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Mixed monotone operator methods for the existence and uniqueness of positive solutions to Riemann-Liouville fractional differential equation boundary value problems

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Abstract

This work is concerned with the existence and uniqueness of positive solutions for the following fractional boundary value problem:

$$\begin{cases} -D_{0+}^{\nu} y(t) = f(t, y(t), y(t)) + g(t, y(t)), & 0 < t < 1, n-1 < \nu \le n, \\ y^{(i)}(0) = 0, & 0 \le i \le n-2, \\ [D_{0+}^{\alpha} y(t)]_{t=1} = 0, & 1 \le \alpha \le n-2, \end{cases}$$

where D_{0+}^{ν} is the standard Riemann-Liouville fractional derivative of order ν, and $n \in N$, $n > 3$. Our analysis relies on two new fixed point theorems for mixed monotone operators with perturbation. Our results can not only guarantee the existence of a unique positive solution, but also be applied to construct an iterative scheme for approximating it. An example is given to illustrate the main result.

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Keywords: Riemann-Liouville fractional derivative; fractional differential equation; positive solution; existence and uniqueness; fixed point theorem for mixed monotone operator

1 Introduction

In this paper, we investigate the existence and uniqueness of positive solutions for the fractional boundary value problem (FBVP for short) of the form:

$$\begin{cases} -D_{0+}^{\nu} y(t) = f(t, y(t), y(t)) + g(t, y(t)), & 0 < t < 1, n-1 < \nu \le n, \\ y^{(i)}(0) = 0, & 0 \le i \le n-2, \\ [D_{0+}^{\alpha} y(t)]_{t=1} = 0, & 1 \le \alpha \le n-2, \end{cases} \tag{1.1}$$

where D_{0+}^{ν} is the standard Riemann-Liouville fractional derivative of order ν, and $n \in N$, $n > 3$.

Fractional differential equations arise in many fields such as physics, mechanics, chemistry, economics, engineering and biological sciences, *etc.*; see [1–6] for example. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives; see the monographs of Miller and Ross [3], Podlubny [5], Kilbas *et al.* [6], and the papers [7–16] and the references therein. In these papers, many authors have investigated the existence of positive solutions for nonlinear fractional differential equation boundary value problems. On the other hand, the uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems has been studied by some authors; see [10, 14, 17] for example.

In [18], Goodrich utilized the Krasnoselskii's fixed point theorem to study a FBVP of the form:

$$\begin{cases} -D_{0^+}^{\nu} y(t) = f(t, y(t)), & 0 < t < 1, n-1 < \nu \leq n, \\ y^{(i)}(0) = 0, & 0 \leq i \leq n-2, \\ [D_{0^+}^{\alpha} y(t)]_{t=1} = 0, & 1 \leq \alpha \leq n-2, \end{cases} \tag{1.2}$$

and established the existence of at least one positive solution for FBVP (1.2). By using the same fixed point theorem, Goodrich [19] considered the existence of a positive solution to the following systems of differential equations of fractional order:

$$\begin{cases} -D_{0^+}^{\nu_1} y_1(t) = \lambda_1 a_1(t) f(y_1(t), y_2(t)), \\ -D_{0^+}^{\nu_2} y_2(t) = \lambda_2 a_2(t) g(y_1(t), y_2(t)), \end{cases}$$

where $t \in (0,1)$, $\nu_1, \nu_2 \in (n-1, n]$ for $n > 3$ and $n \in N$, and $\lambda_1, \lambda_2 > 0$, with the following boundary value conditions:

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0), \quad 0 \leq i \leq n-2,$$
$$\left[D_{0+}^{\alpha} y_1(t)\right]_{t=1} = 0 = \left[D_{0+}^{\alpha} y_2(t)\right]_{t=1}, \quad 1 \leq \alpha \leq n-2,$$

under the assumptions that a_1, a_2, f, g are nonnegative and continuous. But the uniqueness of positive solutions is not treated in these papers.

Different from the works mentioned above, motivated by the work [20], we will use two fixed point theorems for mixed monotone operators with perturbation to show the existence and uniqueness of positive solutions for FBVP (1.1). To our knowledge, there are still very few to utilize the fixed point results on mixed monotone operators with perturbation to study the existence and uniqueness of a positive solution for nonlinear fractional differential equation boundary value problems. So, it is worthwhile to investigate FBVP (1.1) by using our new fixed point theorems in [20]. Our results can not only guarantee the existence of a unique positive solution, but also be applied to construct an iterative scheme for approximating it.

With this context in mind, the outline of this paper is as follows. In Section 2 we recall certain results from the theory of fractional calculus and some definitions, notations and results of mixed monotone operators. In Section 3 we provide some conditions, under which the problem FBVP (1.1) has a unique positive solution. Finally, in Section 4, we provide an example, which explicates the applicability of our result.

2 Preliminaries

For the convenience of the reader, we present here some definitions, lemmas and basic results that will be used in the proofs of our theorems.

Definition 2.1 (See [18]) Let $\nu > 0$ with $\nu \in R$. Suppose that $y : [a, +\infty) \to R$. Then the νth Riemann-Liouville fractional integral is defined to be

$$D_{a^+}^{-\nu} y(t) := \frac{1}{\Gamma(\nu)} \int_a^t y(s)(t-s)^{\nu-1}\, ds,$$

whenever the right-hand side is defined. Similarly, with $\nu > 0$ and $\nu \in R$, we define the νth Riemann-Liouville fractional derivative to be

$$D_{a^+}^{\nu} y(t) := \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_a^t \frac{y(s)}{(t-s)^{\nu+1-n}}\, ds,$$

where $n \in N$ is the unique positive integer satisfying $n-1 \le \nu < n$ and $t > a$.

Lemma 2.2 (See [19]) *Let $g \in C[0,1]$ be given. Then the unique solution to problem $-D_{0^+}^{\nu} y(t) = g(t)$ together with the boundary conditions $y^{(i)}(0) = 0 = [D_{0^+}^{\alpha} y(t)]_{t=1}$, where $1 \le \alpha \le n-2$ and $0 \le i \le n-2$, is*

$$y(t) = \int_0^1 G(t,s) g(s)\, ds, \tag{2.1}$$

where

$$G(t,s) = \begin{cases} \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-1}}{\Gamma(\nu)}, & 0 \le s \le t \le 1, \\ \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}, & 0 \le t \le s \le 1, \end{cases} \tag{2.2}$$

is the Green function for this problem.

Lemma 2.3 (See [19]) *Let $G(t,s)$ be as given in the statement of Lemma* 2.2. *Then we have*

(i) *$G(t,s)$ is a continuous function on the unit square* $[0,1] \times [0,1]$;
(ii) $G(t,s) \ge 0$ *for each* $(t,s) \in [0,1] \times [0,1]$.

Lemma 2.4 *The function $G(t,s)$ defined by* (2.2) *satisfies the following conditions*:

$$\left[1 - (1-s)^{\alpha}\right](1-s)^{\nu-\alpha-1} t^{\nu-1} \le \Gamma(\nu) G(t,s) \le (1-s)^{\nu-\alpha-1} t^{\nu-1}, \quad t,s \in [0,1].$$

Proof Evidently, the right inequality holds. So, we only need to prove the left inequality. If $0 \le s \le t \le 1$, then we have $0 \le t-s \le t-ts = (1-s)t$, and thus

$$(t-s)^{\nu-1} \le (1-s)^{\nu-1} t^{\nu-1}.$$

Hence,

$$\begin{aligned} \Gamma(\nu) G(t,s) &= t^{\nu-1}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-1} \\ &\ge t^{\nu-1}(1-s)^{\nu-\alpha-1} - t^{\nu-1}(1-s)^{\nu-1} \end{aligned}$$

$$= t^{\nu-1}\big[(1-s)^{\nu-\alpha-1}-(1-s)^{\nu-1}\big]$$
$$= \big[1-(1-s)^{\alpha}\big](1-s)^{\nu-\alpha-1}t^{\nu-1}.$$

When $0 \le t \le s \le 1$, we have

$$\begin{aligned}\Gamma(\nu)G(t,s) &= t^{\nu-1}(1-s)^{\nu-\alpha-1}\\ &\ge t^{\nu-1}\big[(1-s)^{\nu-\alpha-1}-(1-s)^{\nu-1}\big]\\ &= \big[1-(1-s)^{\alpha}\big](1-s)^{\nu-\alpha-1}t^{\nu-1}.\end{aligned}$$

So, the proof is complete. □

In the sequel, we present some basic concepts in ordered Banach spaces for completeness and two fixed point theorems which we will be used later. For convenience of readers, we suggest that one refers to [20–22] for details.

Suppose that $(E,\|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, *i.e.*, $x \le y$ if and only if $y-x \in P$. If $x \le y$ and $x \ne y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P$, $\lambda \ge 0 \Rightarrow \lambda x \in P$; (ii) $x \in P$, $-x \in P \Rightarrow x = \theta$.

P is called normal if there exists a constant $N > 0$ such that, for all $x,y \in E$, $\theta \le x \le y$ implies $\|x\| \le N\|y\|$; in this case, N is called the normality constant of P. If $x_1,x_2 \in E$, the set $[x_1,x_2] = \{x \in E \mid x_1 \le x \le x_2\}$ is called the order interval between x_1 and x_2. We say that an operator $A: E \to E$ is increasing (decreasing) if $x \le y$ implies $Ax \le Ay$ ($Ax \ge Ay$).

For all $x,y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \le y \le \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (*i.e.*, $h \ge \theta$ and $h \ne \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

Definition 2.5 (See [20, 22]) $A: P \times P \to P$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in x and decreasing in y, *i.e.*, u_i, v_i $(i = 1,2) \in P$, $u_1 \le u_2$, $v_1 \ge v_2$ imply $A(u_1,v_1) \le A(u_2,v_2)$. Element $x \in P$ is called a fixed point of A if $A(x,x) = x$.

Definition 2.6 An operator $A: P \to P$ is said to be sub-homogeneous if it is satisfies

$$A(tx) \ge tA(x), \quad \forall t \in (0,1), x \in P. \tag{2.3}$$

Definition 2.7 Let $D = P$ and β be a real number with $0 \le \beta < 1$. An operator $A: D \to D$ is said to be β-concave if it satisfies

$$A(tx) \ge t^{\beta}A(x), \quad \forall t \in (0,1), x \in D. \tag{2.4}$$

Lemma 2.8 (See Theorem 2.1 in [20]) *Let $h > \theta$ and $\beta \in (0,1)$. $A: P \times P \to P$ is a mixed monotone operator and satisfies*

$$A\big(tx, t^{-1}y\big) \ge t^{\beta}A(x,y), \quad \forall t \in (0,1), x,y \in P. \tag{2.5}$$

$B: P \to P$ is an increasing sub-homogeneous operator. Assume that

(i) *there is $h_0 \in P_h$ such that $A(h_0,h_0) \in P_h$ and $Bh_0 \in P_h$;*

(ii) *there exists a constant $\delta_0 > 0$ such that $A(x,y) \ge \delta_0 Bx$, $\forall x,y \in P$.*

Then:

(1) $A : P_h \times P_h \to P_h$ *and* $B : P_h \to P_h$;

(2) *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that*

$$rv_0 \le u_0 < v_0, \qquad u_0 \le A(u_0, v_0) + Bu_0 \le A(v_0, u_0) + Bv_0 \le v_0;$$

(3) *the operator equation* $A(x,x) + Bx = x$ *has a unique solution* x^* *in* P_h;

(4) *for any initial values* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ *and* $y_n \to x^*$ *as* $n \to \infty$.

Lemma 2.9 (See Theorem 2.4 in [20]) *Let* $h > \theta$ *and* $\beta \in (0,1)$. $A : P \times P \to P$ *is a mixed monotone operator and satisfies*

$$A\left(tx, t^{-1}y\right) \ge tA(x,y), \quad \forall t \in (0,1), x, y \in P. \tag{2.6}$$

$B : P \to P$ *is an increasing* β*-concave operator. Assume that*

(i) *there is* $h_0 \in P_h$ *such that* $A(h_0, h_0) \in P_h$ *and* $Bh_0 \in P_h$;

(ii) *there exists a constant* $\delta_0 > 0$ *such that* $A(x,y) \le \delta_0 Bx$, $\forall x, y \in P$.

Then:

(1) $A : P_h \times P_h \to P_h$ *and* $B : P_h \to P_h$;

(2) *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that*

$$rv_0 \le u_0 < v_0, \qquad u_0 \le A(u_0, v_0) + Bu_0 \le A(v_0, u_0) + Bv_0 \le v_0;$$

(3) *the operator equation* $A(x,x) + Bx = x$ *has a unique solution* x^* *in* P_h;

(4) *for any initial values* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ *and* $y_n \to x^*$ *as* $n \to \infty$.

Remark 2.10 (i) If we take $B = \theta$ in Lemma 2.8, then the corresponding conclusion is still true (see Corollary 2.2 in [20]); (ii) if we take $A = \theta$ in Lemma 2.9, then the conclusion obtained is also true (see Theorem 2.7 in [23]).

3 Main results

In this section, we apply Lemma 2.8 and Lemma 2.9 to study FBVP (1.1), and we obtain some new results on the existence and uniqueness of positive solutions. The method used here is relatively new to the literature and so are the existence and uniqueness results to the fractional differential equations.

In our considerations, we work in the Banach space $C[0,1] = \{x : [0,1] \to \mathbf{R}$ is continuous$\}$ with the standard norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Notice that this space can be equipped with a partial order given by

$$x, y \in C[0,1], \quad x \le y \quad \Leftrightarrow \quad x(t) \le y(t) \quad \text{for } t \in [0,1].$$

Set $P = \{x \in C[0,1] \mid x(t) \geq 0, t \in [0,1]\}$, the standard cone. It is clear that P is a normal cone in $C[0,1]$ and the normality constant is 1.

Theorem 3.1 *Assume that*

(H_1) $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ *is continuous and* $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous*;

(H_2) $f(t,u,v)$ *is increasing in* $u \in [0,+\infty)$ *for fixed* $t \in [0,1]$ *and* $v \in [0,+\infty)$, *decreasing in* $v \in [0,+\infty)$ *for fixed* $t \in [0,1]$ *and* $u \in [0,+\infty)$, *and* $g(t,u)$ *is increasing in* $u \in [0,+\infty)$ *for fixed* $t \in [0,1]$;

(H_3) $g(t,0) \not\equiv 0$ *and* $g(t,\lambda u) \geq \lambda g(t,u)$ *for* $\lambda \in (0,1)$, $t \in [0,1]$, $u \in [0,+\infty)$, *and there exists a constant* $\beta \in (0,1)$ *such that* $f(t,\lambda u,\lambda^{-1}v) \geq \lambda^{\beta} f(t,u,v)$, $\forall t \in [0,1]$, $\lambda \in (0,1)$, $u,v \in [0,+\infty)$;

(H_4) *there exists a constant* $\delta_0 > 0$ *such that* $f(t,u,v) \geq \delta_0 g(t,u)$, $t \in [0,1]$, $u,v \geq 0$.

Then:

(1) *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that* $rv_0 \leq u_0 < v_0$ *and*

$$u_0(t) \leq \int_0^1 G(t,s)\big[f\big(s,u_0(s),v_0(s)\big) + g\big(s,u_0(s)\big)\big]\,ds, \quad t \in [0,1],$$

$$v_0(t) \geq \int_0^1 G(t,s)\big[f\big(s,v_0(s),u_0(s)\big) + g\big(s,v_0(s)\big)\big]\,ds, \quad t \in [0,1],$$

where $h(t) = t^{\nu-1}$, $t \in [0,1]$ *and* $G(t,s)$ *is given as in* (2.2);

(2) *FBVP* (1.1) *has a unique positive solution* u^* *in* P_h;

(3) *for any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_{n+1}(t) = \int_0^1 G(t,s)\big[f\big(s,x_n(s),y_n(s)\big) + g\big(s,x_n(s)\big)\big]\,ds, \quad n = 0,1,2,\ldots,$$

$$y_{n+1}(t) = \int_0^1 G(t,s)\big[f\big(s,y_n(s),x_n(s)\big) + g\big(s,y_n(s)\big)\big]\,ds, \quad n = 0,1,2,\ldots,$$

we have $\|x_n - u^*\| \to 0$ *and* $\|y_n - u^*\| \to 0$ *as* $n \to \infty$.

Proof To begin with, from Lemma 2.2, FBVP (1.1) has an integral formulation given by

$$u(t) = \int_0^1 G(t,s)\big[f\big(s,u(s),u(s)\big) + g\big(s,u(s)\big)\big]\,ds,$$

where $G(t,s)$ is given as in (2.2).

Define two operators $A : P \times P \to E$ and $B : P \to E$ by

$$A(u,v)(t) = \int_0^1 G(t,s)f\big(s,u(s),v(s)\big)\,ds, \qquad (Bu)(t) = \int_0^1 G(t,s)g\big(s,u(s)\big)\,ds.$$

It is easy to prove that u is the solution of FBVP (1.1) if and only if $u = A(u,u) + Bu$. From (H_1), we know that $A : P \times P \to P$ and $B : P \to P$. In the sequel, we check that A, B satisfy all the assumptions of Lemma 2.8.

Firstly, we prove that A is a mixed monotone operator. In fact, for $u_i, v_i \in P$, $i = 1,2$ with $u_1 \geq u_2$, $v_1 \leq v_2$, we know that $u_1(t) \geq u_2(t)$, $v_1(t) \leq v_2(t)$, $t \in [0,1]$, and by (H_2) and Lemma 2.3,

$$A(u_1, v_1)(t) = \int_0^1 G(t,s)f\big(s, u_1(s), v_1(s)\big)\,ds \geq \int_0^1 G(t,s)f\big(s, u_2(s), v_2(s)\big)\,ds = A(u_2, v_2)(t).$$

That is, $A(u_1, v_1) \geq A(u_2, v_2)$.

Further, it follows from (H_2) and Lemma 2.3 that B is increasing. Next we show that A satisfies the condition (2.5). For any $\lambda \in (0,1)$ and $u, v \in P$, by (H_3) we have

$$\begin{aligned} A\big(\lambda u, \lambda^{-1}v\big)(t) &= \int_0^1 G(t,s)f\big(s, \lambda u(s), \lambda^{-1}v(s)\big)\,ds \\ &\geq \lambda^{\beta} \int_0^1 G(t,s)f\big(s, u(s), v(s)\big)\,ds \\ &= \lambda^{\beta} A(u,v)(t). \end{aligned}$$

That is, $A(\lambda u, \lambda^{-1}v) \geq \lambda^{\beta} A(u,v)$ for $\lambda \in (0,1)$, $u, v \in P$. So, the operator A satisfies (2.5). Also, for any $\lambda \in (0,1)$, $u \in P$, from (H_3) we know that

$$B(\lambda u)(t) = \int_0^1 G(t,s)g\big(s, \lambda u(s)\big)\,ds \geq \lambda \int_0^1 G(t,s)g\big(s, u(s)\big)\,ds = \lambda Bu(t),$$

that is, $B(\lambda u) \geq \lambda Bu$ for $\lambda \in (0,1)$, $u \in P$. That is, the operator B is sub-homogeneous. Now we show that $A(h,h) \in P_h$ and $Bh \in P_h$. On the one hand, from (H_1), (H_2) and Lemma 2.4, for any $t \in [0,1]$, we have

$$\begin{aligned} A(h,h)(t) &= \int_0^1 G(t,s)f\big(s, h(s), h(s)\big)\,ds \\ &= \int_0^1 G(t,s)f\big(s, s^{\nu-1}, s^{\nu-1}\big)\,ds \\ &\leq \frac{1}{\Gamma(\nu)}h(t)\int_0^1 (1-s)^{\nu-\alpha-1}f(s,1,0)\,ds. \end{aligned}$$

On the other hand, also from (H_1), (H_2) and Lemma 2.4, for any $t \in [0,1]$, we obtain

$$\begin{aligned} A(h,h)(t) &= \int_0^1 G(t,s)f\big(s, h(s), h(s)\big)\,ds \\ &= \int_0^1 G(t,s)f\big(s, s^{\nu-1}, s^{\nu-1}\big)\,ds \\ &\geq \frac{1}{\Gamma(\nu)}h(t)\int_0^1 \big[1-(1-s)^{\alpha}\big](1-s)^{\nu-\alpha-1}f(s,0,1)\,ds. \end{aligned}$$

From (H_2), (H_4), we have

$$f(s,1,0) \geq f(s,0,1) \geq \delta_0 g(s,0) \geq 0.$$

Since $g(t,0) \not\equiv 0$, we get

$$\int_0^1 f(s,1,0)\,ds \geq \int_0^1 f(s,0,1)\,ds \geq \delta_0 \int_0^1 g(s,0)\,ds > 0,$$

and in consequence,

$$l_1 := \frac{1}{\Gamma(\nu)} \int_0^1 (1-s)^{\nu-\alpha-1} f(s,1,0)\,ds > 0,$$
$$l_2 := \frac{1}{\Gamma(\nu)} \int_0^1 \bigl[1-(1-s)^{\alpha}\bigr](1-s)^{\nu-\alpha-1} f(s,0,1)\,ds > 0.$$

So, $l_2 h(t) \leq A(h,h)(t) \leq l_1 h(t)$, $t \in [0,1]$; and hence we have $A(h,h) \in P_h$. Similarly,

$$\frac{1}{\Gamma(\nu)} h(t) \int_0^1 \bigl[1-(1-s)^{\alpha}\bigr](1-s)^{\nu-\alpha-1} g(s,0)\,ds \leq Bh(t) \leq \frac{1}{\Gamma(\nu)} h(t) \int_0^1 (1-s)^{\nu-\alpha-1} g(s,1)\,ds,$$

from $g(t,0) \not\equiv 0$, we easily prove $Bh \in P_h$. Hence the condition (i) of Lemma 2.8 is satisfied.

In the following, we show the condition (ii) of Lemma 2.8 is satisfied. For $u, v \in P$, and any $t \in [0,1]$, from (H_4),

$$A(u,v)(t) = \int_0^1 G(t,s) f\bigl(s,u(s),v(s)\bigr)\,ds \geq \delta_0 \int_0^1 G(t,s) g\bigl(s,u(s)\bigr)\,ds = \delta_0 Bu(t).$$

Then we get $A(u,v) \geq \delta_0 Bu$, for $u, v \in P$. Finally, an application of Lemma 2.8 implies: there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 < v_0$, $u_0 \leq A(u_0,v_0) + Bu_0 \leq A(v_0,u_0) + Bv_0 \leq v_0$; the operator equation $A(u,u) + Bu = u$ has a unique solution u^* in P_h; for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1,2,\ldots,$$

we have $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$. That is,

$$u_0(t) \leq \int_0^1 G(t,s)\bigl[f\bigl(s,u_0(s),v_0(s)\bigr) + g\bigl(s,u_0(s)\bigr)\bigr]\,ds, \quad t \in [0,1],$$
$$v_0(t) \geq \int_0^1 G(t,s)\bigl[f\bigl(s,v_0(s),u_0(s)\bigr) + g\bigl(s,v_0(s)\bigr)\bigr]\,ds, \quad t \in [0,1];$$

FBVP (1.1) has a unique positive solution u^* in P_h; for $x_0, y_0 \in P_h$, the sequences

$$x_{n+1}(t) = \int_0^1 G(t,s)\bigl[f\bigl(s,x_n(s),y_n(s)\bigr) + g\bigl(s,x_n(s)\bigr)\bigr]\,ds, \quad n = 0,1,2,\ldots,$$
$$y_{n+1}(t) = \int_0^1 G(t,s)\bigl[f\bigl(s,y_n(s),x_n(s)\bigr) + g\bigl(s,y_n(s)\bigr)\bigr]\,ds, \quad n = 0,1,2,\ldots,$$

satisfy $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$. □

Theorem 3.2 *Assume* (H_1), (H_2) *and*

(H_5) *there exists a constant* $\beta \in (0,1)$ *such that* $g(t,\lambda u) \geq \lambda^{\beta} g(t,u)$, $\forall t \in [0,1]$, $\lambda \in (0,1)$, $u \in [0,+\infty)$, *and* $f(t,\lambda u,\lambda^{-1} v) \geq \lambda f(t,u,v)$ *for* $\lambda \in (0,1)$, $t \in [0,1]$, $u, v \in [0,+\infty)$;

(H_6) *$f(t,0,1) \not\equiv 0$ for $t \in [0,1]$ and there exists a constant $\delta_0 > 0$ such that $f(t,u,v) \leq \delta_0 g(t,u)$, $t \in [0,1]$, $u,v \geq 0$.*

Then:

(1) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 < v_0$ and*

$$u_0(t) \leq \int_0^1 G(t,s)\big[f\big(s,u_0(s),v_0(s)\big) + g\big(s,u_0(s)\big)\big]\,ds, \quad t \in [0,1],$$
$$v_0(t) \geq \int_0^1 G(t,s)\big[f\big(s,v_0(s),u_0(s)\big) + g\big(s,v_0(s)\big)\big]\,ds, \quad t \in [0,1],$$

where $h(t) = t^{\nu-1}$, $t \in [0,1]$ and $G(t,s)$ is given as in (2.2);

(2) *FBVP* (1.1) *has a unique positive solution u^* in P_h;*

(3) *for any $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_{n+1}(t) = \int_0^1 G(t,s)\big[f\big(s,x_n(s),y_n(s)\big) + g\big(s,x_n(s)\big)\big]\,ds, \quad n = 0,1,2,\ldots,$$
$$y_{n+1}(t) = \int_0^1 G(t,s)\big[f\big(s,y_n(s),x_n(s)\big) + g\big(s,y_n(s)\big)\big]\,ds, \quad n = 0,1,2,\ldots,$$

we have $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$.

Sketch of the proof Consider two operators A, B defined in the proof of Theorem 3.1. Similarly, from (H_1), (H_2), we obtain that $A: P \times P \to P$ is a mixed monotone operator and $B: P \to P$ is increasing. From (H_5), we have

$$A\big(\lambda u, \lambda^{-1}v\big) \geq \lambda A(u,v); \qquad B(\lambda u) \geq \lambda^{\beta} Bu, \quad \text{for } \lambda \in (0,1), u,v \in P.$$

From (H_2), (H_6), we have

$$g(s,0) \geq \frac{1}{\delta_0} f(s,0,1), \qquad f(s,1,0) \geq f(s,0,1), \quad s \in [0,1].$$

Since $f(t,0,1) \not\equiv 0$, we get

$$\int_0^1 f(s,1,0)\,ds \geq \int_0^1 f(s,0,1)\,ds > 0,$$
$$\int_0^1 g(s,1)\,ds \geq \int_0^1 g(s,0)\,ds \geq \frac{1}{\delta_0}\int_0^1 f(s,0,1)\,ds > 0,$$

and in consequence,

$$\frac{1}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-\alpha-1} f(s,1,0)\,ds \geq \frac{1}{\Gamma(\nu)}\int_0^1 \big[1-(1-s)^{\alpha}\big](1-s)^{\nu-\alpha-1} f(s,0,1)\,ds > 0,$$
$$\frac{1}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-\alpha-1} g(s,1)\,ds \geq \frac{1}{\Gamma(\nu)}\int_0^1 \big[1-(1-s)^{\alpha}\big](1-s)^{\nu-\alpha-1} g(s,0)\,ds > 0.$$

So, we can easily prove that $A(h,h) \in P_h$, $Bh \in P_h$. For $u,v \in P$, and any $t \in [0,1]$, from (H_6),

$$A(u,v)(t) = \int_0^1 G(t,s) f\big(s,u(s),v(s)\big)\,ds \leq \delta_0 \int_0^1 G(t,s) g\big(s,u(s)\big)\,ds = \delta_0 Bu(t).$$

Then we get $A(u,v) \leq \delta_0 Bu$, for $u, v \in P$. Finally, an application of Lemma 2.9 implies: there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 < v_0$, $u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0$; the operator equation $A(u,u) + Bu = u$ has a unique solution u^* in P_h; for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots,$$

we have $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$. That is,

$$u_0(t) \leq \int_0^1 G(t,s)\bigl[f\bigl(s, u_0(s), v_0(s)\bigr) + g\bigl(s, u_0(s)\bigr)\bigr]\,ds, \quad t \in [0,1],$$
$$v_0(t) \geq \int_0^1 G(t,s)\bigl[f\bigl(s, v_0(s), u_0(s)\bigr) + g\bigl(s, v_0(s)\bigr)\bigr]\,ds, \quad t \in [0,1];$$

FBVP (1.1) has a unique positive solution u^* in P_h; for $x_0, y_0 \in P_h$, the sequences

$$x_{n+1}(t) = \int_0^1 G(t,s)\bigl[f\bigl(s, x_n(s), y_n(s)\bigr) + g\bigl(s, x_n(s)\bigr)\bigr]\,ds, \quad n = 0, 1, 2, \ldots,$$
$$y_{n+1}(t) = \int_0^1 G(t,s)\bigl[f\bigl(s, y_n(s), x_n(s)\bigr) + g\bigl(s, y_n(s)\bigr)\bigr]\,ds, \quad n = 0, 1, 2, \ldots,$$

satisfy $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$. □

From Remark 2.10 and similar to the proofs of Theorems 3.1-3.2, we can prove the following conclusions.

Corollary 3.3 *Let $g \equiv 0$. Assume that f satisfies the conditions of Theorem* 3.1 *and $f(t,0,1) \not\equiv 0$. Then*: (i) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 < v_0$ and*

$$u_0(t) \leq \int_0^1 G(t,s) f\bigl(s, u_0(s), v_0(s)\bigr)\,ds,$$
$$v_0(t) \geq \int_0^1 G(t,s) f\bigl(s, v_0(s), u_0(s)\bigr)\,ds, \quad t \in [0,1],$$

where $h(t) = t^{\nu-1}$, $t \in [0,1]$ and $G(t,s)$ is given as in (2.2); (ii) *the FBVP*

$$\begin{cases} -D_{0^+}^{\nu} y(t) = f(t, y(t), y(t)), & 0 < t < 1, n-1 < \nu \leq n, \\ y^{(i)}(0) = 0, & 0 \leq i \leq n-2, \\ [D_{0^+}^{\alpha} y(t)]_{t=1} = 0, & 1 \leq \alpha \leq n-2, \end{cases}$$

has a unique positive solution u^* in P_h; (iii) *for any $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_{n+1}(t) = \int_0^1 G(t,s) f\bigl(s, x_n(s), y_n(s)\bigr)\,ds,$$
$$y_{n+1}(t) = \int_0^1 G(t,s) f\bigl(s, y_n(s), x_n(s)\bigr)\,ds, \quad n = 0, 1, 2, \ldots,$$

we have $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$.

Corollary 3.4 *Let $f \equiv 0$. Assume that g satisfies the conditions of Theorem* 3.2 *and $g(t,0) \not\equiv 0$ for $t \in [0,1]$. Then*: (i) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \le u_0 < v_0$ and*

$$u_0(t) \le \int_0^1 G(t,s)g\big(s,u_0(s)\big)\,ds, \qquad v_0(t) \ge \int_0^1 G(t,s)g\big(s,v_0(s)\big)\,ds, \quad t \in [0,1],$$

where $h(t) = t^{\nu-1}$, $t \in [0,1]$ and $G(t,s)$ is given as in (2.2); (ii) *the FBVP*

$$\begin{cases} -D_{0^+}^{\nu} y(t) = g(t,y(t)), & 0 < t < 1, n-1 < \nu \le n, \\ y^{(i)}(0) = 0, & 0 \le i \le n-2, \\ [D_{0^+}^{\alpha} y(t)]_{t=1} = 0, & 1 \le \alpha \le n-2, \end{cases}$$

has a unique positive solution u^* in P_h; (iii) *for any $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_{n+1}(t) = \int_0^1 G(t,s)g\big(s,x_n(s)\big)\,ds, \qquad y_{n+1}(t) = \int_0^1 G(t,s)g\big(s,y_n(s)\big)\,ds, \quad n = 0,1,2,\ldots,$$

we have $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$.

4 An example

We now present one example to illustrate Theorem 3.1.

Example 4.1 Consider the following FBVP:

$$\begin{cases} -D_{0^+}^{6.3} u(t) = u^{\frac{1}{4}}(t) + [u(t)+2]^{-\frac{1}{3}} + \frac{u(t)}{1+u(t)}a(t) + b(t) + c, & 0 < t < 1, \\ u^{(i)}(0) = 0, & 0 \le i \le 5, \\ [D_{0^+}^{4.2} u(t)]_{t=1} = 0, \end{cases} \tag{4.1}$$

where $c > 0$ is a constant, $a, b : [0,1] \to [0,\infty)$ are continuous with $a \not\equiv 0$.

Obviously, problem (4.1) fits the framework of FBVP (1.1) with $\nu = 6.3$, $\alpha = 4.2$. (Note that $n = 7$, therefore, in this case.) In this example, we take $0 < d < c$ and let

$$f(t,u,v) = u^{\frac{1}{4}} + [v+2]^{-\frac{1}{3}} + b(t) + d, \qquad g(t,u) = \frac{u}{1+u}a(t) + c - d,$$
$$\beta = \frac{1}{3}, \qquad a_{\max} = \max\big\{a(t) : t \in [0,1]\big\}.$$

Obviously, $a_{\max} > 0$; $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous and $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous with $g(t,0) = c - d > 0$. And $f(t,u,v)$ is increasing in $u \in [0,+\infty)$ for fixed $t \in [0,1]$ and $v \in [0,+\infty)$, decreasing in $v \in [0,+\infty)$ for fixed $t \in [0,1]$ and $u \in [0,+\infty)$, and $g(t,u)$ is increasing in $u \in [0,+\infty)$ for fixed $t \in [0,1]$. Besides, for $\lambda \in (0,1)$, $t \in [0,1]$, $u \in [0,\infty)$, we have

$$g(t,\lambda u) = \frac{\lambda u}{1+\lambda u}a(t) + c - d \ge \frac{\lambda u}{1+u}a(t) + \lambda(c-d) = \lambda g(t,u),$$
$$\begin{aligned} f\big(t,\lambda u,\lambda^{-1}v\big) &= \lambda^{\frac{1}{4}}u^{\frac{1}{4}} + \lambda^{\frac{1}{3}}[v+2\lambda]^{-\frac{1}{3}} + b(t) + d \\ &\ge \lambda^{\frac{1}{3}}\big\{u^{\frac{1}{4}} + [v+2]^{-\frac{1}{3}} + b(t) + d\big\} = \lambda^{\beta} f(t,u,v). \end{aligned}$$

Moreover, if we take $\delta_0 \in (0, \frac{d}{a_{\max}+c-d}]$, then we obtain

$$\begin{aligned} f(t,u,v) &= u^{\frac{1}{4}} + [v+2]^{-\frac{1}{3}} + b(t) + d \geq d = \frac{d}{a_{\max}+c-d} \cdot (a_{\max}+c-d) \\ &\geq \delta_0 \left[\frac{u}{1+u} a(t) + c - d \right] = \delta_0 g(t,u). \end{aligned}$$

Hence all the conditions of Theorem 3.1 are satisfied. An application of Theorem 3.1 implies that problem (4.1) has a unique positive solution in P_h, where $h(t) = t^{\nu-1} = t^{5.3}$, $t \in [0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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