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Extinction properties of solutions for a fast diffusion equation with nonlocal source

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Abstract

In this paper, we investigate extinction properties of nonnegative nontrivial solutions for an initial boundary value problem of a fast diffusion equation with a nonlocal source in bounded domain. By using the super- and sub-solution and the energy methods, we obtain some sufficient conditions for extinction and non-extinction of the weak solutions and give corresponding decay estimates which depend on the initial data, coefficients, and domains.

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Keywords: fast diffusion equation; extinction; non-extinction; decay estimate

1 Introduction

We consider the fast diffusion equation with a nonlocal source

$$u_t = \Delta u^m + au^q \int_{\Omega} u^p(y, t) dy, \quad x \in \Omega, t > 0, \quad (1.1)$$

subject to the homogeneous Dirichlet boundary and initial conditions

$$u(x, t) = 0, \quad x \in \partial\Omega, t > 0, \quad (1.2)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (1.3)$$

where $0 < m < 1, p \geq 0, q > 0, a > 0$, and $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with smooth boundary and $u_0 \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ is a nonnegative function. The symbols $\|\cdot\|_p$ ($p \geq 1$) and $|\Omega|$ denote the L^p -norm and the measure of Ω , respectively.

Equation (1.1) describes the fast diffusion of concentration of some Newtonian fluids through porous media or the density of some biological species in many physical phenomena and biological species theories. It has been known that the nonlocal source term presents a more realistic model for population dynamics, see [1–3]. In the nonlinear diffusion theory, there exist obvious differences among the situations of slow ($m > 1$), fast ($0 < m < 1$), and linear ($m = 1$) diffusions. For example, there is a finite speed propagation in the slow and linear diffusion situations, whereas an infinite speed propagation exists in the fast diffusion situation.

Recently, many scholars have been devoted to the study on blow-up and extinction properties of solutions for nonlinear diffusion equations with nonlocal terms, see [4–10]. Extinction of a function is a phenomenon for which there exists a finite time $T > 0$ such that

the solution is nontrivial on $(0, T)$ and then $u(x, t) \equiv 0$ for all $(x, t) \in \Omega \times [T, +\infty)$. In this case, T is called an extinction time. It is also an important property of solutions to nonlinear parabolic equations which have been studied by many researchers. For example, Kalashnikov [11] studied the Cauchy problem of a semilinear parabolic equation with an absorption term

$$u_t = \Delta u - \lambda u^p, \quad x \in \mathbb{R}^N, t > 0,$$

and obtained extinctions as well as localization and finite propagation properties of the solutions. Evans and Knerr [12] investigated extinction behaviors of the solutions for the Cauchy problem of a semilinear parabolic equation with a fully nonlinear absorption term

$$u_t(x, t) = \Delta u(x, t) - \beta(u(x, t)), \quad x \in \mathbb{R}^N, t > 0.$$

Ferreira and Vazquez [13] studied extinction phenomena of the solutions for the Cauchy problem of a porous medium equation with an absorption term

$$u_t = (u^m)_{xx} - u^p, \quad x \in \mathbb{R}, t > 0,$$

by using the analysis of a self-similar solution. By constructing a suitable comparison function, Li and Wu [14] considered the problem of a porous medium equation with a local source term

$$u_t = \Delta u^m + \lambda u^p, \quad x \in \Omega, t > 0,$$

subject to homogeneous Dirichlet boundary condition (1.2) and initial condition (1.3). They obtained some conditions for extinction and non-extinction of the solutions to the above equation and decay estimates. On extinctions of solutions to the p -Laplacian equation or the doubly degenerate equations, refer to [15, 16] and the references therein.

For equation (1.1) with $p = 0$ and $N > 2$, Han and Gao [8] showed that $q = m$ is a critical exponent for occurrence of extinction or non-extinction. Recently, Fang and Xu [9] considered equation (1.1) with $p = 0$ and a linear absorption term, when the diffusion term was replaced with p -Laplacian operator in the whole dimensional space, and showed that the extinction of the weak solution is determined by competition of source and absorption terms. They also obtained the exponential decay estimates which depend on the initial data, coefficients, and domains. Thereafter, they obtained the same results for a class of nonlocal porous medium equations with strong absorption, see [10].

Motivated by the mentioned works above, we study extinction behaviors of the solutions for problem (1.1)-(1.3) in the whole dimensional space. The main tools we use are the super- and sub-solution and the energy methods to obtain some sufficient conditions for extinction of the solutions, and we give exponential decay estimates which depend on the initial data, coefficients, and domains. In fact, the energy method has a wide application, especially for the equations that do not satisfy the maximum principle (*cf.* [17]).

Our paper is organized as follows. In Section 2, we give preliminary knowledge including lemmas that are required in the proofs of our results. In Section 3, we obtain a critical value for extinction of the solutions to problem (1.1)-(1.3) by using the modified comparison principle and give the decay estimates for the extinctions of the solutions in Section 4.

2 Preliminaries and main results

Due to the singularity of equation (1.1), problem (1.1)-(1.3) has no classical solutions in general, and hence it is reasonable to find a weak solution of the problem. To this end, we first give the following definition of a weak local solution.

Definition 1 A function $u \in L^\infty(Q_T)$ is called a super-solution of problem (1.1)-(1.3) in Q_T if the following conditions hold:

- (i) $u(x, 0) \geq u_0(x)$ in Ω ,
- (ii) $u(x, t) \geq 0$ on $\partial\Omega \times (0, T)$,
- (iii) for every $t \in (0, T)$ and every test function ξ ,

$$\int_{\Omega} u(x, t)\xi(x, t) dx \geq \int_{\Omega} u_0(x)\xi(x, 0) dx + \int_0^t \int_{\Omega} \left\{ u\xi_s + u^m \Delta \xi + au^q \int_{\Omega} u^p(y, t) dy \xi(x, s) \right\} dx ds,$$

where $0 \leq \xi \in C(\overline{Q_T}) \cap C^{1,2}(Q_T)$, $\xi_t, \Delta \xi \in L^2(Q_T)$, $\xi \geq 0$, $\xi_{\partial\Omega \times (0, T)} = 0$, and $Q_T = \Omega \times (0, T)$. A sub-solution can be similarly defined by replacing the inequality sign \geq in the above conditions with \leq . A function is called a local solution of (1.1)-(1.3) if it is both super- and sub-solution for some T .

Let $\varphi(x)$ be the unique positive solution of the following linear elliptic problem:

$$-\Delta\varphi(x) = 1, \quad x \in \Omega; \quad \varphi(x) = 0, \quad x \in \partial\Omega. \tag{2.1}$$

Throughout this paper, the constants M and μ are defined as $M = \sup_{x \in \overline{\Omega}} \varphi(x)$ and $\mu = \int_{\Omega} \varphi^{\frac{p}{m}}(x) dx$. The existence of local solutions can be obtained by utilizing the method of the standard regularization, and the regularities of the solutions can be derived by the argument similar to that in [18]. Since the regularization procedure is important to showing the uniqueness of the solution to problem (1.1)-(1.3) for some special cases, we sketch the outline below.

Consider the regularized problem

$$\begin{aligned} u_t &= \Delta u^m + au^q \int_{\Omega} u^p(y, t) dy, \quad x \in \Omega, 0 < t < T, \\ u(x, t) &= \frac{1}{k}, \quad x \in \partial\Omega, 0 < t < T, \\ u(x, 0) &= u_0(x) + \frac{1}{k}, \quad x \in \Omega, \end{aligned} \tag{2.2}$$

where $T > 0$ can be chosen sufficiently small so that there exists a solution u_k of (2.2) on Q_T for every $k \in \mathbb{N}$ and $\|u_k\|_\infty$ is bounded for all k . Furthermore, $\frac{1}{k} \leq u_l \leq u_k$ for $k < l$, and a super-solution (sub-solution) comparison theory holds for (2.2) (see [19]).

Since the sequence $\{u_k\}$ is monotone and bounded, we may define $U(x, t) \equiv \lim_{k \rightarrow \infty} u_k(x, t)$, and it is easy to see that $U(x, t)$ is a solution of (1.1)-(1.3). Furthermore, if u is a solution

of (1.1)-(1.3), we then have

$$\begin{aligned} \int_{\Omega} (u - u_k)\xi(x, t) \, dx &= \int_0^t \int_{\Omega} \left\{ (u - u_k)\xi_s + (u^m - u_k^m)\Delta\xi \right. \\ &\quad \left. + a\xi(x, s) \left(u^q \int_{\Omega} u^p(y, t) \, dy - u_k^q \int_{\Omega} u_k^p(y, t) \, dy \right) \right\} \, dx \, ds \\ &\quad + \frac{1}{k} \int_0^t \int_{\Omega} \frac{\partial \xi}{\partial n} \, dS_x \, ds - \frac{1}{k} \int_{\Omega} \xi(x, 0) \, dx \\ &\leq \int_0^t \int_{\Omega} \left\{ (u - u_k)\xi_s + (u^m - u_k^m)\Delta\xi \right. \\ &\quad \left. + a\xi(x, s) \left(u^q \int_{\Omega} u^p(y, t) \, dy - u_k^q \int_{\Omega} u_k^p(y, t) \, dy \right) \right\} \, dx \, ds \\ &= \int_0^t \int_{\Omega} \left\{ (u - u_k)\xi_s + (u^m - u_k^m)\Delta\xi + a\xi \int_{\Omega} u^p(y, t) \, dy (u^q - u_k^q) \right. \\ &\quad \left. + a\xi(x, s) u_k^q \int_{\Omega} (u^p(y, t) - u_k^p(y, t)) \, dy \right\} \, dx \, ds. \end{aligned}$$

Here, we have used the fact $\frac{\partial \xi}{\partial n} \leq 0$ on $\partial\Omega$ to derive this inequality.

Define the functions Φ , F , and G as

$$\begin{aligned} \Phi(x, t) &= m \int_0^1 (\theta u + (1 - \theta)u_k)^{m-1} \, d\theta, \\ F(x, t) &= q \int_0^1 (\theta u + (1 - \theta)u_k)^{q-1} \, d\theta, \end{aligned}$$

and

$$G(x, t) = p \int_0^1 (\theta u + (1 - \theta)u_k)^{p-1} \, d\theta,$$

and let $\varphi_u = u^m$, we then have

$$\begin{aligned} \Phi &= \int_0^t \varphi_u(\theta u + (1 - \theta)u_k) \, d\theta \geq \int_0^{\frac{1}{2}} \varphi_u\left(\frac{u_k}{2}\right) \, d\theta \\ &\geq \frac{1}{2} \min_{x \in \Omega, s \in [0, t], \max\{u + u_k(x, t) \in Q_T\} \geq p \geq \frac{1}{2}} \varphi_u(p) \equiv \eta, \end{aligned}$$

and we also have $F, G \in L^\infty$. We can choose sequences of smooth functions Φ_n, F_n , and G_n such that $\Phi_n \rightarrow \Phi, F_n \rightarrow F$, and $G_n \rightarrow G$ in $L^2(Q_{t_1})$ and also find a constant γ for which $\eta < \Phi_n < \gamma$ for all n .

Let $\xi \equiv \xi_n$ denote the solution of the problem

$$\begin{aligned} \xi_s + \Phi_n \Delta \xi + a\xi M^p |\Omega| G_n + aM^q \mu F_n &= 0, \quad x \in \Omega, t \in (0, T), \\ \xi(x, t) &= \chi(x), \quad x \in \Omega, \end{aligned}$$

where χ is a smooth function and has compact support in Ω with $0 \leq \chi \leq 1$. From Lemma 4.2 in [19], we can find some constants C_1 and C_2 such that $0 \leq \xi \leq C_1$ and

$\|\nabla \xi(\cdot, t)\|_2, \|\Delta \xi\|_2, \|\nabla \xi\|_2 \leq C_2$. With this ξ , we have

$$\begin{aligned} & \int_{\Omega} (u - u_k)(x, t) \chi(x, t) \, dx \\ & \leq \int_0^t \int_{\Omega} (u - u_k)(\Phi - \Phi_n) \Delta \xi + a \xi M^p |\Omega| (G - G_n) + a M^q \mu (F - F_n) \, dx \, ds. \end{aligned}$$

Since $u, u_k \in L^\infty(Q_T)$ for some constant M and $u_0 \leq v_0$, we deduce that

$$\begin{aligned} \int_{\Omega} (u - u_k)(x, t) \chi(x, t) \, dx & \leq K (\|\Phi_n - \Phi\|_2 \|\Delta \xi\|_2 + a M^p |\Omega| \|G - G_n\|_2 \|\xi\|_2 \\ & \quad + a M^q |\Omega| \|\mu\| \|F - F_n\|_2) \end{aligned}$$

for some constant K . Letting $n \rightarrow \infty$, we obtain

$$\int_{\Omega} (u - u_k)(x, t) \chi(x, t) \, dx \leq 0.$$

Choosing $\chi = \text{sign}(u - u_k)^+$, we have

$$\int_{\Omega} (u - u_k)^+(x, t) \, dx \leq 0.$$

This implies that $u \leq u_k$ on Q_{t_1} . Since $t_1 < T$ is arbitrary, we have $u \leq u_k \leq U$ on Q_T .

To establish the uniqueness of solution to problem (1.1)-(1.3) for some special cases, it only remains to prove that the reverse inequality is also true. The desired result can be seen in the following proposition.

Proposition 1 *If $p + q = m$ and $a \mu M^{\frac{q}{m}} < 1$, where μ and M are constants defined in (2.1), then the nonnegative solution of (1.1)-(1.3) is unique. Furthermore, if v is a sub-solution of (1.1)-(1.3), we have $v \leq u$.*

Proof We only need to show the uniqueness of solution.

Let u be an arbitrary solution of (1.1)-(1.3), and let u_k denote the solution of (2.2). We then have

$$\begin{aligned} & \int_{\Omega} (u_k - u) \xi(x, t) \, dx \\ & = \int_0^t \int_{\Omega} \left\{ (u_k - u) \xi_s + (u_k^m - u^m) \Delta \xi \right. \\ & \quad \left. + a \xi(x, s) \left(u_k^q \int_{\Omega} u_k^p(y, t) \, dy - u^q \int_{\Omega} u^p(y, t) \, dy \right) \right\} \, dx \, ds \\ & \quad - \frac{1}{k} \int_0^t \int_{\Omega} \frac{\partial \xi}{\partial n} \, dS_x \, ds + \frac{1}{k} \int_{\Omega} \xi(x, 0) \, dx. \end{aligned}$$

Choosing the unique positive solution $\varphi(x)$ of (2.1) as a testing function $\xi(x, t)$ and noticing that $p + q = m$, $a\mu M^{\frac{q}{m}} < 1$, and $0 \leq u \leq u_k \leq M'$, we get

$$\begin{aligned} \int_{\Omega} (u_k - u)\xi(x, t) \, dx &= \int_0^t \int_{\Omega} \left\{ -(u_k - u) + a\mu M^{\frac{q}{m}} (M^{p+q} - 0^{p+q}) \right\} \, dx \, ds \\ &\quad - \frac{1}{k} \int_0^t \int_{\Omega} \frac{\partial \xi}{\partial n} \, dS_x \, ds + \frac{1}{k} \int_{\Omega} \xi(x, 0) \, dx \\ &\leq -\frac{1}{k} \int_0^t \int_{\Omega} \frac{\partial \xi}{\partial n} \, dS_x \, ds + \frac{1}{k} \int_{\Omega} \xi(x, 0) \, dx. \end{aligned}$$

Letting $k \rightarrow \infty$, we obtain

$$\int_{\Omega} (U - u)\xi(x, t) \, dx \leq 0.$$

The above inequality together with the fact $U \geq u$ guarantees that $u \equiv U$. □

The following comparison principle and lemmas will play a crucial role in what follows, but the proofs of them are simple, and so we omit them here (see [19, 20]).

Proposition 2 (Comparison principle) *Let u and v be nonnegative bounded super- and sub-solution of (1.1)-(1.3), respectively, with $u \geq \delta$ for some $\delta > 0$. If $v(x, 0) \leq u(x, 0)$, then $v \leq u$ on Q_T .*

Lemma 1 *Suppose that k and α are positive constants, with $k < 1$. If $y(t)$ is a nonnegative absolutely continuous function on $[0, \infty)$ solving the problem*

$$\frac{dy}{dt} + \alpha y^k \leq 0, \quad t \geq 0; \quad y(0) \geq 0,$$

we then have the decay estimate

$$\begin{aligned} y(t) &\leq [y^{1-k}(0) - \alpha(1-k)t]^{\frac{1}{1-k}}, \quad t \in [0, T_*), \\ y(t) &\equiv 0, \quad t \in [T_*, +\infty), \end{aligned}$$

where $T_ = \frac{y^{1-k}(0)}{\alpha(1-k)}$.*

Lemma 2 [20] *Suppose that $0 < k < m \leq 1$ and $y(t)$ is a nonnegative function solving the problem*

$$\frac{dy}{dt} + \alpha y^k \leq \gamma y^m, \quad t \geq 0; \quad y(0) = y_0 > 0,$$

where α and γ are nonnegative constants. If $\gamma < \alpha y_0^{k-m}$, there exists $\eta > 0$ such that $0 \leq y(t) \leq y_0 e^{-\eta t}$ for all $t \geq 0$.

3 Extinction and non-extinction

In this section, we construct suitable super- and sub-solution to determine whether there exist extinction phenomena for the solutions of problem (1.1)-(1.3).

Theorem 1 *If $p + q = m$ and $a\mu M_1^{\frac{q}{m}} < 1$, then for any nonnegative initial data u_0 , the unique solution $u(x, t)$ of (1.1)-(1.3) vanishes in finite time.*

Proof We will prove this theorem by constructing a proper super-solution. Let $\varphi_1(x)$ be the unique positive solution of the following elliptic problem:

$$-\Delta\varphi_1(x) = 1, \quad x \in \Omega_1; \quad \varphi_1(x) = 0, \quad x \in \partial\Omega_1. \tag{3.1}$$

Let $M_1 = \sup_{x \in \overline{\Omega_1}} \varphi_1(x)$, $\delta = \inf_{x \in \overline{\Omega_1}} \varphi_1(x)$, and let $\mu_1 = \int_{\Omega_1} \varphi_1^{\frac{p}{m}}(x) dx$. We know from the comparison principle of elliptic problems that $\varphi(x) < \varphi_1(x)$ for $x \in \Omega$, and $\delta > 0$ and $\mu < \mu_1$. Since $a\mu M_1^{\frac{q}{m}} < 1$ and $\varphi_1(x)$ is continuous, we can find a domain $\Omega \subset \Omega_1$ such that $a\mu_1 M_1^{\frac{q}{m}} < 1$. Let $g(t)$ be the positive solution of the following problem:

$$g'(t) = -\frac{1 - a\mu_1 M_1^{\frac{q}{m}}}{M_1^{\frac{1}{m}}} g^m(t), \quad g(0) = A,$$

where $A > 0$ is a constant large enough so that $u_0 \leq A\varphi_1^{\frac{1}{m}}(x)$ for all $x \in \overline{\Omega}$. Since $0 < m < 1$, the function $g(t)$ vanishes in a finite time $T_0 > 0$. Set $v(x, t) = g(t)\varphi_1^{\frac{1}{m}}(x)$, and then it can be easily seen that $v(x, t)$ also vanishes from the time T_0 .

On the other hand, one can directly verify that $v(x, t)$ is a super-solution of (1.1)-(1.3) for any fixed T such that $0 < T < T_0$. There exist two positive constants C_1 and C_2 such that $C_1 \leq v(x, t) \leq C_2$. By Proposition 2, we know that $u(x, t) \leq v(x, t)$ for any $(x, t) \in \overline{\Omega} \times [0, T]$. Since $T < T_0$ is arbitrary, one can see that $u(x, T_1) \equiv 0$ for some $T_1 < T_0$. Then it follows from Proposition 1 that $u(x, t) \equiv 0$ for all $t \geq T_1$, which implies that $u(x, t)$ vanishes from the time T_1 . □

Theorem 2 *Suppose that $p + q > m$ and $a \leq \frac{\max_{\Omega} \varphi^{\frac{p+q}{m}}}{2\delta^{p+q-m} |\Omega|}$. Then, for any nonnegative initial data u_0 , the unique solution $u(x, t)$ of (1.1)-(1.3) vanishes in finite time.*

Proof Let $v(x, t) = g(t)\varphi^{\frac{1}{m}}(x)$, where $\varphi(x)$ is the function solving (2.1). Then $v(x, t)$ is a super-solution of (1.1)-(1.3) if and only if the following conditions (3.2), (3.3), and (3.4) hold:

$$g' \varphi^{\frac{1}{m}}(x) \geq g^m(t) \Delta \varphi + a g^{p+q}(t) \varphi^{\frac{p}{m}} \int_{\Omega} \varphi^{\frac{q}{m}} dx, \quad x \in \Omega, 0 < t < T, \tag{3.2}$$

$$g(t) \varphi^{\frac{1}{m}}(x) \geq 0, \quad x \in \partial\Omega, 0 < t < T, \tag{3.3}$$

$$g(0) \varphi^{\frac{1}{m}}(x) \geq 0, \quad x \in \Omega. \tag{3.4}$$

Let $g'(t) = -\frac{g^m(t)}{2H}$, where $\min_{x \in \Omega} \varphi(x) = 1$ and $H = \max_{\Omega} \varphi^{\frac{1}{m}}(x)$. One can see that the following condition is sufficient to guarantee (3.2):

$$g^m(t) \geq -g'(t) \varphi^{\frac{1}{m}}(x) + a g^{p+q}(t) \varphi^{\frac{p}{m}} \int_{\Omega} \varphi^{\frac{q}{m}} dx. \tag{3.5}$$

Since

$$g'(t) \varphi^{\frac{1}{m}}(x) = \frac{g^m(t)}{2H} \varphi^{\frac{1}{m}} \leq \frac{1}{2} g^m(t) \tag{3.6}$$

and $v(x, t) \geq \delta$, we may choose a constant $a \leq \frac{\max \varphi^{\frac{p+q}{m}} |\Omega|}{2\delta^{p+q-m}}$ such that

$$ag^{p+q}(t)\varphi^{\frac{p}{m}} \int_{\Omega} \varphi^{\frac{q}{m}} dx \leq \frac{1}{2}g^m(t). \tag{3.7}$$

Combining (3.6) with (3.7), it can be seen that (3.5) is true. Hence, if $u_0 \leq g(0)\varphi(x)$, the function v is a super-solution of (1.1)-(1.3). From the definition of $g(t)$, we obtain the integral equality

$$\int_{g(t)}^{g_0} \frac{ds}{g^m} = \frac{t}{2H}.$$

Define $\bar{T} = 2H \int_0^{g_0} \frac{1}{g^m} ds$. We then have $v > 0$ for $0 \leq t < \bar{T}$ and $x \in \bar{\Omega}$, and $v(x, \bar{T}) = 0$ for $x \in \Omega$. By Proposition 2, we conclude that $u \leq v$ for $x \in \Omega$ and $0 < t < \bar{T}$. \square

Theorem 3 *Assume that $p + q < m$ or $p + q = m$ and $a\mu M^{\frac{q}{m}} > 1$. Then, for any nonnegative initial data u_0 , the maximal solution $U(x, t)$ of (1.1)-(1.3) does not vanish in finite time.*

Proof We will prove this theorem by constructing a suitable sub-solution. Set $v(x, t) = g(t)\varphi^{\frac{1}{m}}$, where $\varphi(x)$ is the unique positive solution of (2.1). It can be easily verified that $v(x, t)$ is a sub-solution of (1.1)-(1.3) for $p + q < m$, if $g(t)$ solves the problem

$$\begin{aligned} g'(t) &= M^{-\frac{1}{m}} \{-g^m(t) + a\mu M^{\frac{q}{m}} g^{p+q}\}; \\ g(0) &= 0, \quad g(t) > 0 \quad (t > 0). \end{aligned}$$

If $p + q = m$ and $a\mu M^{\frac{q}{m}} > 1$, choose $g(t)$ to be the solution of the problem

$$\begin{aligned} g'(t) &= M^{-\frac{1}{m}} (a\mu M^{\frac{q}{m}} - 1)g^m(t); \\ g(0) &= 0, \quad g(t) > 0 \quad (t > 0). \end{aligned}$$

Then $v(x, t)$ is also a sub-solution of (1.1)-(1.3). Hence, it follows from the sub-solution comparison principle that $U(x, t) > 0$ in $\Omega \times (0, \infty)$. \square

Remark 1 It can be seen from Theorems 1-3 that when $p + q < m$ or $p + q = m$ and $a\mu M^{\frac{q}{m}} > 1$, the maximal solution $U(x, t)$ is positive for all $t > 0$, which means that the effect of the source term is, in some sense, strong and the diffusion term cannot dominate the source term. However, when $p + q > m$ and $a \leq \frac{\max \varphi^{\frac{p+q}{m}} |\Omega|}{2\delta^{p+q-m}}$ or $p + q = m$ and $a\mu M^{\frac{q}{m}} < 1$, the effect of the nonlocal source is a little weak, and the diffusion term may cause the non-negative solution of (1.1)-(1.3) to vanish in finite time, provided that the initial data are sufficiently small.

4 Decay estimates

In Section 3, we have used the super- and sub-solution method to obtain some sufficient conditions on extinction for the solutions of problem (1.1)-(1.3), but we cannot obtain the decay estimates by the method. Therefore, we adopt other approximation techniques, *i.e.*, the energy method and the differential inequality technique to obtain corresponding decay estimates which depend on the initial data, coefficients, and domains.

Theorem 4 Suppose $0 < m < 1$ and $m = p + q$.

(1) When $N = 1$ or 2 , if $|\Omega| < [\frac{4m\gamma_1^{-2}(N,\Omega)}{a(m+1)^2}]^{\frac{2}{3-m}}$ or $a < \frac{4m\gamma_1^{-2}(N,\Omega)}{|\Omega|^{\frac{3-m}{2}}(m+1)^2}$, then for any nonnegative initial data u_0 , the unique solution $u(x, t)$ of (1.1)-(1.3) vanishes in finite time, with the following decay estimates:

$$\begin{aligned} \|u(\cdot, t)\|_2^2 &\leq \left[\|u_0\|_2^{1-m} - C_1 \left(1 - \frac{m+1}{2}\right) t \right]^{\frac{1}{1-\frac{m+1}{2}}}, \quad t \in [0, T_1]; \\ \|u(\cdot, t)\|_2^2 &\equiv 0, \quad t \in [T_1, +\infty), \end{aligned}$$

where $\gamma_1 > 0$ is an embedding constant, and C_1 and T_1 will be determined later.

(2) When $N > 2$,

(a) if $\frac{N-2}{N+2} \leq m < 1$, and $|\Omega| < (\frac{\gamma_2^{-2}}{a})^{\frac{1}{1+\frac{2}{N}}}$ or $a < \frac{\gamma_2^{-2}}{|\Omega|^{1+\frac{2}{N}}}$, then for any nonnegative initial data u_0 , the unique solution $u(x, t)$ of (1.1)-(1.3) vanishes in finite time, with the following decay estimates:

$$\begin{aligned} \|u(\cdot, t)\|_{1+m}^{1+m} &\leq \left[\|u_0\|_{1+m}^{1-m} - C_2 \left(1 - \frac{2m}{1+m}\right) t \right]^{\frac{1}{1-\frac{2m}{1+m}}}, \quad t \in [0, T_2]; \\ \|u(\cdot, t)\|_{1+m}^{1+m} &\equiv 0, \quad t \in [T_2, +\infty), \end{aligned}$$

where $\gamma_2 > 0$ is an embedding constant, and C_2 and T_2 will be determined later;

(b) if $0 < m < \frac{N-2}{N+2}$, and $|\Omega| < [\frac{4ms\gamma_3^{-2}(N,\Omega)}{a(m+s)^2}]^{\frac{1}{1+\frac{2}{N}}}$ or $a < \frac{4ms\gamma_3^{-2}(N,\Omega)}{|\Omega|^{1+\frac{2}{N}}(m+s)^2}$, then for any nonnegative initial data u_0 , the unique solution $u(x, t)$ of (1.1)-(1.3) vanishes in finite time, with the following decay estimates:

$$\begin{aligned} \|u(\cdot, t)\|_{1+s}^{1+s} &\leq \left[\|u_0\|_{1+s}^{1-m} - C_3 \left(1 - \frac{m+s}{1+s}\right) t \right]^{\frac{1}{1-\frac{m+s}{1+s}}}, \quad t \in [0, T_3]; \\ \|u(\cdot, t)\|_{1+s}^{1+s} &\equiv 0, \quad t \in [T_3, +\infty), \end{aligned}$$

where $s = \frac{N}{2}(1-m) - 1$ and $\gamma_3 > 0$ is an embedding constant, and C_3 and T_3 will be determined later.

Proof We first consider the case $N = 1$ or 2 . Multiplying both sides of equation (1.1) by u^s ($s \geq 0$) and integrating the result over Ω , we obtain

$$\frac{1}{s+1} \frac{d}{dt} \int_{\Omega} u^{s+1} dx + \frac{4ms}{(m+s)^2} \int_{\Omega} |\nabla u^{\frac{m+s}{2}}|^2 dx = a \int_{\Omega} u^p dx \int_{\Omega} u^{q+s} dx. \quad (4.1)$$

By Hölder's inequality, we get the inequality

$$\int_{\Omega} u^p dx \int_{\Omega} u^{q+s} dx \leq \|\Omega\|^{\frac{2s_1-p-q-s}{s_1}} \|u\|_{s_1}^{p+q+s},$$

where $s_1 \geq 1$ will be determined later. Choosing $s_1 = s + 1 \geq 1$, we get the inequality

$$\frac{1}{s+1} \frac{d}{dt} \int_{\Omega} u^{s+1} dx + \frac{4ms}{(m+s)^2} \int_{\Omega} |\nabla u^{\frac{m+s}{2}}|^2 dx \leq a |\Omega|^{\frac{2+s-m}{s+1}} \|u\|_{s+1}^{m+s}. \quad (4.2)$$

By using the Sobolev embedding inequality, one can show that there exists an embedding constant $\gamma_1(N, \Omega)$ such that

$$\|u^{\frac{m+s}{2}}\|_{s_2} \leq \gamma_1^2(N, \Omega) \|\nabla u^{\frac{m+s}{2}}\|_2,$$

where $s_2 \geq 2$ will be determined later, *i.e.*,

$$\|u\|_{\frac{s+m}{(m+s)s_2}} \leq \gamma_1^2(N, \Omega) \|\nabla u^{\frac{m+s}{2}}\|_2.$$

Let $C_1 = \frac{4m\gamma_1^{-2}(N, \Omega)}{(m+1)^2} - a|\Omega|^{\frac{3-m}{2}}$, $s_2 = \frac{4}{1+m}$, and let $s = 1$. Then $s_2 \geq 2$, since $0 < m < 1$. From the above inequality with the constant s_2 and s , we obtain the inequality $\frac{d}{dt} \|u\|_2^2 + C_1 \|u\|_2^{m+1} \leq 0$, if $C_1 > 0$, *i.e.*, $|\Omega| < [\frac{4m\gamma_1^{-2}(N, \Omega)}{a(m+1)^2}]^{\frac{2}{3-m}}$. By Lemma 1, we then obtain its decay estimates, with $T_1 = \frac{\|u_0\|_2^{1-m}}{C_1(1-\frac{m+1}{2})}$.

Secondly, we consider the case $N > 2$.

(a) For m such that $\frac{N-2}{N+2} \leq m < 1$, multiplying both sides of equation of (1.1) by u^m and integrating the result over Ω , we obtain

$$\frac{1}{m+1} \frac{d}{dt} \int_{\Omega} u^{m+1} dx + \int_{\Omega} |\nabla u^m|^2 dx = a \int_{\Omega} u^p dx \int_{\Omega} u^{q+m} dx. \tag{4.3}$$

By the Sobolev embedding and Hölder's inequalities, one can show that

$$\frac{1}{m+1} \frac{d}{dt} \|u\|_{1+m}^{(1+m)} + \gamma_2^{-2} |\Omega|^{\frac{N-2}{N} - \frac{2m}{1+m}} \|u\|_{1+m}^{2m} \leq a |\Omega|^{\frac{2}{1+m}} \|u\|_{1+m}^{2m},$$

and hence we get

$$\frac{d}{dt} \|u\|_{1+m}^{1+m} + C_2 \|u\|_{1+m}^{2m} \leq 0,$$

provided $C_2 = (m+1)(\gamma_2^{-2} |\Omega|^{\frac{N-2}{N} - \frac{2m}{1+m}} - a |\Omega|^{\frac{2}{1+m}}) > 0$, *i.e.*, $|\Omega| < (\frac{\gamma_2^{-2}}{a})^{1+\frac{1}{N}}$. By Lemma 1, we then obtain its decay estimates, with $T_2 = \frac{\|u_0\|_{1+m}^{1-m}}{C_2(1-\frac{2m}{1+m})}$.

(b) For m such that $0 < m < \frac{N-2}{N+2}$, multiplying both sides of equation (1.1) by u^s ($s = \frac{N}{2}(1-m) - 1$) and integrating the result over Ω , we obtain

$$\frac{1}{s+1} \frac{d}{dt} \int_{\Omega} u^{s+1} dx + \frac{4ms}{(m+s)^2} \int_{\Omega} |\nabla u^{\frac{m+s}{2}}|^2 dx = a \int_{\Omega} u^p dx \int_{\Omega} u^{q+s} dx. \tag{4.4}$$

By the Sobolev embedding inequality and a suitable s , one can show that

$$\|u\|_{\frac{s+m}{s+1}}^{\frac{s+m}{2}} = \left(\int_{\Omega} u^{\frac{s+m}{2} - \frac{2N}{N-2}} \right) \leq \gamma_3 \|\nabla u^{\frac{s+m}{2}}\|_2. \tag{4.5}$$

Using Hölder's inequality, we have

$$\int_{\Omega} u^p dx \int_{\Omega} u^{q+s} dx \leq \|\Omega\|^{2-\frac{m+s}{s+1}} \|u\|_{\frac{s+m}{s+1}}^{m+s}. \tag{4.6}$$

From inequalities (4.5) and (4.6), we get the inequality

$$\frac{1}{s+1} \frac{d}{dt} \|u\|_{1+s}^{(1+s)} + \frac{4ms}{(m+s)^2} \gamma_3^{-2} \|u\|_{1+s}^{s+m} \leq a |\Omega|^{2-\frac{m+s}{1+s}} \|u\|_{1+s}^{s+m}.$$

Furthermore, we can have $\frac{d}{dt} \|u\|_{1+s}^{1+s} + C_3 \|u\|_{1+s}^{s+m} \leq 0$, provided $C_3 = (s+1)(\gamma_3^{-2} \frac{4ms}{(m+s)^2} - a |\Omega|^{2-\frac{m+s}{1+s}}) > 0$, i.e., $|\Omega| < (\frac{4ms\gamma_3^{-2}}{a(m+s)^2})^{\frac{1}{1+\frac{1}{N}}}$. By Lemma 1, we then obtain its decay estimates, with $T_3 = \frac{\|u_0\|_{1+s}^{1-m}}{C_3 \frac{1-m}{1+s}}$. □

Theorem 5 *Suppose that $0 < m < 1$ and $p + q > m$.*

(1) *When $N = 1$ or 2 , if*

$$|\Omega| < \left[\frac{8m\gamma_4^{-2}(N, \Omega)}{a(\|u_0\|_2^2 e^{-\alpha_1 T_4})^{p+q-m} (m+1)^2} \right]^{\frac{2}{3-p-q}}$$

or

$$a < \frac{8m\gamma_4^{-2}(N, \Omega)}{|\Omega|^{\frac{3-p-q}{2}} (\|u_0\|_2^2 e^{-\alpha_1 T_4})^{p+q-m} (m+1)^2}$$

and u_0 small enough, the unique solution $u(x, t)$ of (1.1)-(1.3) vanishes in finite time, with the following decay estimates:

$$\begin{aligned} \|u(\cdot, t)\|_2^2 &\leq [\|u_0\|_2^2 e^{-\alpha_1 t}], \quad t \in [0, T_4]; \\ \|u(\cdot, t)\|_2^2 &\leq \left[\|u_0\|_2^{1-m} - C_4 \left(1 - \frac{m+1}{2}\right) t \right]^{\frac{1}{1-\frac{m+1}{2}}}, \quad t \in [T_4, T_5]; \\ \|u(\cdot, t)\|_2^2 &\equiv 0, \quad t \in [T_5, +\infty), \end{aligned}$$

where $\gamma_4 > 0$ is an embedding constant, α_1 is a suitable constant, and C_4, T_4, T_5 will be determined later.

(2) *When $N > 2$,*

(a) *for m such that $\frac{N-2}{N+2} \leq m < 1$, if*

$$|\Omega| < \left(\frac{\gamma_5^{-2}}{(a\|u_0\|_{1+m}^{1+m} e^{-\alpha_2 t})^{\frac{p+q-m}{1+m}}} \right)^{\frac{1}{\frac{N+2}{N} + \frac{m-p-q}{1+m}}}$$

or

$$a < \frac{\gamma_5^{-2}}{|\Omega|^{\frac{N+2}{N} + \frac{m-p-q}{1+m}} (\|u_0\|_{1+m}^{1+m} e^{-\alpha_2 t})^{\frac{p+q-m}{1+m}}}$$

and u_0 small enough, the unique solution $u(x, t)$ of (1.1)-(1.3) vanishes in finite time, with the following decay estimates:

$$\begin{aligned} \|u(\cdot, t)\|_{1+m}^{1+m} &\leq [\|u_0\|_{1+m}^{1+m} e^{-\alpha_2 t}], \quad t \in [0, T_6]; \\ \|u(\cdot, t)\|_{1+m}^{1+m} &\leq \left[\|u_0\|_{1+m}^{1-m} - C_5 \left(1 - \frac{2m}{1+m}\right) t \right]^{\frac{1}{1-\frac{2m}{1+m}}}, \quad t \in [T_6, T_7]; \end{aligned}$$

$$\|u(\cdot, t)\|_{1+m}^{1+m} \equiv 0, \quad t \in [T_7, +\infty),$$

where $\gamma_5 > 0$ is an embedding constant, α_2 is a suitable constant, and C_5, T_6, T_7 will be determined later;

(b) for m such that $0 < m < \frac{N-2}{N+2}$, if

$$|\Omega| < \left(\frac{4ms\gamma_6^{-2}}{a(m+s)^2(\|u_0\|_{1+s}^{1+s}e^{-\alpha_3 t})^{\frac{p+q-s}{1+s}}} \right)^{\frac{1}{2-\frac{m+p+q}{1+m}}}$$

or

$$a < \frac{4ms\gamma_6^{-2}}{|\Omega|^{2-\frac{m+p+q}{1+m}}(m+s)^2(\|u_0\|_{1+s}^{1+s}e^{-\alpha_3 t})^{\frac{p+q-s}{1+s}}}$$

and u_0 small enough, the unique solution $u(x, t)$ of (1.1)-(1.3) vanishes in finite time, with the following decay estimates:

$$\begin{aligned} \|u(\cdot, t)\|_{1+s}^{1+s} &\leq [\|u_0\|_{1+s}^{1+s}e^{-\alpha_3 t}], \quad t \in [0, T_8]; \\ \|u(\cdot, t)\|_{1+s}^{1+s} &\leq \left[\|u_0\|_{1+s}^{1-m} - C_6 \left(1 - \frac{s+m}{1+s} \right) t \right]^{1-\frac{s+m}{1+s}}, \quad t \in [T_8, T_9]; \\ \|u(\cdot, t)\|_{1+s}^{1+s} &\equiv 0, \quad t \in [T_9, +\infty), \end{aligned}$$

where $\gamma_6 > 0$ is an embedding constant, α_3 is a suitable constant, and C_6, T_8, T_9 will be determined later.

Proof Multiplying both sides of equation (1.1) by u^s ($s \geq 0$) and integrating the result over Ω , we obtain

$$\frac{1}{s+1} \frac{d}{dt} \int_{\Omega} u^{s+1} dx + \frac{4ms}{(m+s)^2} \int_{\Omega} |\nabla u^{\frac{m+s}{2}}|^2 dx = a \int_{\Omega} u^p dx \int_{\Omega} u^{q+s} dx. \tag{4.7}$$

We first consider the case $p+q \leq 1$.

(1) When $N = 1$ or 2 , setting $s = 1$ in (4.7), we get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx + \frac{4m}{(m+1)^2} \int_{\Omega} |\nabla u^{\frac{m+1}{2}}|^2 dx = a \int_{\Omega} u^p dx \int_{\Omega} u^{q+1} dx.$$

Using Hölder's inequality and the Sobolev embedding inequality, one can show that

$$\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \frac{8m\gamma_4^{-2}}{(m+1)^2} \|u\|_2^{m+1} \leq a|\Omega|^{\frac{3-p-q}{2}} \|u\|_2^{p+q+1}.$$

By Lemma 2, one can see that there exists a constant $\alpha_1 > 0$ such that $0 \leq \|u(\cdot, t)\|_2^2 \leq \|u_0\|_2^2 e^{-\alpha_1 t}$, provided $\|u_0\|_2^2 \leq \left[\frac{8m\gamma_4^{-2}}{a(m+1)^2|\Omega|^{\frac{3-p-q}{2}}} \right]^{\frac{1}{p+q-m}}$. Hence, there exists a constant $T_4 > 0$ such that

$$\frac{8m\gamma_4^{-2}}{(m+1)^2} - a|\Omega|^{\frac{3-p-q}{2}} \|u\|_2^{p+q-m} \geq \frac{8m\gamma_4^{-2}}{(m+1)^2} - a|\Omega|^{\frac{3-p-q}{2}} (\|u_0\|_2^2 e^{-\alpha_1 T_4})^{p+q-m} = C_4 > 0$$

for all $t \in [T_4, +\infty)$, and so $\frac{d}{dt} \|u\|_2^2 + C_4 \|u\|_2^{m+1} \leq 0$. By Lemma 1 we can obtain its decay estimate, with $T_5 = \frac{\|u_0\|_2^{1-m}}{C_4(1-\frac{m+1}{2})} + T_4$.

(2) When $N > 2$,

(a) for m such that $\frac{N-2}{N+2} < m < 1$, we choose $s = m$ in (4.7). We then have the inequality

$$\frac{1}{m+1} \frac{d}{dt} \|u\|_{m+1}^{m+1} + \gamma_5^{-2} |\Omega|^{\frac{N-2}{N} - \frac{2m}{1+m}} \|u\|_{s+1}^{m+s} \leq a |\Omega|^{2 - \frac{p+q+m}{m+1}} \|u\|_{m+1}^{p+q+m} \quad (4.8)$$

by the Sobolev embedding and Hölder's inequalities. By Lemma 2, one can see that there exists a constant $\alpha_2 > 0$ such that $0 \leq \|u(\cdot, t)\|_{m+1}^{m+1} \leq \|u_0\|_{m+1}^{m+1} e^{-\alpha_2 t}$, provided $\|u_0\|_{m+1}^{m+1} \leq [\frac{(m+1)\gamma_5^{-2}}{a|\Omega|^{1+\frac{N}{m+1}}}]^{\frac{1}{p+q-m}}$. Then there exists a constant $T_6 > 0$ such that

$$\begin{aligned} & (1+m) (\gamma_5^{-2} |\Omega|^{\frac{N-2}{N} - \frac{2m}{1+m}} - a |\Omega|^{2 - \frac{m+p+q}{1+m}} \|u\|_{1+m}^{p+q-m}) \\ & \leq (1+m) (\gamma_5^{-2} |\Omega|^{\frac{N-2}{N} - \frac{2m}{1+m}} - a |\Omega|^{2 - \frac{m+p+q}{1+m}} (\|u_0\|_{m+1}^{m+1} e^{-\alpha_2 T_6})^{\frac{p+q-m}{1+m}}) \equiv C_5 \quad (> 0) \end{aligned}$$

for all $t \in [T_6, +\infty)$.

Hence, we get $\frac{d}{dt} \|u\|_{m+1}^{m+1} + C_5 \|u\|_{1+m}^{2m} \leq 0$. By Lemma 1 we can obtain its decay estimate, with $T_7 = \frac{\|u_0\|_{1+m}^{m+1}}{C_6(1-\frac{2m}{m+1})} + T_6$;

(b) for m such that $0 < m < \frac{N-2}{N+2}$, we choose $s = \frac{N}{2}(1-m) - 1$ in (4.4). From the Sobolev embedding inequality, we get

$$\|u\|_s^{\frac{m+s}{2}} = \|u\|_{\frac{2N}{N-2}}^{\frac{m+s}{2}} \leq \gamma_6 \|\nabla u\|_{\frac{m+s}{2}}^{\frac{m+s}{2}}. \quad (4.9)$$

By Hölder's inequality, we obtain the inequality

$$\frac{1}{s+1} \frac{d}{dt} \|u\|_{s+1}^{s+1} + \frac{\gamma_6^{-2} 4ms}{(m+s)^2} \|u\|_{s+1}^{m+s} \leq a |\Omega|^{2 - \frac{p+q+s}{s+1}} \|u\|_{s+1}^{p+q+s}. \quad (4.10)$$

One can see that there exists a constant $\alpha_3 > 0$ such that $0 \leq \|u(\cdot, t)\|_{s+1}^{s+1} \leq \|u_0\|_{s+1}^{s+1} e^{-\alpha_3 t}$ by Lemma 2, provided $\|u_0\|_{s+1}^{s+1} \leq [\frac{4ms\gamma_6^{-2}}{a(m+1)^2|\Omega|^{2-\frac{s+p+q}{s+1}}}]^{\frac{1}{p+q-s}}$. Then there exists a constant $T_8 > 0$ such that

$$\begin{aligned} & (1+m) \left(\frac{4sm\gamma_6^{-2}}{(m+s)^2} - a |\Omega|^{2 - \frac{s+p+q}{1+s}} \|u\|_{1+s}^{p+q-s} \right) \\ & \leq (1+m) \left(\frac{4sm\gamma_6^{-2}}{(m+s)^2} - a |\Omega|^{2 - \frac{s+p+q}{1+s}} (\|u_0\|_{s+1}^{s+1} e^{-\alpha_3 T_8})^{\frac{p+q-s}{1+s}} \right) \equiv C_6 \quad (> 0) \end{aligned}$$

for all $t \in [T_8, +\infty)$.

Hence, $\frac{d}{dt} \|u\|_{s+1}^{s+1} + C_6 \|u\|_{s+1}^{m+s} \leq 0$. By Lemma 1 we can obtain its decay estimate, with $T_9 = \frac{\|u_0\|_{s+1}^{s+1}}{C_6(1-\frac{m+s}{s+1})} + T_8$.

Secondly, we consider the case $p+q > 1$.

It can be easily verified that $k\varphi_1(x)$ is a super-solution of (1.1)-(1.3) for sufficiently small $k > 0$, where $\varphi_1(x)$ is the unique positive solution of (2.1). We then have

$$u(x, t) \leq k\varphi_1^{\frac{1}{m}}(x) \leq kM_1^{\frac{1}{m}}, \quad t > 0$$

by Proposition 2, if $u_0 \leq k\varphi_1^{\frac{1}{m}}(x)$ in Ω .

The above inequality and (4.7) yield the inequality

$$\frac{1}{s+1} \frac{d}{dt} \int_{\Omega} u^{s+1} dx + \frac{4ms}{(m+s)^2} \int_{\Omega} |\nabla u^{\frac{m+s}{2}}|^2 dx \leq ak^{p+q-1} M_1^{\frac{p+q-1}{m}} |\Omega| \int_{\Omega} u^{s+1} dx \quad (4.11)$$

by applying Hölder's inequality to the right-hand side of (4.7).

For m such that $\frac{N-2}{N+2} < m < 1$, we choose $s = m$ in (4.11). It follows from the Sobolev embedding and Hölder's inequalities that

$$\frac{1}{m+1} \frac{d}{dt} \|u\|_{m+1}^{m+1} + \gamma_7^{-2} |\Omega|^{\frac{N-2}{N} - \frac{2m}{1+m}} \|u\|_{m+1}^{2m} \leq ak^{p+q-1} M_1^{\frac{p+q-1}{m}} \|u\|_{m+1}^{1+m}.$$

By Lemma 2, there exists a constant $\alpha_4 > 0$ such that $0 \leq \|u(\cdot, t)\|_{m+1}^{m+1} \leq \|u_0\|_{m+1}^{m+1} e^{-\alpha_4 t}$, provided $\|u_0\|_{m+1}^{m+1} \leq \left[\frac{|\Omega|^{\frac{N-2}{N} - \frac{2m}{1+m}} \gamma_7^{-2}}{ak^{p+q-1} M_1^{\frac{p+q-1}{m}}} \right]^{\frac{1}{1-m}}$. Hence, one can find a constant $T_{10} > 0$ such that

$$\begin{aligned} & (1+m) (\gamma_7^{-2} |\Omega|^{\frac{N-2}{N} - \frac{2m}{1+m}} - ak^{p+q-1} M_1^{\frac{p+q-1}{m}}) \|u\|_{1-m}^{1+m} \\ & \leq (1+m) (\gamma_7^{-2} |\Omega|^{\frac{N-2}{N} - \frac{2m}{1+m}}) - ak^{p+q-1} M_1^{\frac{p+q-1}{m}} (\|u_0\|_{m+1}^{m+1} e^{-\alpha_4 T_{10}})^{\frac{1-m}{1+m}} \equiv C_7 > 0 \end{aligned}$$

for all $t \in [T_{10}, +\infty)$, from which we get

$$\frac{d}{dt} \|u\|_{m+1}^{m+1} + C_7 \|u\|_{1+m}^{2m} \leq 0.$$

By Lemma 1, we then obtain its decay estimates, with $T_{11} = \frac{\|u_0\|_{1-m}^{m+1}}{C_7(1-\frac{2m}{m+1})} + T_{10}$, as follows:

$$\begin{aligned} \|u(\cdot, t)\|_{1+m}^{1+m} & \leq [\|u_0\|_{1+m}^{1+m} e^{-\alpha_4 t}], \quad t \in [0, T_{10}); \\ \|u(\cdot, t)\|_{1+m}^{1+m} & \leq \left[\|u_0\|_{1+m}^{1+m} - C_7 \left(1 - \frac{2m}{1+m} \right) t \right]^{\frac{1}{1-\frac{2m}{1+m}}}, \quad t \in [T_{10}, T_{11}); \\ \|u(\cdot, t)\|_{1+m}^{1+m} & \equiv 0, \quad t \in [T_{11}, +\infty). \end{aligned}$$

For m such that $0 < m < \frac{N-2}{N}$, we choose $s = \frac{N}{2}(1-m) > 1$ in (4.7), and then we can obtain the extinction results by using a similar argument as the one used for the case $p+q \leq 1$, and so the details are omitted. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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