,

$$\frac{1}{p^+}\|u\|^{p^-}\leq G(u)\leq\frac{1}{p^-}\|u\|^{p^+};\tag{3.1}$$

for $\|u\|<1$,

$$\frac{1}{p^+}\|u\|^{p^+}\leq G(u)\leq\frac{1}{p^-}\|u\|^{p^-}.\tag{3.2}$$

Actually, for $\|u\|<1$, set $C_0\geq\frac{1}{p^+}\|u\|^{p^-}-\frac{1}{p^+}\|u\|^{p^+}\geq 0$, then we can obtain

$$G(u)=\int_\Omega\frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\geq\frac{1}{p^+}\|u\|^{p^-}-C_0.$$

Hence we have

$$G(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\,dx \geq \frac{1}{p^+} \|u\|^{p^-} - C_0, \quad \forall u \in X.$$

So, there exists a constant $C_1 \geq 0$ such that

$$\begin{aligned}
\Phi(u,v) &= \int_\Omega \left(\frac{1}{p(x)} |\Delta u|^{p(x)} + \frac{1}{q(x)} |\Delta v|^{q(x)} \right) dx \\
&\geq \frac{1}{p^+} \|u\|^{p^-} + \frac{1}{q^+} \|u\|^{q^-} - C_1
\end{aligned}$$

holds for any $(u,v) \in E$.

$$\begin{aligned}
\lambda J(u,v) &= -\lambda \int_\Omega F(x,u,v)\,dx \\
&\geq -\lambda \int_\Omega C\big(1 + |u|^{\gamma(x)} + |v|^{\beta(x)}\big)\,dx \\
&\geq -\lambda C\big(|\Omega| + |u|^{\gamma^+}_{\gamma(x)} + |u|^{\gamma^-}_{\gamma(x)} + |v|^{\beta^+}_{\beta(x)} + |v|^{\beta^-}_{\beta(x)}\big) \\
&\geq -C_2\big(1 + |u|^{\gamma^+}_{\gamma(x)} + |v|^{\beta^+}_{\beta(x)}\big) \\
&\geq -C_3\big(1 + \|u\|^{\gamma^+} + \|v\|^{\beta^+}\big)
\end{aligned}$$

holds for any $(u,v) \in E$, where constants $C_2 \geq 0$, $C_3 \geq 0$. Here, by using conditions (j3) and (ii) of Proposition 2.1, combining the two inequalities above, we can obtain

$$\Phi(u,v) + \lambda J(u,v) \geq \frac{1}{p^+} \|u\|^{p^-} + \frac{1}{q^+} \|v\|^{q^-} - C_3\big(1 + \|u\|^{\gamma^+} + \|v\|^{\beta^+}\big) - C_1.$$

Due to $\gamma^+ < p^-$, $\beta^+ < q^-$, we get

$$\lim_{\|(u,v)\| \to +\infty} \big(\Phi(u,v) + \lambda J(u,v)\big) = +\infty \quad \forall (u,v) \in E, \lambda \in [0,\infty).$$

Then assumption (2.1) of Proposition 2.5 is fulfilled.

Next, we derive that assumption (2.2) is also fulfilled. It is easy to verify the conditions of Proposition 2.6. Let $(u_0,v_0) = (0,0)$, we can easily have

$$\Phi(u_0,v_0) = -J(u_0,v_0) = 0.$$

Then there exist $\gamma > 0$ and $(u_1,v_1) \in E$ such that $\Phi(u_1,v_1) > \gamma$ and (2.3) is satisfied.

There is a point $x^0 \in \Omega$ since it is a nonempty bounded open set. Let $r_2 > r_1 > 0$, put

$$w(x) = \begin{cases} 0, & x \in \Omega \backslash B(x^0, r_2), \\ \frac{d(3(l^4 - r_2^4) - 4(r_1 + r_2)(l^3 - r_2^3) + 6 r_1 r_2 (l^2 - r_2^2))}{(r_2 - r_1)(r_1 + r_2)}, & x \in B(x^0, r_2) \backslash B(x^0, r_1), \\ d, & x \in B(x^0, r_1), \end{cases}$$

where $B(x,r)$ is the open ball in R^N of radius r centered at x,

$$l = \operatorname{dist}(x,x^0) = \sqrt{\sum_{i=1}^{N}\left(x_i - x_i^0\right)^2}.$$

Let $(u_1(x), v_1(x)) = (w(x), w(x))$, then by (j1) we can derive that

$$-J(u_1,v_1) = -J(w,w) = \int_\Omega F(x,w,w)\,dx > 0.$$

From (j2), $\exists \eta \in [0,1]$, $C_1 > 0$ such that

$$\begin{aligned} F(x,s,t) &< C_1\left(|s|^{p_1(x)} + |t|^{q_1(x)}\right) \\ &< C_1\left(|s|^{p_1^-} + |t|^{q_1^-}\right) \quad \forall (s,t) \in [-\eta,\eta] \times [-\eta,\eta] \text{ a.e. } x \in \Omega. \end{aligned}$$

By (j3), there are nine positive real numbers M_i $(i = 1,2,\ldots,9)$ according to $|s|$, $|t|$ larger or smaller than η and 1. For example, when $|s| > 1$, $|t| < \eta$ some

$$M_i = \sup_{|s|>1,|t|<\eta} \frac{C(1 + |s|^{\gamma^+} + |s|^{\beta^-})}{|s|^{p_1^-} + |t|^{q_1^-}}.$$

Set $M = \max\{C_1, M_1, \ldots, M_9\}$, then

$$F(x,s,t) < M\left(|s|^{p_1^-} + |t|^{q_1^-}\right) \quad \forall (s,t) \in R \times R \text{ a.e. } x \in \Omega.$$

Hence, fix γ such that $0 < \gamma < 1$. And for $\frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} \le \gamma < 1$, by the Sobolev embedding theorem ($X \to L^{p_1^-}(\Omega)$ is continuous), there exist suitable positive constants C_2 and C_3 such that

$$\begin{aligned} -J(u,v) = \int_\Omega F(x,u,v)\,dx &< M\int_\Omega \left(|u|^{p_1^-} + |v|^{q_1^-}\right)dx \\ &\le C_2\left(\|u\|^{p_1^-} + \|v\|^{q_1^-}\right) \\ &\le C_3\left(\gamma^{\frac{p_1^-}{p^+}} + \gamma^{\frac{q_1^-}{q^+}}\right). \end{aligned}$$

Since $p_1^- > p^+$, $q_1^- > q^+$, we have

$$\lim_{\gamma\to 0^+} \frac{\sup_{\frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} \le \gamma} -J(u,v)}{\gamma} = 0. \tag{3.3}$$

We choose $w(x) \in X$ as above such that $-J(w,w) > 0$. Fix γ_0 such that $0 < \gamma < \gamma_0 < \min\{\frac{1}{p^+}, \frac{1}{q^+}\} \cdot \min\{\|w\|^{p^+} + \|w\|^{q^+}, \|w\|^{p^-} + \|w\|^{q^-}, 1\} \le 1$. Then we divide the proof into two cases.

(i) For $\|w\| < 1$, by (3.2) we have

$$\begin{aligned}\Phi(u_1, v_1) &= \Phi(w, w)\\ &= \int_\Omega \left(\frac{1}{p(x)}|\Delta w|^{p(x)} + \frac{1}{q(x)}|\Delta w|^{q(x)}\right) dx\\ &\geq \min\left\{\frac{1}{p^+}, \frac{1}{q^+}\right\} \int_\Omega \left(|\Delta w|^{p(x)} + |\Delta w|^{q(x)}\right) dx\\ &\geq \min\left\{\frac{1}{p^+}, \frac{1}{q^+}\right\} \left(\|w\|^{p^+} + \|w\|^{q^+}\right)\\ &\geq \gamma_0 > \gamma.\end{aligned}$$

By (3.3), we obtain

$$\begin{aligned}\sup_{\frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} \leq \gamma} -J(u, v) &\leq \frac{\gamma}{2} \frac{-J(u_1, v_1)}{\max\{\frac{1}{p^-}, \frac{1}{q^-}\}(\|w\|^{p^-} + \|w\|^{q^-})}\\ &\leq \frac{\gamma}{2} \frac{-J(u_1, v_1)}{\Phi(u_1, v_1)} < \gamma \frac{-J(u_1, v_1)}{\Phi(u_1, v_1)}.\end{aligned}$$

(ii) For $\|w\| \geq 1$, from (3.1) we get

$$\begin{aligned}\Phi(u_1, v_1) &= \Phi(w, w)\\ &= \int_\Omega \left(\frac{1}{p(x)}|\Delta w|^{p(x)} + \frac{1}{q(x)}|\Delta w|^{q(x)}\right) dx\\ &\geq \min\left\{\frac{1}{p^+}, \frac{1}{q^+}\right\} \int_\Omega \left(|\Delta w|^{p(x)} + |\Delta w|^{q(x)}\right) dx\\ &\geq \min\left\{\frac{1}{p^+}, \frac{1}{q^+}\right\} \left(\|w\|^{p^-} + \|w\|^{q^-}\right)\\ &\geq \gamma_0 > \gamma.\end{aligned}$$

From (3.3), we have

$$\begin{aligned}\sup_{\frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} \leq \gamma} -J(u, v) &\leq \frac{\gamma}{2} \frac{-J(u_1, v_1)}{\max\{\frac{1}{p^-}, \frac{1}{q^-}\}(\|w\|^{p^+} + \|w\|^{q^+})}\\ &\leq \frac{\gamma}{2} \frac{-J(u_1, v_1)}{\Phi(u_1, v_1)} < \gamma \frac{-J(u_1, v_1)}{\Phi(u_1, v_1)}.\end{aligned}$$

For any $(u, v) \in \Phi^{-1}((-\infty, \gamma])$, we can obtain $\Phi(u, v) < \gamma$, *i.e.*,

$$\Phi(u, v) = \int_\Omega \left(\frac{1}{p(x)}|\Delta u|^{p(x)} + \frac{1}{q(x)}|\Delta v|^{q(x)}\right) dx \leq \gamma.$$

Then we can have

$$\min\left\{\frac{1}{p^+}, \frac{1}{q^+}\right\} \int_\Omega \left(|\Delta u|^{p(x)} + |\Delta v|^{q(x)}\right) dx \leq \gamma.$$

So,

$$\int_\Omega \left(|\Delta u|^{p(x)} + |\Delta v|^{q(x)}\right) dx < \gamma \cdot \frac{1}{\min\{\frac{1}{p^+}, \frac{1}{q^+}\}} < \gamma_0 \cdot \frac{1}{\min\{\frac{1}{p^+}, \frac{1}{q^+}\}} < 1.$$

This inequality implies

$$\int_\Omega |\Delta u|^{p(x)}\, dx < 1, \qquad \int_\Omega |\Delta v|^{q(x)}\, dx < 1,$$

i.e.,

$$\|u\| < 1, \qquad \|v\| < 1.$$

Therefore we have

$$\frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|u\|^{q^+} < \int_\Omega \left(\frac{1}{p(x)}|\Delta u|^{p(x)} + \frac{1}{q(x)}|\Delta v|^{q(x)}\right) dx \le \gamma.$$

So, we can get that

$$\Phi^{-1}\big((-\infty, \gamma]\big) \subset \left\{(u,v) : (u,v) \in E, \frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|u\|^{q^+} < \gamma\right\}.$$

Then

$$\sup_{(u,v)\in\Phi^{-1}((-\infty,\gamma])} -J(u,v) \le \sup_{\frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|v\|^{q^+} \le \gamma} -J(u,v) < \gamma \frac{-J(u_1,v_1)}{\Phi(u_1,v_1)},$$

that is,

$$\sup_{(u,v)\in\Phi^{-1}((-\infty,\gamma])} -J(u,v) < \gamma \frac{-J(u_1,v_1)}{\Phi(u_1,v_1)}.$$

Hence we can find $\gamma > 0$, $u_1 = v_1 = w$ and $\Phi(w,w) \le \gamma$ satisfying (2.3). Also, we can find ρ satisfying

$$\sup_{(u,v)\in\Phi^{-1}((-\infty,\gamma])} -J(u,v) < \rho < \gamma \frac{-J(u_1,v_1)}{\Phi(u_1,v_1)}.$$

Put $I = [0,\infty)$, moreover, $\Phi(u,v)$, $-J(u,v)$ fulfil the assumption of Proposition 2.6. So, applying Proposition 2.6, we can easily get that (2.2) is fulfilled.

Thus, Φ, J and Ψ fulfil all the assumptions of Proposition 2.5, and our conclusion follows from Proposition 2.5. □

Remark Applying Theorem 2.1 in [23] to the proof of Theorem 1.1, an upper bound of the interval of parameters λ, for which (P) has at least three weak solutions, is obtained. To be precise, in the conclusion of Theorem 1.1, one has

$$\Lambda \subseteq \left[0, \frac{h\gamma}{\inf_{(u,v)\in\Phi^{-1}((-\infty,\gamma])} J(u,v) - \gamma \frac{J(u_1,v_1)}{\Phi(u_1,v_1)}}\right]$$

for each $h > 1$ and (u_1, v_1) as in the proof of Theorem 1.1 (namely, $u_1 = v_1 = w$).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This paper is the result of joint work of all authors who contributed equally to the final version of this paper. All authors read and approved the final manuscript.

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