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Periodic solutions to the Liénard type equations with phase attractive singularities

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Abstract

Sufficient conditions are established guaranteeing the existence of a positive ω-periodic solution to the equation

$$u'' + f(u)u' + g(u) = h(t,u),$$

where $f,g:(0,+\infty) \to \mathbb{R}$ are continuous functions with possible singularities at zero and $h:[0,\omega]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. The results obtained are rewritten for the equation of the type

$$u'' + \frac{cu'}{u^{\mu}} + \frac{g_1}{u^{\nu}} - \frac{g_2}{u^{\gamma}} = h_0(t)u^{\delta},$$

where g_1, g_2, δ are non-negative constants, c, μ, ν, γ are real numbers, and $h_0 \in L([0,\omega];\mathbb{R})$. The last equation also covers the so-called Rayleigh-Plesset equation, frequently used in fluid mechanics to model the bubble dynamics in liquid. In the paper, the case when $\nu > \gamma$, *i.e.*, the case which covers the attractive singularity of the function g, is studied. The results obtained assure that there exists a positive ω-periodic solution to the above-mentioned equation if the power μ or ν is sufficiently large.

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1 Introduction

The topic of singular boundary value problems has been of substantial and rapidly growing interest for many scientists and engineers. The importance of such investigation is emphasized by the fact that numerical simulations of solutions to such problems usually break down near singular points.

On the other hand, problems of this type arise frequently in applied science. Namely, in fluid mechanics, since 1917 the physicists have used the Rayleigh equation,

$$\rho\left[R\ddot{R} + \frac{3}{2}\dot{R}^2\right] = p(R) - p_\infty,$$

to model the bubble dynamics in liquid, where $R(t)$ is the ratio of the bubble at the time t, ρ is the liquid density, p_∞ is the pressure in the liquid at a large distance from the bubble, and $p(R)$ is the pressure in the liquid at the bubble boundary. In 1949, Plesset proposed to use a more exact equation involving the surface-tension constant S and the coefficient of the liquid viscosity μ_*, which was finally improved by adding a term with polytropic coefficient k in 1977, nowadays known as a Rayleigh-Plesset equation (see [1])

$$\rho\left[R\ddot{R}+\frac{3}{2}\dot{R}^2\right]=\left[P_v-p_\infty(t)\right]+P_{g_0}\left(\frac{R_0}{R}\right)^{3k}-\frac{2S}{R}-\frac{4\mu_*\dot{R}}{R}.$$

The transformation $R=u^{\frac{2}{5}}$ in the previous equation leads to the equation

$$\ddot{u}=\frac{5[P_v-p_\infty(t)]}{2\rho}u^{\frac{1}{5}}+\left(\frac{5P_{g_0}R_0^{3k}}{2\rho}\right)\frac{1}{u^{\frac{6k-1}{5}}}-\frac{5S}{u^{\frac{1}{5}}}-4\mu_*\frac{\dot{u}}{u^{\frac{4}{5}}}.$$

Consequently, the class of equations

$$u''(t)+\frac{cu'(t)}{u^\mu(t)}+\frac{g_1}{u^\nu(t)}-\frac{g_2}{u^\gamma(t)}=h_0(t)u^\delta(t)\quad\text{for a.e. } t\in[0,\omega], \tag{1.1}$$

with non-negative constants g_1, g_2, δ, real numbers c, μ, ν, γ, and $h_0\in L([0,\omega];\mathbb{R})$, plays an important role in fluid mechanics. Therefore, the equation

$$u''(t)+f\big(u(t)\big)u'(t)+g\big(u(t)\big)=h\big(t,u(t)\big)\quad\text{for a.e. } t\in[0,\omega], \tag{1.2}$$

subjected to the periodic conditions

$$u(0)=u(\omega),\qquad u'(0)=u'(\omega), \tag{1.3}$$

is investigated in the presented paper. Here, $f,g:(0,+\infty)\to\mathbb{R}$ are continuous, having possible singularities at zero, and $h:[0,\omega]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, *i.e.*, $h(\cdot,x):[0,\omega]\to\mathbb{R}$ is measurable for all $x\in\mathbb{R}$, $h(t,\cdot):\mathbb{R}\to\mathbb{R}$ is continuous for a.e. $t\in[0,\omega]$, and for every $r>0$, there exists a non-negative function $q_r\in L([0,\omega];\mathbb{R})$ such that $|h(t,x)|\le q_r(t)$ for a.e. $t\in[0,\omega]$ and all $|x|\le r$. By a solution to (1.2), (1.3) we understand a function $u:[0,\omega]\to\mathbb{R}$ which is positive, absolutely continuous together with its first derivative, satisfies (1.2) almost everywhere on $[0,\omega]$, and verifies (1.3). In spite of the fact that the problem (1.2), (1.3) was investigated by many mathematicians (see, *e.g.*, [2–27]), most of the mentioned works deal with the repulsive case and/or when f has no singularity. However, the physical model, covered by equation (1.1), justifies considering the types of equations with a singular friction-like term.

A particular case of (1.1) is the equation

$$u''+\frac{1}{u^\nu}=h_0(t) \tag{1.4}$$

studied by Lazer and Solimini. Their results were published in 1987 (see [11]) and they proved, among others, that (1.4), (1.3) has at least one solution if and only if $\overline{h}_0>0$, provided h_0 is bounded. Recently, we have proved (see [28]) that this result cannot be extended to the case when h_0 is a general integrable (and so unbounded) function unless some additional conditions are introduced. In particular, (1.4), (1.3) is solvable for any

$h_0 \in L([0,\omega];\mathbb{R})$ with $\overline{h}_0 > 0$ if $\nu \geq 1$; and, moreover, for any $\nu \in (0,1)$, there exists a function $h_0 \in L([0,\omega];\mathbb{R})$ with $\overline{h}_0 > 0$ such that (1.4), (1.3) has no solution. At this point, we would like to emphasize the important fact that the condition $\nu \geq 1$ can be weakened if (1.4) is generalized to equation (1.1), see Remark 2.2 below.

The structure of the paper is as follows. After the introduction and basic notation, we recall the definition of lower and upper functions to the problem (1.2), (1.3), and we formulate the classical theorem on the existence of a solution to (1.2), (1.3) in the case when there exists a couple of well-ordered lower and upper functions. In Section 2, we establish our main results and their consequences. Sections 3 and 4 are devoted to auxiliary propositions and proofs of the main results, respectively.

For convenience, we finish the introduction with a list of notations which are used throughout the paper:

$\mathbb{N}$ is the set of all natural numbers, $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^+ = (0,+\infty)$, $[x]_+ = \max\{x,0\}$, $[x]_- = \max\{-x,0\}$.

$C([0,\omega];\mathbb{R})$ is the Banach space of continuous functions $u:[0,\omega] \to \mathbb{R}$ with the norm

$$\|u\|_\infty = \max\big\{\big|u(t)\big| : t \in [0,\omega]\big\}.$$

$C(\mathbb{R}^+;\mathbb{R})$, resp. $C(\mathbb{R}^+;\mathbb{R}^+)$, is the set of continuous functions $u:\mathbb{R}^+ \to \mathbb{R}$, resp. $u:\mathbb{R}^+ \to \mathbb{R}^+$.

$C^1(\mathbb{R}^+;\mathbb{R}^+)$ is the set of functions $u:\mathbb{R}^+ \to \mathbb{R}^+$ which are continuous together with their first derivative.

$AC^1([0,\omega];\mathbb{R})$ is a set of all functions $u:[0,\omega] \to \mathbb{R}$ such that u and u' are absolutely continuous.

$L([0,\omega];\mathbb{R})$ is the Banach space of the Lebesgue integrable functions $p:[0,\omega] \to \mathbb{R}$ endowed with the norm

$$\|p\|_1 = \int_0^\omega \big|p(s)\big|\,ds.$$

For a given $p \in L([0,\omega];\mathbb{R})$, its mean value is defined by

$$\overline{p} = \frac{1}{\omega}\int_0^\omega p(s)\,ds.$$

Given $\varphi, \psi \in L([0,\omega];\mathbb{R})$, then

$$\Phi_+ = \int_0^\omega \big[\varphi(s)\big]_+\,ds, \qquad \Phi_- = \int_0^\omega \big[\varphi(s)\big]_-\,ds,$$
$$\Psi_+ = \int_0^\omega \big[\psi(s)\big]_+\,ds, \qquad \Psi_- = \int_0^\omega \big[\psi(s)\big]_-\,ds.$$

The following definitions of lower and upper functions are suitable for us. For more general definitions, one can see, *e.g.*, [12, Definition 8.2].

Definition 1.1 A function $\alpha \in AC^1([0,\omega];\mathbb{R})$ is called a *lower function* to the problem (1.2), (1.3) if $\alpha(t) > 0$ for every $t \in [0,\omega]$ and

$$\alpha''(t) + f\big(\alpha(t)\big)\alpha'(t) + g\big(\alpha(t)\big) \geq h\big(t,\alpha(t)\big) \quad \text{for a.e. } t \in [0,\omega],$$
$$\alpha(0) = \alpha(\omega), \qquad \alpha'(0) \geq \alpha'(\omega).$$

Definition 1.2 A function $\beta \in AC^1([0,\omega];\mathbb{R})$ is called an *upper function* to the problem (1.2), (1.3) if $\beta(t) > 0$ for every $t \in [0,\omega]$ and

$$\beta''(t) + f(\beta(t))\beta'(t) + g(\beta(t)) \leq h(t,\beta(t)) \quad \text{for a.e. } t \in [0,\omega],$$
$$\beta(0) = \beta(\omega), \qquad \beta'(0) \leq \beta'(\omega).$$

The following theorem is well known in the theory of differential equations (see, *e.g.*, [12, Theorem 8.12]).

Theorem 1.1 *Let α and β be lower and upper functions to the problem* (1.2), (1.3) *such that*

$$\alpha(t) \leq \beta(t) \quad \text{for } t \in [0,\omega]. \tag{1.5}$$

Then there exists a solution u to the problem (1.2), (1.3) *such that*

$$\alpha(t) \leq u(t) \leq \beta(t) \quad \text{for } t \in [0,\omega].$$

2 Main results

Theorem 2.1 *Let $\rho_0 \in C^1(\mathbb{R}^+;\mathbb{R}^+)$ and $\rho_1 \in C(\mathbb{R}^+;\mathbb{R}^+)$ be non-decreasing functions, $h_0, h_1 \in L([0,\omega];\mathbb{R})$, and $x_0 > 0$ be such that*

$$h_1(t)\rho_1(x) \leq h(t,x) \leq h_0(t)\rho_0(x) \quad \text{for a.e. } t \in [0,\omega], x \geq x_0, \tag{2.1}$$

and let there exist $c_0, c_1 \in \mathbb{R}$ such that

$$\frac{g(x)}{\rho_0(x)} \leq c_0 < \overline{h}_0 \quad \text{for } x \geq x_0, \tag{2.2}$$

$$\frac{g(x)}{\rho_1(x)} \leq c_1 \leq \overline{h}_1 \quad \text{for } x \geq x_0. \tag{2.3}$$

Let, moreover, there exist $\lambda \in [0,1]$ such that

$$\int_0^1 \frac{ds}{\rho_0^\lambda(s)} < +\infty, \tag{2.4}$$

$$\lim_{x\to 0_+} \frac{g(x)}{\rho_0^\lambda(x)} = +\infty, \tag{2.5}$$

and let either

$$\int_0^1 \left(\frac{[f(s)]_+}{\rho_0^\lambda(s)} + \frac{[g(s)]_+}{\rho_0^{2\lambda}(s)}\right) ds = +\infty, \qquad \int_0^1 \frac{[f(s)]_-}{\rho_0^\lambda(s)} ds < +\infty \tag{2.6}$$

or

$$\int_0^1 \left(\frac{[f(s)]_-}{\rho_0^\lambda(s)} + \frac{[g(s)]_+}{\rho_0^{2\lambda}(s)}\right) ds = +\infty, \qquad \int_0^1 \frac{[f(s)]_+}{\rho_0^\lambda(s)} ds < +\infty. \tag{2.7}$$

Furthermore, let us suppose that ρ_0 fulfills at least one of the following conditions:

(a) *there exists a sequence* $\{y_n\}_{n=1}^{+\infty}$ *of positive numbers such that*

$$\lim_{n\to+\infty} y_n = +\infty, \qquad \lim_{n\to+\infty} \frac{\rho_0^{1-\lambda}(y_n)}{\sigma(y_n)} = 0, \tag{2.8}$$

and there exist $\varepsilon_0 > 0$, $\varepsilon_1 \in (0, \varepsilon_0]$, *and* $n_0 \in \mathbb{N}$ *such that*

$$\frac{\rho_0^{1-\lambda}((1+\varepsilon_0)y_n)}{\rho_0^{1-\lambda}(y_n)} \Phi_- \le \Phi_+ - \varepsilon_0 \quad \text{for } n \ge n_0, \tag{2.9}$$

$$(1+\varepsilon_1)\sigma(y_n) \le \sigma\big((1+\varepsilon_0)y_n\big) \quad \text{for } n \ge n_0, \tag{2.10}$$

where $\varphi(t) = h_0(t) - c_0$ *for almost every* $t \in [0, \omega]$ *and*

$$\sigma(x) = \int_0^x \frac{ds}{\rho_0^\lambda(s)}; \tag{2.11}$$

(b) *the function* $\frac{\rho_0^{1-\lambda}(x)}{\sigma(x)}$ *is non-increasing and*

$$\frac{\omega}{4} \Phi_+ \Phi_- \frac{\rho_0^{1-\lambda}(x_0)}{\sigma(x_0)} < \Phi_+ - \Phi_-, \tag{2.12}$$

where $\varphi(t) = h_0(t) - c_0$ *for almost every* $t \in [0, \omega]$ *and* σ *is given by* (2.11).

Besides, let us suppose that ρ_1 *fulfills at least one of the following conditions*:

(c) *there exists a sequence* $\{z_n\}_{n=1}^{+\infty}$ *of positive numbers such that*

$$\lim_{n\to+\infty} z_n = +\infty, \qquad \lim_{n\to+\infty} \frac{\rho_1(z_n)}{z_n} = 0,$$

and there exist $\varepsilon_2 > 0$ *and* $n_1 \in \mathbb{N}$ *such that*

$$\frac{\rho_1(z_n(1+\varepsilon_2))}{\rho_1(z_n)} \Psi_- \le \Psi_+ \quad \text{for } n \ge n_1,$$

where $\psi(t) = h_1(t) - c_1$ *for almost every* $t \in [0, \omega]$;

(d) *the function* $\frac{\rho_1(x)}{x}$ *is non-increasing and*

$$\frac{\omega}{4} \Psi_+ \Psi_- \frac{\rho_1(x_0)}{x_0} \le \Psi_+ - \Psi_-,$$

where $\psi(t) = h_1(t) - c_1$ *for almost every* $t \in [0, \omega]$.

Then there exists at least one solution to the problem (1.2), (1.3).

Remark 2.1 Note that there exists a suitable ε_1 such that (2.10) holds, *e.g.*, if

$$\limsup_{x\to+\infty} \frac{\rho_0^\lambda((1+\varepsilon_0)x)}{\rho_0^\lambda(x)} < 1 + \varepsilon_0.$$

For equation (1.1), from Theorem 2.1 we get the following assertion.

Corollary 2.1 *Let* $g_1 > 0$, $g_2 \geq 0$, $0 \leq \delta < 1$, $\nu > \gamma$, $\nu + \delta > 0$ *and*

$$\text{either} \quad (\mu + \delta)\operatorname{sgn}|c| \geq 1 \quad \text{or} \quad \nu + 2\delta \geq 1.$$

If

$$\overline{h}_0 > -\lim_{x \to +\infty} \frac{g_2}{x^{\gamma+\delta}},$$

then (1.1), (1.3) *has at least one solution.*

Remark 2.2 In [28], it is proved, among others, that the equation

$$u'' + \frac{1}{u^\nu} = h_0(t), \tag{2.13}$$

with $h_0 \in L([0,\omega];\mathbb{R})$ and $\overline{h}_0 > 0$, has a positive ω-periodic solution if $\nu \geq 1$. Moreover, there is also an example introduced showing that for any $\nu \in (0,1)$, there exists $h_0 \in L([0,\omega];\mathbb{R})$ with $\overline{h}_0 > 0$ such that (2.13), (1.3) has no positive solution.

Corollary 2.1 says that if a friction-like term or sub-linear term are added to (2.13), the condition $\nu \geq 1$ can be weakened. For example,

$$u'' + \frac{u'}{u^\mu} + \frac{1}{u^\nu} = h_0(t)$$

has a positive solution satisfying (1.3) for any $\nu > 0$ if $\mu \geq 1$, provided $\overline{h}_0 > 0$. Also, the equation

$$u'' + \frac{1}{u^\nu} = h_0(t)u^\delta$$

subjected to the boundary conditions (1.3) is solvable for any $\nu > 0$ if $\delta \in [1/2, 1)$, provided $\overline{h}_0 > 0$.

Example 2.1 As it was mentioned in the introduction, the particular case of (1.1) is the so-called Rayleigh-Plesset equation frequently used in fluid mechanics. This equation has the following form:

$$u'' + \frac{cu'}{u^{\frac{4}{5}}} + \frac{g_1}{u^{\frac{1}{5}}} - \frac{g_2}{u^\gamma} = h_0(t)u^{\frac{1}{5}}, \tag{2.14}$$

where $h_0 \in L([0,\omega];\mathbb{R})$, c, g_1, g_2 are positive constants and $\gamma \in \mathbb{R}$ (see [9, 10]).

The results dealing with the existence of positive ω-periodic solutions of (2.14) were established in [10] provided h_0 is bounded from above (see [10, Theorems 4.4, 4.6, 4.7]). However, Corollary 2.1 says that in the case when $\gamma < 1/5$, the problem (2.14), (1.3) is solvable if one of the following items is fulfilled:

1. $\gamma > -1/5$ and $\overline{h}_0 > 0$;
2. $\gamma = -1/5$ and $\overline{h}_0 > -g_2$;
3. $\gamma < -1/5$.

Thus, Corollary 2.1 assures that the boundedness of h_0 is not necessary.

Corollary 2.2 *Let $g_1 > 0$, $g_2 \geq 0$, $\nu > \gamma$, $\nu + 1 > 0$. Let, moreover, either $g^* = -\infty$ or*

$$\overline{h}_0 > g^* > -\infty, \qquad \frac{\omega}{4}\Phi_+\Phi_- < \Phi_+ - \Phi_-, \tag{2.15}$$

where $\varphi(t) = h_0(t) - g^*$ for almost every $t \in [0,\omega]$, and

$$g^* = -\lim_{x\to+\infty}\frac{g_2}{x^{\gamma+1}}.$$

Then the problem (1.1), (1.3) *with $\delta = 1$ has at least one solution.*

Remark 2.3 According to [29] and Theorem 1.1, it can be easily verified that the problem

$$u'' + \frac{g_1}{u^\nu} = h_0(t)u; \qquad u(0) = u(\omega), \qquad u'(0) = u'(\omega), \tag{2.16}$$

with $g_1 > 0$ and $\nu > 0$, has a positive solution if and only if the inclusion $\mathcal{L}[0,-h_0] \in V^-$ holds (see notation in [29]).

Indeed, according to [29, Definition 1.1], the inclusion $\mathcal{L}[0,-h_0] \in V^-$ implies the existence of a positive solution ν to the problem

$$\nu'' = h_0(t)\nu - g_1; \qquad u(0) = u(\omega), \qquad u'(0) = u'(\omega).$$

Therefore there exist constants $x > 0$ and $y > 0$ such that $x^{1+\nu} \leq \nu(t) \leq y^{1+\nu}$ for $t \in [0,\omega]$. By setting

$$\alpha(t) \stackrel{\text{def}}{=} \frac{\nu(t)}{y^\nu}, \qquad \beta(t) \stackrel{\text{def}}{=} \frac{\nu(t)}{x^\nu} \quad \text{for } t \in [0,\omega],$$

one can easily realize that α and β are, respectively, lower and upper functions to (2.16) satisfying (1.5). Now, the existence of a positive solution to (2.16) follows from Theorem 1.1.

On the other hand, the existence of a positive solution to (2.16) implies the inclusion $\mathcal{L}[0,-h_0] \in V^-$ (see [29, Theorem 2.1]).

However, one of the optimal effective conditions guaranteeing such an inclusion is $h_0 \not\equiv 0$ and

$$\frac{\omega}{4}\int_0^\omega \big[h_0(s)\big]_+\,ds\int_0^\omega \big[h_0(s)\big]_-\,ds \leq \int_0^\omega \big[h_0(s)\big]_+\,ds - \int_0^\omega \big[h_0(s)\big]_-\,ds$$

(see [29, Corollary 2.5]). Therefore, the condition (2.15) is natural in a certain sense.

When the right-hand side of equation (1.3) does not depend on u, *i.e.*, when $h(t,x) \equiv h_0(t)$, then (1.3) has the form

$$u''(t) + f\big(u(t)\big)u'(t) + g\big(u(t)\big) = h_0(t) \quad \text{for a.e. } t \in [0,\omega]. \tag{2.17}$$

From Theorem 2.1, for equation (2.17) we get the following assertion.

Corollary 2.3 *Let there exist $x_0 > 0$ and $c_0 \in \mathbb{R}$ such that*

$$g(x) \leq c_0 < \overline{h}_0 \quad \textit{for } x \geq x_0$$

and let

$$\lim_{x\to 0_+} g(x) = +\infty.$$

Let, moreover, either

$$\int_0^1 \left([f(s)]_+ + [g(s)]_+\right) ds = +\infty, \qquad \int_0^1 [f(s)]_- \, ds < +\infty$$

or

$$\int_0^1 \left([f(s)]_- + [g(s)]_+\right) ds = +\infty, \qquad \int_0^1 [f(s)]_+ \, ds < +\infty.$$

Then there exists at least one solution to the problem (2.17), (1.3).

In the following result, the assumptions do not depend on the friction-like term. On the other hand, a certain smallness of oscillation of the primitive to h_0 is supposed. Clearly, Theorems 2.1 and 2.2 are independent.

Theorem 2.2 *Let* $\rho_0 \in C^1(\mathbb{R}^+;\mathbb{R}^+)$ *and* $\rho_1 \in C(\mathbb{R}^+;\mathbb{R}^+)$ *be non-decreasing functions,* $h_0, h_1 \in L([0,\omega];\mathbb{R})$, *and* $0 < x_0 \le x_1 < +\infty$ *be such that*

$$h(t,x) \le h_0(t)\rho_0(x) \quad \text{for a.e. } t \in [0,\omega], 0 < x \le x_0, \tag{2.18}$$

$$h(t,x) \ge h_1(t)\rho_1(x) \quad \text{for a.e. } t \in [0,\omega], x \ge x_1. \tag{2.19}$$

Let, moreover,

$$\frac{\omega}{8}\|h_0 - \overline{h}_0\|_1 < \int_0^{x_0} \frac{ds}{\rho_0(s)} < +\infty, \tag{2.20}$$

$$\frac{g(x)}{\rho_0(x)} \ge \overline{h}_0 \quad \text{for } 0 < x \le x_0, \tag{2.21}$$

and let there exist $c_1 \in \mathbb{R}$ *such that*

$$\frac{g(x)}{\rho_1(x)} \le c_1 \le \overline{h}_1 \quad \text{for } x \ge x_1. \tag{2.22}$$

Besides, let us suppose that ρ_1 *fulfills at least one of the conditions* (c) *or* (d) *of Theorem* 2.1. *Then there exists at least one solution to the problem* (1.2), (1.3).

In the particular case, when equation (1.2) has the form (1.1), the following assertion immediately follows from Theorem 2.2.

Corollary 2.4 *Let* $0 \le \delta < 1$, *and let* $0 < x_0 \le x_1 < +\infty$ *be such that*

$$(1-\delta)\frac{\omega}{8}\|h_0 - \overline{h}_0\|_1 < x_0^{1-\delta}.$$

Let, moreover,

$$\frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} \geq \overline{h}_0 \quad \textit{if } 0 < x \leq x_0,$$
$$\frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} \leq \overline{h}_0 \quad \textit{if } x \geq x_1.$$

Then the problem (1.1), (1.3) *has at least one solution.*

Remark 2.4 The consequence of Theorem 2.2 for the problem (2.17), (1.3) coincides with the result obtained in [10, Theorem 3.6].

3 Auxiliary propositions

In what follows, we will show the existence of a solution to the equation

$$u''(t) + f\big(u(t)\big)u'(t) + g\big(u(t)\big) = h_0(t)\rho_0\big(u(t)\big) \quad \text{for a.e. } t \in [0,\omega], \tag{3.1}$$

satisfying the boundary conditions (1.3). Here, $\rho_0 \in C(\mathbb{R}^+;\mathbb{R}^+)$ is a non-decreasing function, $h_0 \in L([0,\omega];\mathbb{R})$, and $f, g \in C(\mathbb{R}^+;\mathbb{R})$. Together with (3.1), for every $k \in \mathbb{N}$, consider the auxiliary equation

$$u''(t) + f\big(u(t)\big)u'(t) + g\big(u(t)\big) = h_{0k}(t)\rho_0\big(u(t)\big) \quad \text{for a.e. } t \in [0,\omega], \tag{3.2}$$

where

$$h_{0k}(t) = \begin{cases} k & \text{if } h_0(t) > k, \\ h_0(t) & \text{if } h_0(t) \leq k \end{cases} \quad \text{for a.e. } t \in [0,\omega], k \in \mathbb{N}. \tag{3.3}$$

Obviously,

$$h_{0k}(t) \leq h_{0m}(t) \leq h_0(t) \quad \text{for a.e. } t \in [0,\omega], k \leq m, \tag{3.4}$$

and

$$\lim_{k\to+\infty} \overline{h}_{0k} = \overline{h}_0. \tag{3.5}$$

The following three results can be found in [10].

Lemma 3.1 (see [10, Corollary 2.17]) *Let $x_0 > 0$ and $c \in \mathbb{R}$ be such that*

$$\frac{g(x)}{\rho_0(x)} \leq c \leq \overline{h}_0 \quad \textit{for } x \geq x_0. \tag{3.6}$$

Let, moreover, there exist a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive numbers such that

$$\lim_{n\to+\infty} y_n = +\infty, \qquad \lim_{n\to+\infty} \frac{\rho_0(y_n)}{y_n} = 0, \tag{3.7}$$

and let there exist $\varepsilon > 0$ *and* $n_0 \in \mathbb{N}$ *such that*

$$\frac{\rho_0((1+\varepsilon)y_n)}{\rho_0(y_n)}\Phi_- \le \Phi_+ \quad \textit{for } n \ge n_0,$$

where $\varphi(t) = h_0(t) - c$ *for almost every* $t \in [0,\omega]$. *Then there exists an upper function* β *to the problem* (3.1), (1.3) *satisfying*

$$\beta(t) \ge x_0 \quad \textit{for } t \in [0,\omega]. \tag{3.8}$$

Lemma 3.2 (see [10, Corollary 2.18]) *Let* $x_0 > 0$ *and* $c \in \mathbb{R}$ *be such that* (3.6) *holds. If* $\frac{\rho_0(x)}{x}$ *is a non-increasing function such that*

$$\frac{\omega}{4}\Phi_+\Phi_-\frac{\rho_0(x_0)}{x_0} \le \Phi_+ - \Phi_-,$$

where $\varphi(t) = h_0(t) - c$ *for almost every* $t \in [0,\omega]$, *then there exists an upper function* β *to the problem* (3.1), (1.3) *satisfying* (3.8).

Lemma 3.3 (see [10, Corollary 2.11]) *Let* $x_0 > \frac{\omega}{8}\|h_0 - \overline{h}_0\|_1$ *be such that*

$$g(x) \ge \overline{h}_0 \quad \textit{for } 0 < x \le x_0.$$

Then there exists a lower function α *to the problem* (2.17), (1.3) *with*

$$0 < \alpha(t) \le x_0 \quad \textit{for } t \in [0,\omega]. \tag{3.9}$$

Lemma 3.4 *Let* $x_0 > 0$ *and* $c_0 \in \mathbb{R}$ *be such that* (2.2) *holds. Let, moreover, there exist a sequence* $\{y_n\}_{n=1}^{+\infty}$ *of positive numbers such that* (3.7) *is fulfilled, and let there exist* $\varepsilon > 0$ *and* $n_0 \in \mathbb{N}$ *such that*

$$\frac{\rho_0((1+\varepsilon)y_n)}{\rho_0(y_n)}\Phi_- \le \Phi_+ - \varepsilon \quad \textit{for } n \ge n_0, \tag{3.10}$$

where $\varphi(t) = h_0(t) - c_0$ *for almost every* $t \in [0,\omega]$. *Then there exist* $k_0 \in \mathbb{N}$ *and an upper function* β *to the problems* (3.2), (1.3) *for* $k \ge k_0$ *satisfying* (3.8).

Proof Put

$$\varphi_k(t) = h_{0k}(t) - c_0 \quad \text{for a.e. } t \in [0,\omega], \tag{3.11}$$

$$\Phi_{k+} = \int_0^\omega [\varphi_k(s)]_+ \, ds, \qquad \Phi_{k-} = \int_0^\omega [\varphi_k(s)]_- \, ds. \tag{3.12}$$

Then, obviously, in view of (3.3), we have

$$\lim_{k\to+\infty}\Phi_{k+} = \Phi_+, \qquad \lim_{k\to+\infty}\Phi_{k-} = \Phi_- \tag{3.13}$$

and, consequently, on account of (2.2), (3.5), (3.10), and (3.13), there exists $k_0 \in \mathbb{N}$ such that

$$\frac{g(x)}{\rho_0(x)} \le c_0 \le \overline{h}_{0k_0} \le \overline{h}_0 \quad \text{for } x \ge x_0, \tag{3.14}$$

$$\frac{\rho_0((1+\varepsilon)y_n)}{\rho_0(y_n)} \Phi_{k_0-} \le \Phi_{k_0+} \quad \text{for } n \ge n_0. \tag{3.15}$$

Therefore, according to Lemma 3.1, there exists an upper function β to (3.2), (1.3) with $k = k_0$ satisfying (3.8). Obviously, in view of (3.4) and the non-negativity of ρ_0, it follows that β is also an upper function to (3.2), (1.3) for $k \ge k_0$. □

Lemma 3.5 *Let $x_0 > 0$ and $c_0 \in \mathbb{R}$ be such that* (2.2) *holds. If $\frac{\rho_0(x)}{x}$ is a non-increasing function such that*

$$\frac{\omega}{4} \Phi_+ \Phi_- \frac{\rho_0(x_0)}{x_0} < \Phi_+ - \Phi_-, \tag{3.16}$$

where $\varphi(t) = h_0(t) - c_0$ for almost every $t \in [0,\omega]$, then there exist $k_0 \in \mathbb{N}$ and an upper function β to the problems (3.2), (1.3) *for $k \ge k_0$ satisfying* (3.8).

Proof Define φ_k, Φ_{k+}, and Φ_{k-} by (3.11) and (3.12). Then, obviously, in view of (3.3), we have that (3.13) holds and, consequently, on account of (2.2), (3.5), (3.13), and (3.16), there exists $k_0 \in \mathbb{N}$ such that (3.14) is valid and

$$\frac{\omega}{4} \Phi_{k_0+} \Phi_{k_0-} \frac{\rho_0(x_0)}{x_0} \le \Phi_{k_0+} - \Phi_{k_0-}. \tag{3.17}$$

Therefore, according to Lemma 3.2, there exists an upper function β to (3.2), (1.3) with $k = k_0$ satisfying (3.8). Obviously, in view of (3.4) and the non-negativity of ρ_0, it follows that β is also an upper function to (3.2), (1.3) for $k \ge k_0$. □

Lemma 3.6 *Let*

$$\liminf_{x \to 0+} g(x) > -\infty, \tag{3.18}$$

and let either

$$\int_0^1 \big[f(s)\big]_+ \, ds < +\infty \tag{3.19}$$

or

$$\int_0^1 \big[f(s)\big]_- \, ds < +\infty. \tag{3.20}$$

Then, for every $K > 0$, there exists a constant $K_1 > 0$ such that for any $k \in \mathbb{N}$ and any positive solution u of (3.2), (1.3) *with*

$$\|u\|_\infty \le K, \tag{3.21}$$

we have the estimate

$$\|u'\|_\infty \le K_1. \tag{3.22}$$

Proof Assume that (3.20) is fulfilled. Let u be a positive solution to (3.2), (1.3) satisfying (3.21). Then there exist $t_0, t_1 \in [0,\omega]$ such that

$$u(t_0) = \min\{u(t) : t \in [0,\omega]\}, \qquad u(t_1) = \max\{u(t) : t \in [0,\omega]\}. \tag{3.23}$$

Define the operator ϑ of ω-periodic prolongation by

$$\vartheta(v)(t) = \begin{cases} v(t) & \text{if } t \in [0,\omega], \\ v(t-\omega) & \text{if } t \in (\omega, 2\omega]. \end{cases} \tag{3.24}$$

Then, obviously, from (3.2) and (1.3) it follows that

$$\begin{aligned} &\vartheta(u)''(t) + f\big(\vartheta(u)(t)\big)\vartheta(u)'(t) + g\big(\vartheta(u)(t)\big) \\ &\quad = \vartheta(h_{0k})(t)\rho_0\big(\vartheta(u)(t)\big) \quad \text{for a.e. } t \in [0,2\omega]. \end{aligned} \tag{3.25}$$

The integration of (3.25) from t_0 to t, on account of (3.23), yields

$$\begin{aligned} \vartheta(u)'(t) = &- \int_{t_0}^{t} f\big(\vartheta(u)(s)\big)\vartheta(u)'(s)\,ds - \int_{t_0}^{t} g\big(\vartheta(u)(s)\big)\,ds \\ &+ \int_{t_0}^{t} \vartheta(h_{0k})(s)\rho_0\big(\vartheta(u)(s)\big)\,ds \quad \text{for } t \in [t_0, t_0+\omega]. \end{aligned} \tag{3.26}$$

From (3.21), (3.23), and (3.24) it follows that

$$0 < \vartheta(u)(t_0) \le \vartheta(u)(t) \le K \quad \text{for } t \in [t_0, t_0+\omega]. \tag{3.27}$$

Put

$$\mu = \sup\{[g(s)]_- : s \in (0,K]\}. \tag{3.28}$$

According to (3.18), we have

$$0 \le \mu < +\infty. \tag{3.29}$$

Thus, using (3.3), (3.20), (3.21), and (3.27)-(3.29) in (3.26), we arrive at

$$\vartheta(u)'(t) \le \int_0^K [f(s)]_-\,ds + \omega\mu + \|h_0\|_1\rho_0(K) \quad \text{for } t \in [t_0, t_0+\omega]. \tag{3.30}$$

Put

$$K_1 = \int_0^K [f(s)]_-\,ds + \omega\mu + \|h_0\|_1\rho_0(K).$$

Then, on account of (3.24) and (3.30), we have

$$u'(t) \le K_1 \quad \text{for } t \in [0, \omega]. \tag{3.31}$$

On the other hand, the integration of (3.25) from t to $t_1 + \omega$, with respect to (3.23), results in

$$\begin{aligned} \vartheta(u)'(t) = {} & \int_t^{t_1+\omega} f\big(\vartheta(u)(s)\big)\vartheta(u)'(s)\,ds + \int_t^{t_1+\omega} g\big(\vartheta(u)(s)\big)\,ds \\ & - \int_t^{t_1+\omega} \vartheta(h_{0k})(s)\rho_0\big(\vartheta(u)(s)\big)\,ds \quad \text{for } t \in [t_1, t_1+\omega]. \end{aligned} \tag{3.32}$$

Now, using (3.3), (3.20), (3.21), and (3.27)-(3.29) in (3.32), we obtain

$$-\vartheta(u)'(t) \le K_1 \quad \text{for } t \in [t_1, t_1+\omega]. \tag{3.33}$$

Therefore, in view of (3.24), from (3.33) we get

$$-u'(t) \le K_1 \quad \text{for } t \in [0, \omega]. \tag{3.34}$$

Consequently, (3.31) and (3.34) result in (3.22).

Now, suppose that (3.19) is fulfilled. Put

$$v(t) = u(\omega - t) \quad \text{for } t \in [0, \omega]. \tag{3.35}$$

Then, according to (3.2), we have

$$v''(t) - f\big(v(t)\big)v'(t) + g\big(v(t)\big) = \widetilde{h}_{0k}(t)\rho_0\big(v(t)\big) \quad \text{for a.e. } t \in [0, \omega], \tag{3.36}$$

where

$$\widetilde{h}_{0k}(t) = h_{0k}(\omega - t) \quad \text{for a.e. } t \in [0, \omega].$$

Analogously to the above-proved, using (3.19) instead of (3.20), we obtain

$$\|v'\|_\infty \le K_1 \tag{3.37}$$

with

$$K_1 = \int_0^K \big[f(s)\big]_+\,ds + \omega\mu + \|h_0\|_1\rho_0(K).$$

Thus, (3.35) and (3.37) yield (3.22). □

Lemma 3.7 *Let*

$$\lim_{x \to 0_+} g(x) = +\infty \tag{3.38}$$

and let either

$$\int_0^1 \big([f(s)]_+ + [g(s)]_+\big)\,ds = +\infty, \qquad \int_0^1 [f(s)]_-\,ds < +\infty \tag{3.39}$$

or

$$\int_0^1 \big([f(s)]_- + [g(s)]_+\big)\,ds = +\infty, \qquad \int_0^1 [f(s)]_+\,ds < +\infty. \tag{3.40}$$

Then, for every $K > 0$, *there exists a constant* $a > 0$ *such that for any* $k \in \mathbb{N}$ *and any positive solution* u *of* (3.2), (1.3) *satisfying* (3.21), *we have the estimate*

$$a \le u(t) \quad \text{for } t \in [0,\omega]. \tag{3.41}$$

Proof Let u be a positive solution to (3.2), (1.3) satisfying (3.21). Thus, the integration of (3.2) from 0 to ω, in view of (1.3) and (3.4), yields

$$\int_0^\omega g\big(u(s)\big)\,ds \le \|h_0\|_1\rho_0(K). \tag{3.42}$$

On the other hand, (3.38) implies the existence of $x_0 \in (0,+\infty)$ such that

$$g(x) > \frac{\|h_0\|_1\rho_0(K)}{\omega} \ge 0 \quad \text{for } x \in (0,x_0). \tag{3.43}$$

Let $t_m \in [0,\omega]$ be such that

$$u(t_m) = \min\big\{u(t) : t \in [0,\omega]\big\}. \tag{3.44}$$

Obviously, either

$$u(t_m) \ge x_0$$

or

$$u(t_m) < x_0. \tag{3.45}$$

Obviously, it is sufficient to show the estimate (3.41) is valid just in the case when (3.45) is fulfilled. Let, therefore, (3.45) hold.

If $u(t) < x_0$ for $t \in [0,\omega]$, then applying (3.43) in (3.42) we obtain a contradiction. Thus, there exist points $t_1, t_2 \in (t_m, t_m+\omega)$ such that

$$\vartheta(u)(t) < x_0 \quad \text{for } t \in [t_m,t_1),\ \vartheta(u)(t_1) = x_0, \tag{3.46}$$

$$\vartheta(u)(t) < x_0 \quad \text{for } t \in (t_2,t_m+\omega],\ \vartheta(u)(t_2) = x_0, \tag{3.47}$$

where ϑ is the operator defined by (3.24). Obviously, (3.25) holds.

Assume that (3.39) holds. Then, according to Lemma 3.6, there exists $K_1 > 0$ such that (3.22) holds. The integration of (3.25) from t_m to t_1, in view of (3.4), (3.21), (3.22), (3.39), (3.43), (3.44), and (3.46), results in

$$\begin{aligned} &\vartheta(u)'(t_1) + \int_{\vartheta(u)(t_m)}^{x_0} \big[f(s)\big]_+ \, ds + \frac{1}{K_1} \int_{\vartheta(u)(t_m)}^{x_0} \big[g(s)\big]_+ \, ds \\ &\quad \le \int_0^{x_0} \big[f(s)\big]_- \, ds + \|h_0\|_1 \rho_0(K). \end{aligned} \tag{3.48}$$

Note that in view of (3.46), we have $\vartheta(u)'(t_1) \ge 0$. Consequently, from (3.48) we obtain

$$\int_{\vartheta(u)(t_m)}^{x_0} \big(\big[f(s)\big]_+ + \big[g(s)\big]_+\big) \, ds \le K_2, \tag{3.49}$$

where

$$K_2 = (K_1 + 1)\left(\int_0^{x_0} \big[f(s)\big]_- \, ds + \|h_0\|_1 \rho_0(K)\right).$$

Note that K_2 does not depend on k. Therefore, if we apply (3.39) in (3.49), it can be easily seen, with respect to (3.44), that there exists a constant $a > 0$ such that (3.41) holds.

If (3.40) holds, we integrate (3.25) from t_2 to $t_m + \omega$ and apply similar steps as above, just using (3.47) instead of (3.46). Finally, we arrive at

$$\int_{\vartheta(u)(t_m+\omega)}^{x_0} \big(\big[f(s)\big]_- + \big[g(s)\big]_+\big) \, ds \le K_2$$

with

$$K_2 = (K_1 + 1)\left(\int_0^{x_0} \big[f(s)\big]_+ \, ds + \|h_0\|_1 \rho_0(K)\right).$$

Therefore, also in this case, there exists a constant $a > 0$ such that (3.41) holds. □

Lemma 3.8 *Let* $x_0 > 0$ *and* $c_0 \in \mathbb{R}$ *be such that* (2.2) *holds. Let, moreover,* (3.38) *be fulfilled, and let either* (3.39) *or* (3.40) *be valid. Let, in addition, there exist a sequence* $\{y_n\}_{n=1}^{+\infty}$ *of positive numbers such that* (3.7) *holds, and let there exist* $\varepsilon > 0$ *and* $n_0 \in \mathbb{N}$ *such that* (3.10) *is fulfilled, where* $\varphi(t) = h_0(t) - c_0$ *for almost every* $t \in [0, \omega]$. *Then there exists a positive solution* u *to* (3.1), (1.3).

Proof According to Lemma 3.4, there exist $k_0 \in \mathbb{N}$ and an upper function β to the problems (3.2), (1.3) for $k \ge k_0$ satisfying (3.8). On the other hand, in view of (3.4) and (3.38), there exists $x_k \in (0, x_0]$ for $k \ge k_0$ such that

$$g(x_k) \ge h_{0k}(t)\rho_0(x_k) \quad \text{for a.e. } t \in [0, \omega].$$

Thus, if we put $\alpha_k(t) = x_k$ for $t \in [0, \omega]$, according to Theorem 1.1, there exists a solution u_k to (3.2), (1.3) for $k \ge k_0$ satisfying

$$0 < \alpha_k(t) \le u_k(t) \le \beta(t) \quad \text{for } t \in [0, \omega]. \tag{3.50}$$

Moreover, according to Lemmas 3.6 and 3.7, in view of (3.50), there exist constants $K > 0$, $K_1 > 0$, and $a > 0$, not depending on k, such that

$$\|u_k\|_\infty \le K, \qquad \|u_k'\|_\infty \le K_1 \quad \text{for } k \ge k_0, \tag{3.51}$$

$$a \le u_k(t) \quad \text{for } t \in [0,\omega], k \ge k_0, \tag{3.52}$$

$$|u_k''(t)| \le f_0 K_1 + g_0 + |h_0(t)|\rho_0(K) \quad \text{for a.e. } t \in [0,\omega], k \ge k_0, \tag{3.53}$$

where

$$f_0 = \max\{|f(x)| : x \in [a,K]\}, \qquad g_0 = \max\{|g(x)| : x \in [a,K]\}.$$

Therefore, according to the Arzelà-Ascoli theorem, there exist $u_0 \in C([0,\omega];\mathbb{R})$ and $v_0 \in C([0,\omega];\mathbb{R})$ such that

$$\lim_{k\to+\infty} \|u_k - u_0\|_\infty = 0, \qquad \lim_{k\to+\infty} \|u_k' - v_0\|_\infty = 0. \tag{3.54}$$

Moreover, since u_k are solutions to (3.2), (1.3), in view of (3.3), (3.52), and (3.54), we have $u_0 \in AC^1([0,\omega];\mathbb{R})$, $u_0' \equiv v_0$, and u_0 is a positive solution to (3.1), (1.3). □

The following assertion can be proved analogously to Lemma 3.8, just Lemma 3.5 is used instead of Lemma 3.4.

Lemma 3.9 *Let $x_0 > 0$ and $c_0 \in \mathbb{R}$ be such that* (2.2) *holds. Let, moreover,* (3.38) *be fulfilled, and let either* (3.39) *or* (3.40) *be valid. Let, in addition, $\frac{\rho_0(x)}{x}$ be a non-increasing function and let* (3.16) *be fulfilled, where $\varphi(t) = h_0(t) - c_0$ for almost every $t \in [0,\omega]$. Then there exists a positive solution u to* (3.1), (1.3).

Lemma 3.10 *Let $\rho_0 \in C^1(\mathbb{R}^+;\mathbb{R}^+)$ be non-decreasing, $x_0 > 0$, and $c_0 \in \mathbb{R}$ be such that* (2.2) *holds. Let, moreover, there exist $\lambda \in [0,1]$ such that* (2.4) *and* (2.5) *are valid, and let either* (2.6) *or* (2.7) *be fulfilled. Let, in addition, there exist a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive numbers such that* (2.8) *holds and let there exist $\varepsilon_0 > 0$, $\varepsilon_1 \in (0,\varepsilon_0]$, and $n_0 \in \mathbb{N}$ such that* (2.9) *and* (2.10) *are fulfilled, where $\varphi(t) = h_0(t) - c_0$ for almost every $t \in [0,\omega]$ and σ is given by* (2.11). *Then there exists a lower function α to the problem* (3.1), (1.3).

Proof Because ρ_0 is a positive function, from (2.4) and (2.11) we obtain that σ is a positive increasing function. Therefore, there exists an inverse function σ^{-1} to σ which is also increasing. Moreover, in view of (2.4) and (2.11), it follows that

$$\begin{aligned} &\lim_{x\to0_+} \sigma(x) = 0, \qquad \lim_{x\to0_+} \sigma^{-1}(x) = 0, \\ &\lim_{x\to+\infty} \sigma(x) = +\infty, \qquad \lim_{x\to+\infty} \sigma^{-1}(x) = +\infty. \end{aligned} \tag{3.55}$$

Consider the auxiliary equation

$$\begin{aligned} &u''(t) + f\big(\sigma^{-1}(u(t))\big)u'(t) + \frac{g(\sigma^{-1}(u(t)))}{\rho_0^\lambda(\sigma^{-1}(u(t)))} \\ &\quad = h_0(t)\rho_0^{1-\lambda}\big(\sigma^{-1}(u(t))\big) \quad \text{for a.e. } t \in [0,\omega]. \end{aligned} \tag{3.56}$$

Put $z = \sigma(x)$, $z_0 = \sigma(x_0)$. Then from (2.2) we get

$$\frac{g(\sigma^{-1}(z))}{\rho_0(\sigma^{-1}(z))} \le c_0 < \overline{h}_0 \quad \text{for } z \ge z_0 \tag{3.57}$$

and, in view of (3.55), from (2.5) we have

$$\lim_{z \to 0_+} \frac{g(\sigma^{-1}(z))}{\rho_0^\lambda(\sigma^{-1}(z))} = +\infty. \tag{3.58}$$

Furthermore, the substitution $r = \sigma(s)$ in (2.6), resp (2.7), with respect to (2.11) yields

$$\int_0^1 \left(\left[f\left(\sigma^{-1}(r)\right) \right]_+ + \frac{[g(\sigma^{-1}(r))]_+}{\rho_0^\lambda(\sigma^{-1}(r))} \right) dr = +\infty, \qquad \int_0^1 \left[f\left(\sigma^{-1}(r)\right) \right]_- dr < +\infty, \tag{3.59}$$

resp.

$$\int_0^1 \left(\left[f\left(\sigma^{-1}(r)\right) \right]_- + \frac{[g(\sigma^{-1}(r))]_+}{\rho_0^\lambda(\sigma^{-1}(r))} \right) dr = +\infty, \qquad \int_0^1 \left[f\left(\sigma^{-1}(r)\right) \right]_+ dr < +\infty. \tag{3.60}$$

Moreover, put $z_n = \sigma(y_n)$ for $n \in \mathbb{N}$. Then from (2.8), in view of (3.55), we get

$$\lim_{n \to +\infty} z_n = +\infty, \qquad \lim_{n \to +\infty} \frac{\rho_0^{1-\lambda}(\sigma^{-1}(z_n))}{z_n} = 0. \tag{3.61}$$

Finally, (2.10) results in

$$\sigma^{-1}\left((1+\varepsilon_1)z_n\right) \le (1+\varepsilon_0)y_n \quad \text{for } n \ge n_0,$$

and so, since ρ_0 is a non-decreasing function, from (2.9) we obtain

$$\frac{\rho_0^{1-\lambda}(\sigma^{-1}((1+\varepsilon_1)z_n))}{\rho_0^{1-\lambda}(\sigma^{-1}(z_n))} \Phi_- \le \Phi_+ - \varepsilon_1 \quad \text{for } n \ge n_0. \tag{3.62}$$

Therefore, applying Lemma 3.8, according to (3.57)-(3.62), there exists a positive solution u to the problem (3.56), (1.3).

Now, we put $\alpha(t) = \sigma^{-1}(u(t))$ for $t \in [0, \omega]$, *i.e.*, in view of (2.11),

$$u(t) = \int_0^{\alpha(t)} \frac{ds}{\rho_0^\lambda(s)} \quad \text{for } t \in [0, \omega].$$

Obviously, $\alpha \in AC^1([0, \omega]; \mathbb{R})$ is a positive function and

$$u'(t) = \frac{\alpha'(t)}{\rho_0^\lambda(\alpha(t))} \quad \text{for } t \in [0, \omega],$$

$$u''(t) = \frac{\alpha''(t)}{\rho_0^\lambda(\alpha(t))} - \frac{\lambda \alpha'^2(t) \rho_0'(\alpha(t))}{\rho_0^{1+\lambda}(\alpha(t))} \le \frac{\alpha''(t)}{\rho_0^\lambda(\alpha(t))} \quad \text{for a.e. } t \in [0, \omega].$$

Thus, it can be easily seen that α is a lower function to the problem (3.1), (1.3). □

Analogously to the proof of Lemma 3.10, one can prove the following assertion applying Lemma 3.9 instead of Lemma 3.8.

Lemma 3.11 *Let $\rho_0 \in C^1(\mathbb{R}^+;\mathbb{R}^+)$ be non-decreasing, $x_0 > 0$, and $c_0 \in \mathbb{R}$ be such that* (2.2) *holds. Let, moreover, there exist $\lambda \in [0,1]$ such that* (2.4) *and* (2.5) *are valid, and let either* (2.6) *or* (2.7) *be fulfilled. Let, in addition, $\frac{\rho_0^{1-\lambda}(x)}{\sigma(x)}$ be a non-increasing function and let* (2.12) *be fulfilled, where $\varphi(t) = h_0(t) - c_0$ for almost every $t \in [0,\omega]$ and σ is given by* (2.11). *Then there exists a lower function α to the problem* (3.1), (1.3).

4 Proofs of the main results

Proof of Theorem 2.1 According to Lemmas 3.1, 3.2, 3.10, and 3.11, the conditions of the theorem guarantee a well-ordered couple of lower and upper functions, therefore the result is a direct consequence of Theorem 1.1. □

Proof of Corollary 2.1 It follows from Theorem 2.1 with $h_1 \equiv h_0$, $\rho_0(x) = \rho_1(x) = x^\delta$, $\lambda = 1$, and $c_0 = c_1$ such that

$$\overline{h}_0 > c_0 > -\lim_{x\to+\infty} \frac{g_2}{x^{\gamma+\delta}}.$$

Then items (a) and (c) of Theorem 2.1 are fulfilled. □

Proof of Corollary 2.2 It follows from Theorem 2.1 with $h_1 \equiv h_0$, $\rho_0(x) = \rho_1(x) = x$, and $\lambda < 1$ such that $\nu + \lambda > 0$, $\nu + 2\lambda \geq 1$. Then items (b) and (d) of Theorem 2.1 are fulfilled. □

Proof of Corollary 2.3 It immediately follows from Theorem 2.1 with $h_1 \equiv h_0$, $\rho_i(x) \equiv 1$ $(i = 0,1)$. □

Proof of Theorem 2.2 Put

$$\sigma(x) = \int_0^x \frac{ds}{\rho_0(s)} \quad \text{for } x \geq 0. \tag{4.1}$$

Because ρ_0 is a positive function, from (2.20) and (4.1) we obtain that σ is an increasing function. Therefore, there exists an inverse function σ^{-1} to σ which is also increasing.

Consider the auxiliary equation

$$u''(t) + f\big(\sigma^{-1}\big(u(t)\big)\big)u'(t) + \frac{g(\sigma^{-1}(u(t)))}{\rho_0(\sigma^{-1}(u(t)))} = h_0(t) \quad \text{for a.e. } t \in [0,\omega]. \tag{4.2}$$

Put $z = \sigma(x)$, $z_0 = \sigma(x_0)$. Then from (2.20) and (2.21), in view of (4.1), we get

$$\frac{\omega}{8}\|h_0 - \overline{h}_0\|_1 < z_0,$$
$$\frac{g(\sigma^{-1}(z))}{\rho_0(\sigma^{-1}(z))} \geq \overline{h}_0 \quad \text{for } 0 < z \leq z_0.$$

Therefore, according to Lemma 3.3, there exists a lower function w to the problem (4.2), (1.3) satisfying

$$0 < w(t) \leq \sigma(x_0) \quad \text{for } t \in [0,\omega]. \tag{4.3}$$

Now, we put $\alpha = \sigma^{-1}(w(t))$ for $t \in [0,\omega]$, *i.e.*, in view of (4.1),

$$w(t) = \int_0^{\alpha(t)} \frac{ds}{\rho_0(s)} \quad \text{for } t \in [0,\omega].$$

Obviously, with respect to (4.3), $\alpha \in AC^1([0,\omega];\mathbb{R})$ is a positive function satisfying (3.9), and

$$w'(t) = \frac{\alpha'(t)}{\rho_0(\alpha(t))} \quad \text{for } t \in [0,\omega],$$

$$w''(t) = \frac{\alpha''(t)}{\rho_0(\alpha(t))} - \frac{\alpha'^2(t)\rho_0'(\alpha(t))}{\rho_0^2(\alpha(t))} \le \frac{\alpha''(t)}{\rho_0(\alpha(t))} \quad \text{for a.e. } t \in [0,\omega].$$

Thus, on account of (2.18), (3.9), and (4.2), it can be easily seen that α is a lower function to the problem (1.2), (1.3).

The existence of an upper function β to (1.2), (1.3) satisfying

$$\beta(t) \ge x_1 \quad \text{for } t \in [0,\omega] \tag{4.4}$$

follows from (2.19) and Lemma 3.1, resp. 3.2.

Obviously, in view of (3.9) and (4.4), we have that (1.5) holds. Thus the theorem follows from Theorem 1.1. □

Proof of Corollary 2.4 It follows from Theorem 2.2 with $h_1 \equiv h_0$, $\rho_0(x) = \rho_1(x) = x^\delta$, and $c_1 = \overline{h}_0$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
RH and MZ obtained the results in a joint research. Both authors read and approved the final manuscript.

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