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Boundedness of fractional oscillatory integral operators and their commutators on generalized Morrey spaces

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Abstract
In this paper, it is proved that both oscillatory integral operators and fractional oscillatory integral operators are bounded on generalized Morrey spaces $M_{p,\varphi}$. The corresponding commutators generated by BMO functions are also considered.

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1 Introduction and main results

The classical Morrey spaces, were introduced by Morrey [1] in 1938, have been studied intensively by various authors and together with weighted Lebesgue spaces play an important role in the theory of partial differential equations; they appeared to be quite useful in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces.

Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ are defined as the set of all functions $f \in L_p(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x,r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

Under this definition, $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ becomes a Banach space; for $\lambda = 0$, it coincides with $L_p(\mathbb{R}^n)$ and for $\lambda = 1$ with $L_\infty(\mathbb{R}^n)$.

We also denote by $W\mathcal{M}_{p,\lambda}$ the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where WL_p denotes the weak L_p-space.

Definition 1 Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} \equiv \|f\|_{M_{p,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also, by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$, we denote the weak generalized Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} \equiv \|f\|_{WM_{p,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \left|B(x,r)\right|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the spaces $M_{p,\lambda}$ and $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}} = M_{p,\lambda},$$
$$WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}} = WM_{p,\lambda}.$$

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operator, Riesz potential and the singular integral operators *etc.*, from one weighted Lebesgue space to another one is well studied. Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The fractional maximal operator M_α and the Riesz potential I_α are defined by

$$M_\alpha f(x) = \sup_{t>0} \left|B(x,t)\right|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} \left|f(y)\right| dy, \quad 0 \le \alpha < n,$$
$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)\, dy}{|x-y|^{n-\alpha}}, \quad 0 < \alpha < n.$$

If $\alpha = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator. In [2], Chiarenza and Frasca obtained the boundedness of M on $M_{p,\lambda}(\mathbb{R}^n)$. In [3], Adams established the boundedness of I_α on $M_{p,\lambda}(\mathbb{R}^n)$.

Here and subsequently, C will denote a positive constant which may vary from line to line but will remain independent of the relevant quantities.

The Calderón-Zygmund singular integral operator is defined by

$$\widetilde{T} f(x) = p.v. \int_{\mathbb{R}^n} K(x-y) f(y)\, dy, \tag{1.1}$$

where K is a Calderón-Zygmund kernel (CZK). We say a kernel $K \in C^1(\mathbb{R}^n \setminus \{0\})$ is a CZK if it satisfies

$$\left|K(x)\right| \le \frac{C}{|x|^n}, \tag{1.2}$$

$$\left|\nabla K(x)\right| \le \frac{C}{|x|^{n+1}} \tag{1.3}$$

and

$$\int_{a<|x|<b} K(x)\, dx = 0, \tag{1.4}$$

for all a, b with $0 < a < b$. Chiarenza and Frasca [2] showed the boundedness of $\widetilde{T}$ on $M_{p,\lambda}(\mathbb{R}^n)$.

It is worth pointing out that the kernel in (1.1) is convolution kernel. However, there were many kinds of operators with non-convolution kernels, such as Fourier transform and Radon transform [4], which both are versions of oscillatory integrals. The object we consider in this paper is a class of oscillatory integrals due to Ricci and Stein [5]

$$Tf(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) f(y)\, dy, \tag{1.5}$$

where $P(x,y)$ is a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$, and K is a CZK.

It is well known that the oscillatory factor $e^{iP(x,y)}$ makes it impossible to establish the L_p norm inequalities of (1.5) by the method as in the case of Calderón-Zygmund operators or fractional integrals. In [6], Chanillo and Christ established the weak $(1,1)$ type estimate of T.

A distribution kernel K is called a standard Calderón-Zygmund kernel (SCZK) if it satisfies the following hypotheses:

$$\left|K(x,y)\right| \le \frac{C}{|x-y|^n}, \quad x \ne y \tag{1.6}$$

and

$$\left|\nabla_x K(x,y)\right| + \left|\nabla_y K(x,y)\right| \le \frac{C}{|x-y|^{n+1}}, \quad x \ne y. \tag{1.7}$$

The corresponding Calderón-Zygmund integral operator $\widetilde{S}$ and oscillatory integral operator S are defined by

$$\widetilde{S}f(x) = p.v. \int_{\mathbb{R}^n} K(x,y) f(y)\, dy \tag{1.8}$$

and

$$Sf(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} K(x,y) f(y)\, dy, \tag{1.9}$$

where $P(x,y)$ is a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$. In [7], Lu and Zhang proved that S is bounded on L_p with $1 < p < \infty$. In [5], Ricci and Stein also introduced the standard fractional Calderón-Zygmund kernel (SFCZK) K_α with $0 < \alpha < n$, where the conditions (1.6) and (1.7) were replaced by

$$\left|K_\alpha(x,y)\right| \le \frac{C}{|x-y|^{n-\alpha}}, \quad x \ne y \tag{1.10}$$

and

$$\left|\nabla_x K_\alpha(x,y)\right| + \left|\nabla_y K_\alpha(x,y)\right| \le \frac{C}{|x-y|^{n+1-\alpha}}, \quad x \ne y. \tag{1.11}$$

The corresponding fractional oscillatory integral operator is defined by (see [8])

$$S_\alpha f(x) = \int_{\mathbb{R}^n} e^{iP(x,y)} K_\alpha(x,y) f(y)\, dy, \tag{1.12}$$

where $P(x,y)$ is also a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$. Obviously, when $\alpha = 0$, $S_0 = S$ and $K_0 = K$. Partly motivated by the idea from [9, 10] and the results of [11], we now give the results of this paper in the following.

Theorem 1.1 *Let* $1 \le p < \infty$, *and* (φ_1, φ_2) *satisfies the condition*

$$\int_r^\infty \frac{\operatorname{ess\,sup}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} \, dt \le C\varphi_2(x,r), \tag{1.13}$$

where C *does not depend on* x *and* t. *If* K *is a SCZK and the operator* $\widetilde{S}$ *is of type* $(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n))$, *then for* $1 < p < \infty$ *and any polynomial* $P(x,y)$ *the operator* S *is bounded from* M_{p,φ_1} *to* M_{p,φ_2}.

Moreover, for $p = 1$ *and* K *is a CZK operator, the operator* T *is bounded from* M_{1,φ_1} *to* WM_{1,φ_2}.

Theorem 1.2 *Let* $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $P(x,y)$ *is a polynomial, and* (φ_1, φ_2) *satisfies the condition*

$$\int_r^\infty \frac{\operatorname{ess\,sup}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} \, dt \le C\varphi_2(x,r), \tag{1.14}$$

where C *does not depend on* x *and* t. *Then for* $p > 1$ *the operator* S_α *is bounded from* M_{p,φ_1} *to* M_{q,φ_2} *and for* $p = 1$ *the operator* S_α *is bounded from* M_{1,φ_1} *to* WM_{q,φ_2}.

For a locally integrable function b, the commutator operator formed by S (or S_α) and b are defined by

$$S_b f(x) = b(x) Sf(x) - S(bf)(x)$$

and

$$S_{\alpha,b} f(x) = b(x) S_\alpha f(x) - S_\alpha(bf)(x).$$

Theorem 1.3 *Let* $1 < p < \infty$, $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* (φ_1, φ_2) *satisfies the condition*

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname{ess\,sup}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} \, dt \le C\varphi_2(x,r), \tag{1.15}$$

where C *does not depend on* x *and* t. *If* K *is a SCZK and the operator* $\widetilde{S}$ *is of type* $(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n))$, *then for any polynomial* $P(x,y)$ *the operator* S_b *is bounded from* M_{p,φ_1} *to* M_{p,φ_2}.

Theorem 1.4 *Let* $1 < p < \infty$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $P(x,y)$ *is a polynomial, and* (φ_1, φ_2) *satisfies the condition*

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname{ess\,sup}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} \, dt \le C\varphi_2(x,r), \tag{1.16}$$

where C *does not depend on* x *and* t. *Then the operator* $S_{b,\alpha}$ *is bounded from* M_{p,φ_1} *to* M_{q,φ_2}.

2 Some known results in generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$

In [9, 10, 12, 13] and [14], there were obtained sufficient conditions on weights φ_1 and φ_2 for the boundedness of the singular operator T from $\mathcal{M}_{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\varphi_2}(\mathbb{R}^n)$.

The following statements were proved by Nakai [14].

Theorem A *Let $1 \le p < \infty$ and $\varphi(x,r)$ satisfy the conditions*

$$c^{-1}\varphi(x,r) \le \varphi(x,t) \le c\varphi(x,r) \tag{2.1}$$

whenever $r \le t \le 2r$, where c (≥ 1) does not depend on t, r and $x \in \mathbb{R}^n$ and

$$\int_r^\infty \varphi(x,t)^p \frac{dt}{t} \le C\varphi(x,r)^p, \tag{2.2}$$

where C does not depend on x and r. Then for $p > 1$ the operators M and T are bounded in $\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ and for $p = 1$, M and T are bounded from $\mathcal{M}_{1,\varphi}(\mathbb{R}^n)$ to $W\mathcal{M}_{1,\varphi}(\mathbb{R}^n)$.

Theorem B *Let $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\varphi(x,t)$ satisfy the conditions* (2.1) *and*

$$\int_r^\infty \varphi(x,t)^p \frac{dt}{t} \le C\varphi(x,r)^p, \tag{2.3}$$

where C does not depend on x and r. Then for $p > 1$, the operators M_α and I_α are bounded from $\mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\varphi}(\mathbb{R}^n)$ and for $p = 1$, M_α and I_α are bounded from $\mathcal{M}_{1,\varphi}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\varphi}(\mathbb{R}^n)$.

The following statements, containing Nakai results obtained in [13, 14] was proved by Guliyev in [9, 10] (see also [15, 16]).

Theorem C *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_t^\infty \varphi_1(x,r) \frac{dr}{r} \le C\varphi_2(x,t), \tag{2.4}$$

where C does not depend on x and t. Then the operators M and T are bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}.

Theorem D *Let $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty t^\alpha \varphi_1(x,t) \frac{dt}{t} \le C\varphi_2(x,r), \tag{2.5}$$

where C does not depend on x and r. Then the operators M_α and I_α are bounded from M_{p,φ_1} to M_{q,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p = 1$.

The following statements, containing Guliyev results obtained in [9, 10] was proved by Guliyev *et al.* in [11, 12].

Theorem E *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition* (2.4). *Then the operators M and T are bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}.*

Theorem F *Let* $1 \leq p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* (φ_1, φ_2) *satisfy the condition* (1.14). *Then the operators* M_α *and* I_α *are bounded from* M_{p,φ_1} *to* M_{q,φ_2} *for* $p > 1$ *and from* M_{1,φ_1} *to* WM_{q,φ_2} *for* $p = 1$.

Note that integral conditions of type (2.3) after the paper [17] of 1956 are often referred to as Bary-Stechkin or Zygmund-Bary-Stechkin conditions; see also [18]. The classes of almost monotonic functions satisfying such integral conditions were later studied in a number of papers, see [19–21] and references therein, where the characterization of integral inequalities of such a kind was given in terms of certain lower and upper indices known as Matuszewska-Orlicz indices. Note that in the cited papers the integral inequalities were studied as $r \to 0$. Such inequalities are also of interest when they allow to impose different conditions as $r \to 0$ and $r \to \infty$; such a case was dealt with in [22, 23].

3 The fractional oscillatory integral operators in the spaces $M_{p,\varphi}(\mathbb{R}^n)$

In this section, we are going to use the following statement on the boundedness of the Hardy operator:

$$(Hg)(t) := \frac{1}{t} \int_0^t g(r)\, dr, \quad 0 < t < \infty.$$

Theorem G [24] *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)Hg(t) \leq c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all non-negative and non-increasing g *on* $(0,\infty)$ *if and only if*

$$A := \sup_{t>0} \frac{w(t)}{t} \int_0^t \frac{dr}{\operatorname{ess\,inf}_{0<s<r} v(s)} < \infty,$$

and $c \approx A$.

Lemma 3.1 *Let* $1 \leq p < \infty$, *and* K *is a SCZK and the Calderón-Zygmund singular integral operator* $\widetilde{S}$ *is of type* $(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n))$. *Then for* $1 < p < \infty$ *and any polynomial* $P(x,y)$ *the inequality*

$$\|Sf\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{-1-\frac{n}{p}}\, dt$$

holds for any ball $B(x_0,r)$ *and for all* $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover, for $p = 1$ *and* K *is a CZK*

$$\|Tf\|_{WL_1(B(x_0,r))} \lesssim r^n \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{-1-n}\, dt \tag{3.1}$$

holds for any ball $B(x_0,r)$ *and for all* $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof Let $p \in (1,\infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and radius r, $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), f_2(y) = f(y)\chi_{(2B)^{\complement}}(y)$$

and have

$$\|Sf\|_{L_p(B)} \le \|Sf_1\|_{L_p(B)} + \|Sf_2\|_{L_p(B)}.$$

It is known that (see [5], see also [7, 25, 26]), if K is a SCZK and the operator $\widetilde{S}$ is of type $(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n))$, then for $1 < p < \infty$ and any polynomial $P(x,y)$ the operator S is bounded on $L_p(\mathbb{R}^n)$. Since $f_1 \in L_p(\mathbb{R}^n)$, $Sf_1 \in L_p(\mathbb{R}^n)$ and boundedness of S in $L_p(\mathbb{R}^n)$ (see [5]) it follows that

$$\|Sf_1\|_{L_p(B)} \le \|Sf_1\|_{L_p(\mathbb{R}^n)} \le C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f_1\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f.

It is clear that $x \in B$, $y \in (2B)^{\complement}$ implies $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$. We get

$$\left|Sf_2(x)\right| \le c_0 \int_{(2B)^{\complement}} \frac{|f(y)|}{|x_0 - y|^n}\, dy.$$

By Fubini's theorem and applying Hölder inequality, we have

$$\begin{aligned}
\int_{(2B)^{\complement}} \frac{|f(y)|}{|x_0 - y|^n}\, dy &\approx \int_{(2B)^{\complement}} \left|f(y)\right| \int_{|x_0-y|}^{\infty} t^{-1-n}\, dt\, dy \\
&\approx \int_{2r}^{\infty} \int_{2r<|x_0-y|<t} \left|f(y)\right| dy t^{-1-n}\, dt \\
&\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} \left|f(y)\right| dy t^{-1-n}\, dt \\
&\lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{-1-\frac{n}{p}}\, dt. \qquad (3.2)
\end{aligned}$$

Moreover, for all $p \in [1,\infty)$ the inequality

$$\|Sf_2\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{-1-\frac{n}{p}}\, dt \qquad (3.3)$$

is valid. Thus,

$$\|Sf\|_{L_p(B)} \lesssim \|f\|_{L_p(2B)} + r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{-1-\frac{n}{p}}\, dt.$$

On the other hand,

$$\begin{aligned}
\|f\|_{L_p(2B)} &\approx r^{\frac{n}{p}} \|f\|_{L_p(2B)} \int_{2r}^{\infty} t^{-1-\frac{n}{p}}\, dt \\
&\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{-1-\frac{n}{p}}\, dt. \qquad (3.4)
\end{aligned}$$

Hence,

$$\|Sf\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{-1-\frac{n}{p}}\, dt.$$

Let $p = 1$. From the weak (1,1) boundedness of T (see [6]) and (3.4), it follows that:

$$\begin{aligned}\|Tf_1\|_{WL_1(B)} &\le \|Tf_1\|_{WL_1(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)} \\ &= \|f\|_{L_1(2B)} \lesssim r^n \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|\,dy \frac{dt}{t^{n+1}}.\end{aligned} \tag{3.5}$$

Then by (3.4) and (3.5), we get the inequality (3.1). □

Proof of Theorem 1.1 By Lemma 3.1 and Theorem G, we get

$$\begin{aligned}\|Sf\|_{M_{p,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L_p(B(x,t))} t^{-1-\frac{n}{p}}\,dt \\ &\approx \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-\frac{n}{p}}} \|f\|_{L_p(B(x,t^{-\frac{p}{n}}))}\,dt \\ &= \sup_{x\in\mathbb{R}^n, r>0} \varphi_2\bigl(x,r^{-\frac{p}{n}}\bigr)^{-1} \int_0^{r} \|f\|_{L_p(B(x,t^{-\frac{p}{n}}))}\,dt \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1\bigl(x,r^{-\frac{p}{n}}\bigr)^{-1} r \|f\|_{L_p(B(x,r^{-\frac{p}{n}}))} = \|f\|_{M_{p,\varphi_1}}\end{aligned}$$

if $p \in (1,\infty)$, and

$$\begin{aligned}\|Tf\|_{WM_{1,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L_1(B(x,t))} t^{-1-n}\,dt \\ &\approx \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-n}} \|f\|_{L_1(B(x,t^{-\frac{1}{n}}))}\,dt \\ &= \sup_{x\in\mathbb{R}^n, r>0} \varphi_2\bigl(x,r^{-\frac{1}{n}}\bigr)^{-1} \int_0^{r} \|f\|_{L_1(B(x,t^{-\frac{1}{n}}))}\,dt \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1\bigl(x,r^{-\frac{1}{n}}\bigr)^{-1} r \|f\|_{L_1(B(x,r^{-\frac{1}{n}}))} = \|f\|_{M_{1,\varphi_1}}\end{aligned}$$

if $p = 1$. □

Proof of Theorem 1.2 The proof of Theorem 1.2 follows from Theorem F and the following observation:

$$|S_\alpha f(x)| \le I_\alpha(|f|)(x).$$ □

4 Commutators of fractional oscillatory integral operators in the spaces $M_{p,\varphi}(\mathbb{R}^n)$

Let T be a Calderón-Zygmund singular integral operator and $b \in \mathrm{BMO}(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [27] states that the commutator operator $[b,T]f = T(bf) - bTf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [2, 28, 29]).

First, we recall the definition of the space $\mathrm{BMO}(\mathbb{R}^n)$.

Definition 2 Suppose that $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, let

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right| dy < \infty,$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy.$$

Define

$$\mathrm{BMO}(\mathbb{R}^n) = \left\{ f \in L_1^{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_* < \infty \right\}.$$

If one regards two functions whose difference is a constant as one, then space $\mathrm{BMO}(\mathbb{R}^n)$ is a Banach space with respect to norm $\|\cdot\|_*$.

Remark 1 (1) The John-Nirenberg inequality: there are constants $C_1, C_2 > 0$, such that for all $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $\beta > 0$

$$\left|\left\{ x \in B : \left|f(x) - f_B\right| > \beta \right\}\right| \le C_1 |B| e^{-C_2\beta/\|f\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) The John-Nirenberg inequality implies that

$$\|f\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right|^p dy \right)^{\frac{1}{p}} \tag{4.1}$$

for $1 < p < \infty$.

(3) Let $f \in \mathrm{BMO}(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$|f_{B(x,r)} - f_{B(x,t)}| \le C\|f\|_* \ln\frac{t}{r} \quad \text{for } 0 < 2r < t, \tag{4.2}$$

where C is independent of f, x, r and t.

Lemma 4.1 *Let $1 \le p < \infty$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, K is a SCZK and the Calderón-Zygmund singular integral operator $\widetilde{S}$ is of type $(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n))$. Then for $1 < p < \infty$ and any polynomial $P(x,y)$ the inequality*

$$\|S_b f\|_{L_p(B(x_0,r))} \lesssim \|b\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{-1-\frac{n}{p}}\, dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof Let $p \in (1,\infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and radius r, $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), f_2(y) = f(y)\chi_{(2B)^{\complement}}(y)$$

and have

$$\|S_b f\|_{L_p(B)} \leq \|S_b f_1\|_{L_p(B)} + \|S_b f_2\|_{L_p(B)}.$$

It is known that (see [5], see also [7, 25, 26]), if K is a SCZK and the operator $\widetilde{S}$ is of type $(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n))$, then for $1 < p < \infty$ and any polynomial $P(x,y)$ the commutator operator S_b is bounded on $L_p(\mathbb{R}^n)$. Since $f_1 \in L_p(\mathbb{R}^n)$, $Sf_1 \in L_p(\mathbb{R}^n)$ and boundedness of S_b in $L_p(\mathbb{R}^n)$ (see [5]) it follows that

$$\|S_b f_1\|_{L_p(B)} \leq \|S_b f_1\|_{L_p(\mathbb{R}^n)} \leq C\|b\|_* \|f_1\|_{L_p(\mathbb{R}^n)} = C\|b\|_* \|f_1\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f.

For $x \in B$, we have

$$\begin{aligned} \left|S_b f_2(x)\right| &\lesssim \int_{\mathbb{R}^n} \frac{|b(y) - b(x)|}{|x-y|^n} \left|f(y)\right| dy \\ &\approx \int_{\complement_{(2B)}} \frac{|b(y) - b(x)|}{|x_0-y|^n} \left|f(y)\right| dy. \end{aligned}$$

Then

$$\begin{aligned} \|S_b f_2\|_{L_p(B)} &\lesssim \left(\int_B \left(\int_{\complement_{(2B)}} \frac{|b(y) - b(x)|}{|x_0-y|^n} \left|f(y)\right| dy\right)^p dx\right)^{\frac{1}{p}} \\ &\lesssim \left(\int_B \left(\int_{\complement_{(2B)}} \frac{|b(y) - b_B|}{|x_0-y|^n} \left|f(y)\right| dy\right)^p dx\right)^{\frac{1}{p}} \\ &\quad + \left(\int_B \left(\int_{\complement_{(2B)}} \frac{|b(x) - b_B|}{|x_0-y|^n} \left|f(y)\right| dy\right)^p dx\right)^{\frac{1}{p}} \\ &= I_1 + I_2. \end{aligned}$$

Let us estimate I_1.

$$\begin{aligned} I_1 &\approx r^{\frac{n}{p}} \int_{\complement_{(2B)}} \frac{|b(y) - b_B|}{|x_0-y|^n} \left|f(y)\right| dy \\ &\approx r^{\frac{n}{p}} \int_{\complement_{(2B)}} \left|b(y) - b_B\right| \left|f(y)\right| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}}\, dy \\ &\approx r^{\frac{n}{p}} \int_{2r}^{\infty} \int_{2r \leq |x_0-y| \leq t} \left|b(y) - b_B\right| \left|f(y)\right| dy \frac{dt}{t^{n+1}} \\ &\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} \left|b(y) - b_B\right| \left|f(y)\right| dy \frac{dt}{t^{n+1}}. \end{aligned}$$

Applying Hölder's inequality and by (4.1), (4.2), we get

$$\begin{aligned} I_1 &\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} \left|b(y) - b_{B(x_0,t)}\right| \left|f(y)\right| dy \frac{dt}{t^{n+1}} \\ &\quad + r^{\frac{n}{p}} \int_{2r}^{\infty} \left|b_{B(x_0,r)} - b_{B(x_0,t)}\right| \int_{B(x_0,t)} \left|f(y)\right| dy \frac{dt}{t^{n+1}} \end{aligned}$$

$$\begin{aligned}
&\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \left(\int_{B(x_0,t)} \left| b(y) - b_{B(x_0,t)} \right|^{p'} dy \right)^{\frac{1}{p'}} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\quad + r^{\frac{n}{p}} \int_{2r}^{\infty} \left| b_{B(x_0,r)} - b_{B(x_0,t)} \right| \|f\|_{L_p(B(x_0,t))} t^{-1-\frac{n}{p}} \, dt \\
&\lesssim \|b\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r} \right) \|f\|_{L_p(B(x_0,t))} t^{-1-\frac{n}{p}} \, dt.
\end{aligned}$$

In order to estimate I_2 note that

$$I_2 = \left(\int_B \left| b(x) - b_B \right|^p dx \right)^{\frac{1}{p}} \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^n} \, dy.$$

By (4.1), we get

$$I_2 \lesssim \|b\|_* r^{\frac{n}{p}} \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^n} \, dy.$$

Thus, by (3.2)

$$I_2 \lesssim \|b\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} t^{-1-\frac{n}{p}} \, dt.$$

Summing up I_1 and I_2, for all $p \in (1, \infty)$ we get

$$\|S_b f_2\|_{L_p(B)} \lesssim \|b\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r} \right) \|f\|_{L_p(B(x_0,t))} t^{-1-\frac{n}{p}} \, dt. \tag{4.3}$$

Finally,

$$\|S_b f\|_{L_p(B)} \lesssim \|b\|_* \|f\|_{L_p(2B)} + \|b\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r} \right) \|f\|_{L_p(B(x_0,t))} t^{-1-\frac{n}{p}} \, dt,$$

and statement of Lemma 4.1 follows by (3.4). □

Proof of Theorem 1.3 The statement of Theorem 1.3 follows by Lemma 4.1 and Theorem G in the same manner as in the proof of Theorem G. □

Proof of Theorem 1.4 The proof of Theorem 1.4 follows from the Theorem 7.4 in [11] and the following observation:

$$\left| S_{\alpha,b} f(x) \right| \le I_{\alpha,b}\left(|f| \right)(x).$$ □

Competing interests
The author declares that they have no competing interests.

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References

1. Morrey, CB: On the solutions of quasi-linear elliptic partial differential equations. Trans. Am. Math. Soc. **43**, 126-166 (1938)
2. Chiarenza, F, Frasca, M: Morrey spaces and Hardy-Littlewood maximal function. Rend. Mat. Appl. **7**, 273-279 (1987)
3. Adams, DR: A note on Riesz potentials. Duke Math. J. **42**, 765-778 (1975)
4. Phong, DH, Stein, EM: Singular integrals related to the Radon transform and boundary value problems. Proc. Natl. Acad. Sci. USA **80**, 7697-7701 (1983)
5. Ricci, F, Stein, EM: Harmonic analysis on nilpotent groups and singular integrals I: oscillatory integrals. J. Funct. Anal. **73**, 179-194 (1987)
6. Chanillo, S, Christ, M: Weak $(1,1)$ bounds for oscillatory singular integral. Duke Math. J. **55**, 141-155 (1987)
7. Lu, SZ, Zhang, Y: Criterion on L^p-boundedness for a class of oscillatory singular integrals with rough kernels. Rev. Mat. Iberoam. **8**, 201-219 (1992)
8. Ding, Y: L_p-Boundedness for fractional oscillatory integral operator with rough kernel. Approx. Theory Appl. **12**, 70-79 (1996)
9. Guliyev, VS: Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$. Doctor of Sciences, Mat. Inst. Steklova, Moscow (1994), 329 pp. (in Russian)
10. Guliyev, VS: Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces. J. Inequal. Appl. **2009**, Article ID 503948 (2009)
11. Guliyev, VS, Aliyev, SS, Karaman, T, Shukurov, PS: Boundedness of sublinear operators and commutators on generalized Morrey space. Integral Equ. Oper. Theory **71**, 327-355 (2011)
12. Akbulut, A, Guliyev, VS, Mustafayev, R: On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces. Math. Bohem. **137**(1), 27-43 (2012)
13. Mizuhara, T: Boundedness of some classical operators on generalized Morrey spaces. In: Igari, S (ed.) Harmonic Analysis. ICM 90 Satellite Proceedings, pp. 183-189. Springer, Tokyo (1991)
14. Nakai, E: Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces. Math. Nachr. **166**, 95-103 (1994)
15. Sawano, Y, Sugano, S, Tanaka, H: A note on generalized fractional integral operators on generalized Morrey spaces. Bound. Value Probl. **2009**, Article ID 835865 (2009)
16. Softova, L: Singular integrals and commutators in generalized Morrey spaces. Acta Math. Sin. Engl. Ser. **22**(3), 757-766 (2006)
17. Bary, NK, Stechkin, SB: Best approximations and differential properties of two conjugate functions. Tr. Mosk. Mat. Obŝ. **5**, 483-522 (1956) (in Russian)
18. Guseinov, AI, Mukhtarov, KS: Introduction to the Theory of Nonlinear Singular Integral Equations. Nauka, Moscow (1980) (in Russian)
19. Karapetiants, NK, Samko, NG: Weighted theorems on fractional integrals in the generalized Hölder spaces $H_0^\omega(\rho)$ via the indices m_ω and M_ω. Fract. Calc. Appl. Anal. **7**(4), 437-458 (2004)
20. Samko, N: Singular integral operators in weighted spaces with generalized Hölder condition. Proc. A. Razmadze Math. Inst. **120**, 107-134 (1999)
21. Samko, N: On non-equilibrated almost monotonic functions of the Zygmund-Bary-Stechkin class. Real Anal. Exch. **30**(2), 727-745 (2004/2005)
22. Kokilashvili, V, Samko, S: Operators of harmonic analysis in weighted spaces with non-standard growth. J. Math. Anal. Appl. **352**, 15-34 (2009)
23. Samko, N, Samko, S, Vakulov, B: Weighted Sobolev theorem in Lebesgue spaces with variable exponent. J. Math. Anal. Appl. **335**, 560-583 (2007)
24. Carro, M, Pick, L, Soria, J, Stepanov, VD: On embeddings between classical Lorentz spaces. Math. Inequal. Appl. **4**(3), 397-428 (2001)
25. Lu, SZ: A class of oscillatory integrals. Int. J. Appl. Math. Sci. **2**(1), 42-58 (2005)
26. Lu, SZ, Ding, Y, Yan, DY: Singular Integrals and Related Topics. World Scientific, Singapore (2007)
27. Coifman, R, Rochberg, R, Weiss, G: Factorization theorems for Hardy spaces in several variables. Ann. Math. **103**(2), 611-635 (1976)
28. Chiarenza, F, Frasca, M, Longo, P: Interior $W^{2,p}$-estimates for nondivergence elliptic equations with discontinuous coefficients. Ric. Mat. **40**, 149-168 (1991)
29. Fazio, GD, Ragusa, MA: Interior estimates in Morrey spaces for strong solutions to nodivergence form equations with discontinuous coefficients. J. Funct. Anal. **112**, 241-256 (1993)

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